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Robust optimality of linear saturated control in uncertain linear network flows

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***Abstract*— We propose a novel approach that, given a linear saturated feedback control policy, asks for the objective function that makes robust optimal such a policy. The approach is specialized to a linear network flow system with unknown but bounded demand and politopic bounds on controlled flows. All results are derived via the Hamilton-Jacobi-Isaacs and viscosity theory.**

Keywords: Optimal control, Robust optimization, Inventory control, Viscosity solutions.

I. Introduction

Consider the problem of driving a continuous time state $z(t) \in \mathbb{R}^m$ within a target set $\mathcal{T} = \{\xi \in \mathbb{R}^m : |\xi| \leq \epsilon\}$ in a finite time $T \geq 0$ with $\epsilon \geq 0$ a-priori chosen and keeping the state within $\mathcal{T}$ from time T on. Such a problem is shortly referred to as the ϵ-*stabilizability* problem of $z(t)$. Define $u(t) \in \mathbb{R}^m$ the controlled flow vector, $w(t) \in \mathbb{R}^n$ an *Unknown But Bounded (UBB)* exogenous input (disturbance/demand) with $n < m$, and let $D \in \mathbb{R}^{n \times m}$ a given matrix, $\mathcal{U} = \{\mu \in \mathbb{R}^m : u^- \leq \mu \leq u^+\}$ and $\mathcal{W} = \{\eta \in \mathbb{R}^n : w^- \leq \eta \leq w^+\}$ be two hyper-boxes with assigned u^+, u^-, w^+ and w^-. Also, let σ be a binary state such that $\sigma(t) = 0$ if $z(t) \notin \mathcal{T}$ and $\sigma(t) = 1$ if $z(t) \in \mathcal{T}$. The robust counterpart of the problem takes on the form

$$\min_{u \in U} \max_{w \in W} \; J(\zeta, u(.), w(.)) = \int_0^\infty e^{-\lambda(\sigma)t} g^\sigma(z(t), u(t)) dt \quad (1)$$

$$\dot{z}(t) = u(t) - Dw(t), \; z(0) = \zeta \quad \text{for all } t \geq 0 \quad (2)$$

$$z(t) \in \mathcal{T} \quad \text{for all } t \geq T, \quad (3)$$

where we denote by $U = \{u : [0, +\infty[\to \mathcal{U}\}$ and by $W = \{w : [0, +\infty[\to \mathcal{W}\}$ the sets of measurable controls and demands respectively. From a game theoretic standpoint we will consider two players, player 1 playing u and player 2 playing w. The state $z(t)$ has initial value ζ and integrates the discrepancy between the controlled flow $u(t)$ and $Dw(t)$ as described in (2). Controls $u(t)$ and demand $w(t)$ are bounded within hyperboxes by their definitions. Condition (3) guarantees the reachability of the target set from time T on. Among all controls satisfying the above conditions (call it admissible controls or solution), we wish to find the one that minimizes the objective function (1) under the worst demand. The objective function is defined on an infinite horizon with discount factor $e^{-\lambda(\sigma)t}$ depending on σ. The reason for such a dependency on σ will be clearer later on. The integrand in (1) is a function of z and u and its structure depends on σ as follows

$$g^\sigma(z(t), u(t)) = \begin{cases} \tilde{g}(z(t), u(t)) & \text{if } \sigma = 0 \; (z(t) \notin \mathcal{T}) \\ \hat{g}(z(t), u(t)) & \text{if } \sigma = 1 \; (z(t) \in \mathcal{T}) \end{cases} \quad (4)$$

where $\tilde{g}(.)$ and $\hat{g}(.)$ have to be designed as explained below.

In a previous work [2], it has been shown that under certain conditions on the matrix D (recalled below), the following (linear) saturated control policy drives the state z within $\mathcal{T}$:

$$\begin{aligned} & u(t) = sat_{[u^-,u^+]}(-kz(t)) := \\ & := \big(sat_{[u^-,u^+]}(-kz_1(t)), ..., sat_{[u^-,u^+]}(-kz_n(t))\big) \in \mathbb{R}^n, \end{aligned} \quad (5)$$

with $k > 0$ and where

$$sat_{[\alpha,\beta]}(\xi_i) = \begin{cases} \beta, & \text{if } \; \xi_i > \beta, \\ \xi_i, & \text{if } \; \alpha \leq \xi_i \leq \beta, \\ \alpha, & \text{if } \; \xi_i < \alpha. \end{cases}$$

Then, we deduce that the saturated control policy returns an admissible solution for problem (1)-(3). In the light of this consideration, we focus on the following problem.

Problem 1: We wish to design the integrand $g^\sigma(.)$ of the objective function (1) in (4) such that the saturated control turns optimal for the min-max problem (1)-(3).

