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Robust Dynamic Cooperative Games*

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Abstract Classical cooperative game theory is no longer a suitable tool for those situations where the values of coalitions are not known with certainty. We consider a dynamic context where at each point in time the coalitional values are *unknown but bounded* by a polyhedron. However, the average value of each coalition in the long run is known with certainty. We design "robust" allocation rules for this context, which are allocation rules that keep the coalition excess bounded while guaranteeing each player a certain average allocation (over time). We also present a joint replenishment application to motivate our model.

Keywords cooperative games, dynamic games, joint replenishment.

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1 Introduction

Classical cooperative game theory is no longer a suitable tool for those situations where the values of coalitions are not known with certainty (see, e.g., Suijs and Borm (1999) [19], Suijs *et al.* (1999) [20], Timmer *et al.* (2003) [22], Timmer *et al.* (2005) [23]. In this paper we consider a sequence of games, where, differently from Filar and Petrosjan (2000) [8] and Haurie (1975) [10], the *average* coalitions' values (over time) are known with certainty but the instantaneous values are *unknown but bounded* by a polyhedron. This model may be seen as a dynamic extension of the recently introduced cooperative interval games (cf. Alparslan Gök *et al.* (2008) [1–3]) where a coalition value is a closed interval on the real line.

At each point in time a certain revenue is allocated to each player. In general, these revenues will not meet the actual instantaneous value of the coalitions. To keep track of this, an excess vector stores the difference between the instantaneous value of each coalition and the sum of the allocated revenues to all its players. (This excess is different from the coalitional excess that appears, e.g., in the definition of the nucleolus [15]).) We may interpret this excess vector as the state variable describing the history of our dynamic system. Under the assumption that the only information available at each time is the excess of the coalitions, our goal is to design "robust" allocation rules, i.e., allocation rules that i) keep the excess vector bounded within a pre-defined threshold ϵ at each time (we will refer to such rules as ϵ-stabilizing), while ii) guaranteeing a certain average allocation vector over time. Justification for keeping the excess vector bounded follows from the observation that a fair allocation should not allocate maximum excess to the same coalition each time. Our problem of interest may arise in a number of real life situations as, for instance, in joint replenishment applications (cf. Section 2.3). One may notice that our problem is similar in spirit to classical problems in machine learning (cf. Cesa-Bianchi *et al.* (2006) [6], Cesco (1998) [7] and Lehrer (2002) [12]). Therefore, after introducing our allocation rule (or algorithm, since it is an iterative procedure), we compare it to the algorithms proposed in [7, 12].

This paper is organized as follows. In Section 2 we describe the problem. In Section 3 we design the allocation rule. In Section 4 we compare our algorithm to some existing algorithms. In Section 5 we consider allocation rules based on the Shapley value. Finally, in Section 6 we draw some conclusions.

2 Problem statement

2.1 Family of balanced games

We start by introducing the definition of a family of games with coalitions' values lying on pre-assigned closed intervals. Let a game in coalitional form $< N, v >$ be given where $N = \{1, \dots, n\}$ is a set of n players and v is the characteristic function returning the value of each coalition $S \subseteq N$. Henceforth let the inclusion $S \subseteq N$ mean "all coalitions of N except the empty set $\emptyset$". Denote by $m = 2^n - 1$ the number of all coalitions of N except the empty set $\emptyset$ and, with a little abuse of notation, let also $v \in \mathbb{R}^m$ be the vector of coalitions' values, namely, $v = [v(S)]_{S \subseteq N}$.

Definition 1 *A* family of games $< N, \mathcal{V} >$ *is the set of games* $< N, v >$ *obtained when* v *varies within a polyhedron* $\mathcal{V} = \{v \in \mathbb{R}^m : v_{min} \leq v \leq v_{max}\}$, *where the bounds* v_{min} *and* v_{max} *are given.*

For the sake of simplicity, throughout this paper we always assume $v \geq 0$. Also, for the sake of notation, we henceforth denote by 2^N the family of subsets of N. Let us recall the definition of a balanced map and a balanced game for games $< N, v >$ (see, e.g., Tijs (2003) [21, Def. 11.5]). A map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}^+$ is called a *balanced map* if $\sum_{S \subseteq N} \lambda(S) e^S = e^N$. Here, $\mathbb{R}^+$ is the set of nonnegative real numbers and $e^S \in \mathbb{R}^n$ is the *characteristic vector* for coalition S with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \setminus S$. Also, an n-person game $< N, v >$ is called a *balanced game* if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}^+$,

$$\sum_{S \subseteq N} \lambda(S) v(S) \leq v(N). \tag{1}$$

If the above condition is satisfied for each game $v \in \mathcal{V}$, we say that the polyhedron $\mathcal{V}$ describes a family of *balanced* games, as established more formally in the next definition.