A. Literature and main results

In this work, we add new results concerning the optimality of the saturated control policy, which is proved to solve the ϵ-stabilizability problem in [2]. Our interest for the saturated control is also due to the fact that it represents the simplest form of a piece-wise linear control [6]. The idea of modeling the demand as unknown but bounded variable is in line with some recent literature on robust optimization [3], [5], [10] though the "unknown but bounded" approach has a long history in control [4]. The conservative approach of Section V reminds the Soyster decomposition [12], used in robust linear programming. Also, the notion of *feedback* in control, present in this work, reminds the notion of *recourse* used in robust optimization [7]. Concerning the nature of the problem, we wish to emphasize our reversed perspective: given a solution (the saturated control policy) we ask for the objective function that makes the solution optimal. Similarly to the *dual mode control* in [11] we provide a solution approach which decomposes the problem into two subproblems, within and without a predefined neighborhood of the origin. All results are derived via the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs equations and the related viscosity solutions theory (see Bardi-Capuzzo Dolcetta [1] as a general reference).

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B. Some basic facts about Hamilton-Jacobi equations, optimal control and differential games

A Hamilton-Jacobi equation is a first order partial differential equation of the form

$$F(x, v(x), \nabla v(x)) = 0 \text{ in } \Omega,$$

where $\Omega \subseteq \mathbb{R}^m$, is open, $F : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is continuous. A viscosity solution of it is a continuous function $v : \Omega \to \mathbb{R}$ such that, for every $x \in \Omega$ and for every differentiable function $\varphi : \Omega \to \mathbb{R}$, the following holds

$$\begin{array}{ll} i) & x \text{ is local max for } v - \varphi \Rightarrow F(x, v(x), \nabla\varphi(x)) \leq 0; \\ ii) & x \text{ is local min for } v - \varphi \Rightarrow F(x, v(x), \nabla\varphi(x)) \geq 0. \end{array}$$

The idea is hence to substitute the derivatives of v, which usually do not exist, with the derivatives of the test function φ, and to require that the equation is "semi-verified" in the point of maximum for $v - \varphi$ and (oppositely) "semi-verified" in the point of minimum for $v - \varphi$. If a function satisfies i) only (for every test functions) then it is called a subsolution, whereas it is called a supersolution in the other case. Such a notion of solution goes back to Crandall-Evans-Lions [8]. Obviously, this is a weak definition of solution, and in particular, if a function v is a classical solution (i.e. it is differentiable and satisfies the equation by equality), then it is also a viscosity solution.

Let us consider an optimal control problem

$$\max_{\beta} J(x, \beta) \ \left(= \max_{\beta} \int_0^{\infty} e^{-t} \ell(y(t), \beta(t)) dt \right),$$
$$\text{subject to } \dot{y}(t) = f(y(t), \beta(t)), \quad y(0) = x,$$

where $\beta : [0, +\infty[\to B$ is the measurable control, with B a compact set. Under rather general hypotheses, the value function of the problem, $U(x) = \sup_{\beta} J(x, \beta)$, is a viscosity solution of the Hamilton-Jacobi-Bellman equation

$$U(x) + \min_{b \in B} \{-f(x, b) \cdot \nabla U(x) - \ell(x, b)\} = 0.$$

Such an equation holds in the whole $\mathbb{R}^m$ if the control problem is without state-constraints (i.e. the state $y(\cdot)$ is free to move in $\mathbb{R}^m$); otherwise, if the problem is confined in the closure $\overline{\Omega}$ of an open set Ω, the equation must be coupled with suitable boundary conditions on $\partial\Omega$, usually given by an exit cost ψ from $\overline{\Omega}$. The problem is then

$$\max_{\beta} J(x, \beta) \quad \left(= \max_{\beta} \int_0^{t_x(\beta)} e^{-t} \ell(y(t), \beta(t)) dt + \right.$$
$$\left. + e^{-t_x(\beta)} \psi(y(t)) \right),$$

where $t_x(\beta)$ is the first exit time from $\overline{\Omega}$ for the trajectory starting from x with control β (with the convention $t_x(\beta) = +\infty$ if the trajectory never exit from $\overline{\Omega}$).

Under some hypotheses on the regularity of $\partial\Omega$ and on the existence of inner suitable fields on the points of the boundary, the value function turns out to satisfy the boundary condition $U = \psi$ in the so-called "viscosity sense". This means that on the point x of the boundary which are of local maximum (respectively local minimum) for $U - \varphi$ (when restricted to the closure of Ω), we must have $U(x) \leq \psi(x)$ (resp. $U(x) \geq \psi(x)$) or $U(x) + min_{b \in B} \{-f(x, b) \cdot \nabla\varphi(x) - \ell(x, b)\} \leq 0$ (resp. ≥ 0), i.e. the equation holds with the "right" sign.

Under general hypotheses (on the regularity of Ω, and some "compatibility conditions" for the exit-cost ψ), the value function is characterized as the unique bounded uniformly continuous viscosity solution of the boundary value problem for the Hamilton-Jacobi-Bellman equation (note that if Ω is $\mathbb{R}^m$, then there are not boundary conditions).

Now we consider a differential game with state equation

$$y'(t) = f(y(t), \alpha(t), \beta(t)), \quad y(0) = x,$$

and cost functional

$$J(x, \alpha, \beta) = \int_0^{+\infty} e^{-t} \ell(y(t), \alpha(t), \beta(t)) dt,$$

for the infinite horizon case (i.e. without restriction to $\overline{\Omega}$), or

$$\begin{array}{rcl} J(x, \alpha, \beta) & = & \int_0^{t_x(\alpha,\beta)} e^{-t} \ell(y(t), \alpha(t), \beta(t)) dt + \\ & + & e^{-t_x(\alpha,\beta)} \psi(y(t)), \end{array} \qquad (6)$$

for the exit-time problem. The measurable control $\alpha \in \mathcal{A} = \{\alpha : [0, +\infty[\to A, \text{ measurable}\}$ is governed by the first player who wants to minimize the cost, whereas and the second player, by choosing the measurable control $\beta \in \mathcal{B} = \{\beta : [0, +\infty[\to B, \text{ measurable}\}$, wants to maximize the cost. We define the non-anticipative strategies (see, e.g., [9]) for the first player

$$\Gamma = \left\{ \gamma : \mathcal{B} \to \mathcal{A}, \beta \mapsto \gamma[\beta] \middle| \beta_1 = \beta_2 \text{ in } [0, s] \Longrightarrow \gamma[\beta_1] = \gamma[\beta_2] \text{ in } [0, s] \right\}. \qquad (7)$$

Hence the (lower) value function for the minimization/maximization problem is defined as

$$V(x) = \min_{\gamma \in \Gamma} \max_{\beta \in \mathcal{B}} J(x, \gamma[\beta], \beta).$$

Under rather general hypothesis, the value function V is the unique bounded uniformly continuous viscosity solution of the following Hamilton-Jacobi-Isaacs equation

$$V(x) + \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot \nabla V(x) - \ell(x, a, b)\} = 0,$$

which also in this case must be coupled with appropriate boundary conditions for the exit-time problem.

II. Solution approach

We will pursue the idea of decomposing the infinite horizon (1) into a *finite horizon* problem with $\sigma(t) = 0$ and an infinite *infinite horizon* problem with $\sigma(t) = 1$ as expressed below

$$\begin{array}{c} J(\zeta, u(.), w(.)) = [\int_0^T \tilde{g}(z(t), u(t), w(t)) dt + \\ \int_T^{\infty} e^{-t} \hat{g}(z(t), u(t), w(t)) dt]. \end{array} \qquad (8)$$

We can do such a decomposition as once the state enters the target $\mathcal{T}$ it will remain in it for the rest of the time [2].

Let us now explain more in details the notion of optimality of a saturated control mentioned in Problem 1. Let U and W be the sets of measurable controls and demands as in the

Introduction (after equation (3)), and let Γ be the set of non-anticipative strategies for the player 1 (see (7), replacing $\mathcal{B}$ by W, $\mathcal{A}$ by U, and β by u). The (lower) value function for the differential game is then

$$V(\zeta) = \inf_{\gamma \in \Gamma} \sup_{w \in W} J(\zeta, \gamma[w], w),$$

where ζ is the initial state. Now, V must be the unique viscosity solution of the Hamilton-Jacobi-Isaacs (HJI) equation

$$\sigma V(\zeta) + H(\zeta, \nabla V(\zeta)) = 0, \tag{9}$$

where the Hamiltonian H is, for every $(\zeta, p) \in \mathbb{R}^m \times \mathbb{R}^m$:

$$H(\zeta, p) := \min_{\omega \in \mathcal{W}} \max_{\mu \in \mathcal{U}} \{-(\mu - D\omega) \cdot p - g^\sigma(\zeta, \mu, \omega)\}. \tag{10}$$

Observe that the above equation depends on function $g^\sigma(.)$ and on σ. Hence when dealing with the infinite horizon $\sigma = 1$ and so in the left hand side of the equation there is the presence of the addend $+V(\zeta)$. We can look at the saturated control as a special non anticipative strategy γ_0, namely, for every $w \in W$ we define

$$\gamma_0[w](t) = sat_{[u^-, u^+]}(-kz),$$

where z is the state trajectory of (2) under the saturated control as choice for u and under the choice of w. Given this, we wish to find a function $g^\sigma(.)$ such that the worst cost returned by the saturated control equals the value function V. This corresponds to imposing

$$\tilde{V}(\zeta) := \sup_{w \in W} J(\zeta, \gamma_0[w], w) = V(\zeta),$$

i.e. we get the robust optimality of the saturated control if

$$\tilde{V} = V, \tag{11}$$

where $\tilde{V}$ is obtained by maximizing over w

$$\tilde{J}(\zeta, w) = \int_0^T e^{-\sigma(t)t} g^\sigma(z(t), sat_{[u^-, u^+]}(-kz(t)), w(t)) dt,$$

(in the infinite horizon the extremes are T and ∞) subject to the controlled dynamics

$$\dot{z}(t) = sat_{[u^-, u^+]}(-kz(t)) - Dw(t), \quad z(0) = \zeta \quad (\text{or } z(T) = \zeta)$$

Now, $\tilde{V}$ must be the unique viscosity solution of the Hamilton-Jacobi-Bellman (HB) equation:

$$\sigma \tilde{V}(\zeta) + \tilde{H}(\zeta, \nabla \tilde{V}(\zeta)) = 0, \tag{12}$$

where the Hamiltonian $\tilde{H}$ is, for every $(\zeta, p) \in \mathbb{R}^m \times \mathbb{R}^m$

$$\begin{aligned} \tilde{H}(\zeta, p) := \min\nolimits_{\omega \in \mathcal{W}} \{ & -(sat_{[u^-, u^+]}(-k\zeta) - D\omega) \cdot p + \\ & -g^\sigma(\zeta, sat_{[u^-, u^+]}(-k\zeta), \omega)\}. \end{aligned}$$

In the following, we will look for suitable cost g in order to get the optimality of the saturated control for the corresponding problems. We will prove such an optimality (i.e. (11)) in two different ways: i) directly computing the functions V, $\tilde{V}$ and checking their equality, 2) writing the two corresponding Hamilton-Jacobi equations and checking they have the same unique solution.

Remark 1: A trivial choice is $g^\sigma(\zeta, \mu, \omega) = |sat(-k\zeta) - \mu|$. It penalizes any control u different from the saturated control. However, such a choice makes the game (and the mathematical problem) without interest.

III. Minimum time problem outside the target set

Let us start by observing that we can always choose $\tilde{g}(.)$ big enough in comparison with $\hat{g}(.)$ such that, for all u and w, the second contribution $\int_T^\infty \hat{g}(z(t), u(t), w(t)) dt$ in (1) can be neglected if compared with the first contribution $\int_0^T \tilde{g}(z(t), u(t), w(t)) dt$. In particular this is true if we choose $\tilde{g}(\zeta, \mu, \omega) = M$ with $M > 0$ big enough. With the above choice the problem outside the target $\mathcal{T}$ is equivalent to a minimum time problem with $\tilde{g}(\zeta, \mu, \omega) \equiv 1$. With this in mind, take without loss of generality $\mathcal{U} = \{\mu \in \mathbb{R}^m \Big| -1 \leq \mu_i \leq 1 \ \forall i = 1, .., m\}$, $\mathcal{W} = \{\omega \in \mathbb{R}^n \Big| -1 \leq \omega_j \leq 1 \ \forall j = 1, ..., n\}$, and D an $m \times n$ matrix satisfying $\mathcal{U} \supset D\mathcal{W}$. We denote by D_{ij} the entries of the matrix. The target is $\mathcal{T} = \{\xi \in \mathbb{R}^m \Big| |\xi_i| \leq (1/k) \ \forall i = 1, ..., m\}$, and the saturated control policy is $u(t) = sat_{[-1,1]}(-kz(t))$. Hence, the two Hamiltonians become, for all $\zeta, p \in \mathbb{R}^m$ (recall that we are considering $\tilde{g} \equiv 1$),

$$\begin{aligned} H(\zeta, p) &= -\sum_{j=1}^n \left| \sum_{i=1}^m p_i D_{ij} \right| + \sum_{i=1}^m |p_i| - 1, \\ \tilde{H}(\zeta, p) &= -\sum_{j=1}^n \left| \sum_{i=1}^m p_i D_{ij} \right| - \sum_{i=1}^m sat_{[-1,1]}(-k\zeta_i) p_i - 1 \end{aligned} \tag{13}$$

By our hypotheses, the controllable set is $\mathbb{R}^n \setminus \mathcal{T}$, and hence V and $\tilde{V}$ are, respectively, the unique solutions of

$$\begin{array}{l} \left\{ \begin{array}{ll} H(\zeta, \nabla V(\zeta)) = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{T} \\ V = 0 & \text{on } \partial \mathcal{T}; \end{array} \right. \\ \left\{ \begin{array}{ll} \tilde{H}(\zeta, \nabla \tilde{V}(\zeta)) = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{T} \\ \tilde{V} = 0 & \text{on } \partial \mathcal{T}; \end{array} \right. \end{array} \tag{14}$$

The question is then to prove that such two problems have the same solution (note that we do not a priori know V and $\tilde{V}$). Anyway, in this case, due to the structure of the system and to other hypotheses, we can easy guess that the saturated control is an optimal choice for player 1. Indeed, since, whatever $w(t)$ is, for every i-th component, $(Dw(t))_i$ cannot change the sign of $sat_{[-1,1]}(-kz_i(t)) - (Dw(t))_i$ (when the initial point satisfies $|\zeta_i| > (1/k)$), and since that is the "good sign" for steering ζ to the target, then any controller will use such a control (or non anticipative strategy).

In the light of the above considerations, for the value function, it is reasonable to consider the following expression

$$V(\zeta) = \max_{i=1,...,m} \left\{ \frac{\max\left\{0, |\zeta_i| - \frac{1}{k}\right\}}{1 - \sum_{j=1}^n |D_{ij}|} \right\}. \tag{15}$$

That is V is the time requested for steering all the components in the interval $[-1/k, 1/k]$, under the worst scenario concerning the demand w.

Let i^* be the solution of the above maximization (the last component to reach the target set), the generic component of the costate is $p_{i^*} = \frac{1}{1 - \sum_{j=1}^n |D_{i^*j}|}$ and $p_j = 0$ for all $j \neq i^*$. The optimal choice for w is $w_j = sign(D_{i^*j})$. It is easy to check that the two Hamilton-Jacobi problems in (14) are both satisfied by (15) where such a function is differentiable. On the other hand, on the points where it is not differentiable (i.e. the point where the maximizing index in (15) changes), the definition of viscosity solution applies. We can also note that, on such points of non-differentiability (which are located on some portion of hyperplanes), we can only have test function φ such that $V - \varphi$ has a minimum, and also that the i-th component of the gradient $\nabla\varphi$ has the same sign of ζ_i. Hence the left-hand side of the equations in (14) are the same (see also (13)).

It must be noted that while the saturated control is unique optimal for the component i^*, this is no longer true for all the other components $j \neq i^*$. Actually, all z_j with $j \neq i^*$ once reached the target may exit and enter again several times following an infinite number of different trajectories and this until also z_{i^*} reaches the target set.

IV. A "QUADRATIC COST" WITHIN THE TARGET SET

Within the target set we consider the following quadratic cost depending on ζ, μ, ω for fixed $k > 0$:

$$\hat{g}(\zeta,\mu,\omega) = \frac{k+1}{2}\|\zeta + \frac{D\omega}{k}\|^2 + \frac{1}{2k}\|\mu - D\omega\|^2 + C\|D(\omega - \bar{\omega})\|^2,$$

where, $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$, $\overline{\omega} \in \mathcal{W}$ is a generic vertex, a-priori chosen, of $\mathcal{W}$ and $C \geq 0$ is a suitable constant, which will be fixed later. Our guess is that, inside the target $\mathcal{T}$, the saturated control is the unique optimal strategy for the min-max problem related to the cost $\hat{g}(.)$, and suitable exit-cost from the closed set $\mathcal{T}$.

Since we are decoupling the initial problem in two problems, outside and inside the target $\mathcal{T}$, in this section we may consider the infinite horizon problem with initial time $T = 0$.

First of all, let us consider the maximization problem over $w \in W$, with control u equal to the linear saturated one:

$$\begin{aligned} &\tilde{V}(\zeta) = \sup_{w \in w} \int_0^\infty e^{-t}\hat{g}(z(t), -kz(t), \omega), \\ &\text{subject to } \dot{z}(t) = -kz(t) - D\omega,\ z(0) = \zeta \in \mathcal{T}. \end{aligned} \tag{16}$$

Note that here we are not imposing an exit cost from $\mathcal{T}$. Indeed, it is without meaning in this case since, whichever the control w is, the trajectory can not exit from $\mathcal{T}$.

We now specialize the constant $C \geq 0$ in the definition of the cost $\hat{g}$. We choose a vertex $\overline{\omega}$ of $\mathcal{W}$, and $C \geq 0$ such that, for all $\zeta \in \mathcal{T}$, the maximum over $\omega \in \mathcal{W}$ of the expression

$$\frac{2k+1}{2}\left\|\zeta + \frac{D\omega}{k}\right\|^2 - \zeta \cdot \frac{D\omega}{k} + \frac{D\omega}{k} \cdot \frac{D\overline{\omega}}{k} + C\|D(\omega - \bar{\omega})\|^2, \tag{17}$$

is always taken in $-\overline{\omega}$ (note that the first addendum of such expression is just the sum of the two first addenda of g when $\mu = -k\zeta$). This is possible by choosing $\overline{\omega}$ equal to one of the two opposite vertices which strictly maximize the norm of $D\omega$ (which exist since we may suppose the matrix D have positive entries), and then taking C such that

$$C \geq \max_{\zeta \in \mathcal{T}, \omega \in vert\mathcal{W}, \omega \neq -\overline{\omega}} \frac{\frac{2k+1}{k}\left(\left\|\zeta + \frac{D\omega}{k}\right\|^2 - \left\|\zeta - \frac{D\overline{\omega}}{2}\right\|^2\right) + -\zeta\cdot(D\omega + D\overline{\omega}) + \frac{D\overline{\omega}\cdot(D\omega + D\overline{\omega})}{k^2}}{4\|D\overline{\omega}\|^2 - \|D\omega - D\overline{\omega}\|^2}. \tag{18}$$

Now, if we fix $w(t) \equiv -\overline{\omega}$ for all t, then, since $u(t) = -kz(t)$, the trajectory is given by

$$z(t) = e^{-kt}\left(\zeta - \frac{D\overline{\omega}}{k}\right) + \frac{D\overline{\omega}}{k}.$$

Hence, the cost associated to such a choice of w is (after simple calculation)

$$\tilde{J}(\zeta) = \frac{1}{2}\left\|\zeta - \frac{D\overline{\omega}}{k}\right\|^2 + 4C\|D\overline{\omega}\|^2,$$

We guess, not surprisingly, that $\tilde{J}$ is indeed the value function $\tilde{V}$ of the maximization problem. This can be done, for instance, by proving that $\tilde{J}$ solves the corresponding Hamilton-Jacobi equation

$$\tilde{J}(\zeta) + \min_{\omega \in \mathcal{W}}\left\{-(-k\zeta - D\omega)\cdot\nabla\tilde{J}_\omega(\zeta) - \hat{g}(\zeta, -k\zeta, \omega)\right\} = 0. \tag{19}$$

This can be easily checked, since $\tilde{J}$ is differentiable and hence a classical solution of (19) (when we put the gradient of $\tilde{J}$ inside the equation, by the hypothesis about the maximization of (17) we immediately get that the minimum in the left-hand side is reached in $-\overline{\omega}$, and hence we conclude). By uniqueness of the solution of (19), $\tilde{J}$ must coincide with the value function $\tilde{V}$.

We now consider the differential game, subject to (2), with running cost $\hat{g}$, and exit-cost from $\mathcal{T}$ given by

$$\psi(\zeta) = \frac{1}{2}\|\zeta - \frac{D\overline{\omega}}{k}\|^2 + 4C\|D\overline{\omega}\|^2,$$

Following the solution approach explained in Section II, we guess that the (lower) value function V for such a problem, coincides with the function $\tilde{V} = \tilde{J}$ already found.

By the general results, as explained in the Introduction, the lower value function V is the unique bounded continuous viscosity solution of the boundary value problem

$$\begin{cases} V(\zeta) + \min\limits_{\omega \in \mathcal{W}}\max\limits_{\mu \in \mathcal{U}} & \{-(\mu - D\omega)\cdot\nabla V(\zeta) + \\ & -\hat{g}(\zeta,\mu,\omega)\} = 0 \text{ in } \mathcal{T}, \\ V = \psi \text{ on } \partial\mathcal{T}, & \end{cases} \tag{20}$$

where the boundary condition are in the viscosity sense.

If now we specialize a little bit more the constant $C \geq 0$ in the definition of $\hat{g}$, we may get that $\tilde{V}$ is also a solution of (20) (note that it satisfies the boundary condition in the classical way, and hence also in the viscosity sense). This is possible by the following observations. Let us put $\tilde{V}$ (which is differentiable) and its gradient in (20). For every $\omega \in \mathcal{W}$, let $\mu_\omega \in \mathcal{U}$ reach the maximum in the left-hand side. Now, note that our condition on C is only a lower bound. Hence we may take C larger than its lower bound. In particular, since, for every $\zeta \in \mathcal{T}$, $\mu \in \mathcal{U}$, $\omega \in \mathcal{W}$, the difference $|\hat{g}(\zeta, -k\zeta, \omega) - \hat{g}(\zeta,\mu,\omega)|$, which is

$$\frac{1}{2k}\left|\|k\zeta + D\omega\|^2 - \|\mu - D\omega\|^2\right|,$$

is small of order $1/k$, we can take C a little bit larger than its lower bound such that it is also true that, for every $\zeta \in \mathcal{T}$, the minimum with respect to $\omega \in \mathcal{W}$ of the expression

$$-(\mu_\omega - D\omega) \cdot \nabla V(\zeta) - g(\zeta, \mu_\omega, \omega),$$

is taken in $-\overline{\omega}$. But, as standard calculations show, the only possibility is $\mu_{-\overline{\omega}} = -k\zeta$ and hence $\tilde{V}$ solves (20). By uniqueness, we then get $\tilde{V} = V$, and $u(t) = -kz(t)$ is the unique possibility for optimality.

Remark 2: Argument of future works is searching a suitable running cost $\hat{g}$ which leaves the demand free to switch (at least) between two opposite vertices.

V. A conservative approximation

In this section, we propose a conservative approach that allows us to solve the original problem without decomposing it into the finite an infinite horizon problem. Let us split the demand $w(t)$ into m independent demands $w^{(i)}(t)$ each one acting on a different component. This corresponds to considering m decoupled one-dimensional dynamics of type

$$\dot{z}_i(t) = u_i(t) - \sum_{j=1}^{n} D_{ij} w_j^{(i)}(t).$$

In the rest of the section, we focus on the one dimensional version of our problem and drop the index i where possible. In the one dimensional context, it is natural to think (and we will prove it in the sequel, for a suitable cost) that the optimal choice for the player 2 is to use

$$w(t) = -sign(z(t)) \arg\max_{\omega} |\sum_{j=1}^{n} D_{ij}\omega_j| \in [-1,1]^n, \quad (21)$$

where, if $z(t) = 0$, then $w(t)$ may be any value from $[-1,1]^n$. Now, consider the following objective function

$$g(\zeta, \mu) = \max\{|sat_{[-1,1]}(-k\zeta)|, |\mu|\},$$

and the corresponding infinite horizon game with cost

$$J(\zeta, u(.), w(.)) = \int_0^\infty e^{-t} g(z(t), u(t)) dt,$$

where e^{-t} is a given discount term. We want to show that the saturated strategy, is the optimal one for the first player. Hence, first of all, let us prove the (non surprising) optimality of (21) for the second player in the corresponding maximizing optimal control problem when $u(t) = sat_{[-1,1]}(-kz(t))$, that is when the cost is just equal to $|sat_{[-1,1]}(-kz(t))|$. Defining $c = \sum_{j=1}^{n} |D_{ij}|$, the system is (recall $0 < c < 1$)

$$\dot{z}(t) = sat_{[-1,1]}(-kz(t)) + c(sign(z(t))), \quad z(0) = \zeta. \quad (22)$$

Let us suppose $-(1/k) \le \zeta < 0$. Then the trajectory is (recall that if $z(t) \in [-(1/k), 0[$, then $sat_{[-1,1]}(-kz(t)) = -kz(t)$, and $sign(z(t)) = -1$)

$$z(t) = e^{-kt}\left(\zeta + \frac{c}{k}\right) - \frac{c}{k} \quad \forall t \ge 0$$

Note that such a trajectory is always negative and hence it is exactly the solution of the system (22) with the second member given by $-kz(t) - c$. Moreover, observe that it is increasing if $\zeta < -(c/k)$ and decreasing if $-(c/k) < \zeta < 0$. Hence, in any case, it converges (for $t \to +\infty$) to the equilibrium $-(c/k)$. Note that such an equilibrium point is just obtained from $(c\omega^\zeta)/k$ when $\omega^\zeta = -1$, that is when ω^ζ solves the problem $\omega^\zeta(c\zeta) = \max_{\omega \in [-1,1]} \omega(c\zeta)$. The cost of such a controlled trajectory is

$$\tilde{V}(\zeta) = \int_0^\infty e^{-t}(-kz(t))dt = \frac{k(c-\zeta)}{k+1}$$

Let us note that $\tilde{V}$ has a continuous derivative in $[-()1/k), 0[$, given by the negative constant $\tilde{V}'(\zeta) \equiv -k/(k+1)$. Moreover we have that $\tilde{V}(\zeta) \to \frac{kc}{k+1}$, for $\zeta \to 0^-$ and $\tilde{V}(\zeta) \to \frac{kc+1}{k+1}$, for $\zeta \to \frac{-}{1}k$.

If instead $\zeta \le -(1/k)$, then the system has the right hand side equal to $1 - c$ until z reaches the value $-(1/k)$ and after, again, equal to $-kz - c$. Since the reaching time is $\tau = -(k\zeta + 1)/(k - kc)$, dividing the system in the two intervals of time $[0, \tau]$ (with initial point ζ), and $[\tau, +\infty]$ (with initial point $-(1/k)$), we obtain the trajectory

$$z(t) = \begin{cases} \zeta + (1-c)t & 0 \le t \le \tau, \\ e^{-kt+k\tau}\left(\dfrac{c-1}{k}\right) - \dfrac{c}{k} & \end{cases}$$

Again, such a trajectory is increasing and converges to $-(c/k)$. The corresponding cost and cost derivative in $]-\infty, -(1/k)[$ are

$$\tilde{V}(\zeta) = -e^{-\tau}(1-c)\frac{k}{k+1} + 1, \quad \tilde{V}(\zeta)' = e^{-\tau}\frac{k}{k+1}.$$

Note that $\tilde{V}$ is then continuous and derivable, since $\tilde{V}(\zeta) \to \frac{kc+1}{k+1}$, for $\zeta \to -\frac{1}{k}^-$ and $\tilde{V}(\zeta)' \to -\frac{k}{k+1}$, for $\zeta \to -\frac{1}{k}^-$.

If instead $\zeta > 0$, a similar analysis as before gives

$$\begin{array}{ll} \tilde{V}(\zeta) = \frac{k}{k+1}(\zeta + c) & 0 < \zeta < (1/k) \\ \tilde{V}(\zeta) = -e^{-\tau}(1-c)\dfrac{k}{k+1} + 1 & (1/k) \le \zeta, \end{array}$$

where $\tau = (1 - k\zeta)/(k - kc)$ is the reaching time of the value $1/k$. Also in this case c/k is an attracting equilibrium point. Then, the value $\tilde{V}$ is continuous and derivable in $]-\infty, 0[\cup]0, +\infty[$. Moreover, it is continuous in $\zeta = 0$ where it is equal to $kc/(k+1)$ but is not derivable in $\zeta = 0$. The left limit of derivatives is $-k/(k+1)$ whereas the right limit is $k/k+1$. Hence, in the view of the viscosity solutions approach, we can say that there are not test functions φ such that $\tilde{V} - \varphi$ has a local maximum in $\zeta = 0$, whereas the set of the derivatives in $\zeta = 0$ of all test functions φ such that $\tilde{V} - \varphi$ has a local minimum in $\zeta = 0$ is exactly the interval $[-k/(k+1), k/(k+1)]$.

The function $\tilde{V}$ is the optimal value for the problem of maximizing, among $w \in W$, the following cost

$$\tilde{J}(\zeta, w(.)) = \int_0^\infty |sat_{[-1,1]}(-kz(t))|dt,$$

subject to the dynamics

$$\dot{z}(t) = sat_{[-1,1]}(-kz(t)) - cw(t), \quad z(0) = \zeta,$$

if and only if it is a viscosity solution in $\mathbb{R}$ of the problem

$$\tilde{V}(\zeta) + \min_{\omega\in[-1,1]} \quad \{-(sat_{[-1,1]}(-k\zeta) - c\omega)\tilde{V}(\zeta)' + \\ -|sat_{[-1,1]}(-k\zeta)|\} = 0$$

Here we do not have boundary conditions since the problem is not restricted to a subset, but it is treated in the whole $\mathbb{R}$. A direct calculation shows that $\tilde{V}$ is a viscosity solution. Now we guess that $\tilde{V}$ is also a viscosity solution of the Isaacs equation for the differential game given by the cost

$$J(\zeta, u(.), w(.)) = \int_0^\infty e^{-t} \max\{|sat_{[-1,1]}(-kz(t))|, |u(t)|\}dt$$

subject to the dynamics (2). Hence, we have to prove that $\tilde{V}$ is a viscosity solution of

$$\tilde{V}(\zeta) + \quad \min_{\omega\in[-1,1]} \max_{\mu\in[-1,1]}\{-(\mu - c\omega)\tilde{V}(\zeta)' + \\ - \max(|sat_{[-1,1]}(-k\zeta)|, |\mu|)\} = 0$$

and hence we will get, by uniqueness, that it is equal to the lower value function of the game.

To this end, we have to split the analysis in the following cases: a) $\zeta < -(1/k)$, b) $-(1/k) \leq \zeta < 0$, c) $\zeta = 0$, d) $0 < \zeta \leq 1/k$, e) $\zeta > 1/k$. A careful analysis of all these cases brings the desired result, and also the fact that the linear saturated control is the unique optimal choice for the first player. In particular, outside the target, the optimal choice for the first player is $\mu = 1$ if $\zeta < 0$ ($\mu = -1$ if $\zeta > 0$) which is exactly the linear saturated control, and corresponds to the optimal choice for a minimum time problem.

VI. Numerical Illustrations

Consider dynamics $\dot{x} = Bu - w$ where B is the incidence matrix of the network with $n = 5$ nodes and $m = 9$ arcs in Fig. 1. Table I lists the upper bounds on u (lower bounds are all 0), the demand bounds, and the long–term average demands. Now, given the nominal demand $\bar{w} = [0\ 1\ 2\ 1\ 1]$ and the nominal balancing flow $\bar{u} = [1\ 1\ 1\ 0\ 0\ 1\ 1\ 3\ 2]' \in \mathcal{U}$ (which is $\bar{w} = B\bar{u}$) we translate the variables by setting $\delta u \doteq u - \bar{u}$ and $\delta w = w - \bar{w}$. We choose matrix D as in (40) of [2], obtained via constraint generation, (see [2] Section 5.2). We simulate the system under the saturated linear state feedback control (5) (we initialize $x(0) = 0$, $y(0) = 0$, and set $k = 4$). The time plot of $z(t)$ in Fig. 2 shows that $z(t)$ converges to the interval $[-\delta u^+/k, -\delta u^-/k]$ (dotted line in Fig. 2).

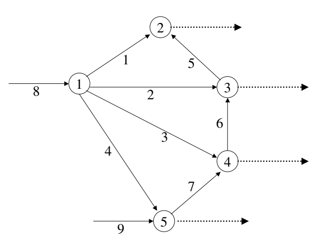

Fig. 1. Example of a system with 5 nodes and 9 arcs.

arcs	1	2	3	4	5	6	7	8	9
upper bounds	3	2	3	3	3	3	3	5	5
nodes	1	2	3	4	5				
upper bounds	0	2	3	2	2				
averages	0	1	2	1	1				

TABLE I

Controlled flows constraints and demand bounds

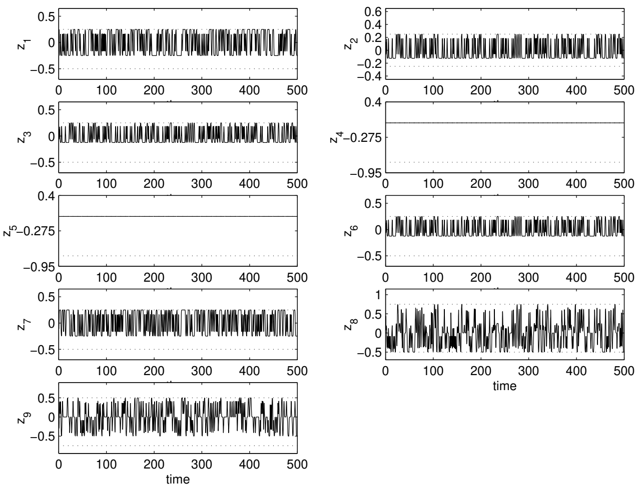

Fig. 2. The variable $z(t)$ with saturated linear feedback control (5) with $k = 4$.

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