Definition 2 *A* family of balanced games $< N, \mathcal{V}_b >$ *is the set of games* $< N, v >$ *obtained when* v *varies within a polyhedron*

$$\mathcal{V}_b = \{v \in \mathcal{V} : \textit{ condition (1) holds}\},$$

where the bounds v_{min} *and* v_{max} *are given.*

Sets of balanced games can also be found in the work of Kranich *et al.* (2005) [11] and Lehrer (2002) [12].

Next, let us revisit the notions of core and allocation rules. Indicate with Δ^n the simplex in $\mathbb{R}^n$ and recall that a game is balanced if and only if the core is nonempty [5,18]. By definition each game $< N, v >$ with $v \in \mathcal{V}_b$ is balanced, and so the core $C(v)$,

$$C(v) = \{a \in \mathbb{R}^n : \frac{a}{v(N)} \in \Delta^n, \sum_{i \in S} a_i \geq v(S) \text{ for all } S \subseteq N\},$$

is nonempty. This means that there exists an allocation $a \in C(v)$ of $v(n)$ with the interpretation that no coalition has an incentive to split off from the coalition N. Now, the problem is to find an allocation rule $a(v)$ such that $a(v) \in C(v)$ for all games $v \in \mathcal{V}_b$. To solve this, first observe that the core is a convex set described by linear equations and inequalities. For our purpose it is useful to change all inequalities into equations. Therefore, we first introduce a vector of nonnegative surplus variables $s = [s_1, \ldots, s_{m-1}]'$ where ζ' denotes the transposed of a given vector ζ. Each surplus variable corresponds to a coalition of players and describes the difference between the allocated value and the coalitional value, $\sum_{i \in S} a_i - v(S)$. Notice that we only need $m-1$ surplus variables because $\sum_{i \in N} a_i = v(N)$ due to the efficiency condition of the core. Further, we introduce an incidence matrix $B \in \mathbb{R}^{m \times n}$ with the characteristic vectors e^S as rows, and an augmented matrix $A \in \mathbb{R}^{m\times(n+m-1)}$ defined by

$$A = \left[B \left| \begin{array}{c} -I \\ ----- \\ 0\dots 0 \end{array} \right. \right], \tag{2}$$

where I is the $(m-1)$-dimensional identity matrix. Now, finding an allocation rule a in the core $C(v)$ corresponds to finding a so-called *allocation vector* $u \in \mathbb{R}^{n+m-1}$ in the set

$$\mathcal{U}(v) = \{u : Au = v,\ u \geq 0\} \tag{3}$$

because if $u \in \mathcal{U}(v)$ then $u = \left[\begin{smallmatrix} a \\ s \end{smallmatrix}\right]$ for some $a \in C(v)$. Observe that, in general, $\mathcal{U}(v)$ is a polyhedron of dimension $n-1$.

2.2 Dynamic system

Given the definition of family of balanced games, we now consider a sequence of games that fluctuates in the bounded polyhedron $\mathcal{V}_b$. We denote this by

$$\mathbf{v}(t),\ t = 1, 2, \dots \text{ with } \mathbf{v}(t) \in \mathcal{V}_b \text{ for all } t \tag{4}$$

and $\mathbf{v}(t) = [v(t, S)]_{S\subseteq N}$ is the vector of coalitional values. The values of $\mathbf{v}(t)$ and $\mathbf{v}(t+1)$ are not correlated, which means that we cannot describe transitions from $\mathbf{v}(t)$ to $\mathbf{v}(t+1)$. This also implies that we cannot take $\mathbf{v}(t)$ as a state variable and define dynamics (neither deterministic nor stochastic) on it. On the contrary, it is realistic to assume that we know with certainty the *average vector of coalitions' values* $\bar{\mathbf{v}}$, being defined by

$$\bar{\mathbf{v}} = \lim_{T\longrightarrow\infty} \frac{1}{T} \sum_{k=0}^{T} \mathbf{v}(t). \tag{5}$$

Obviously, $\bar{\mathbf{v}}$ characterizes the sequence of games under consideration.

Further, assume that allocations to players are made at a higher rate than the rate of change of the coalitional values, which equals 1. Allowing different rates means that he who allocates the revenues provides a faster response in reply to the excesses of the coalitions. We will show later on that faster allocations allow for lower excesses. More precisely, let the integer number $1/\Theta$ be the rate of allocations. Then Θ is the time between two successive allocations. To facilitate our analysis, we stretch the time scale by the rate $1/\Theta$ and consider a new sequence of games, namely

$$v(k) = \mathbf{v}(t)\Theta, \quad k = \frac{t-1}{\Theta} + 1, \ \dots, \frac{t}{\Theta}, \quad t = 1, 2, \dots. \tag{6}$$

This new sequence of games has the following interpretation. In the original time interval $(t-1, t]$ the vector of coalitional values equals $\mathbf{v}(t)$. We distribute these values equally over the $1/\Theta$ allocations that occur in this time period, so this results in values $\mathbf{v}(t)\Theta$ for each point in time where allocations are made. This way we can ensure that the total amount allocated to the players in the new interval $((t-1)/\Theta, t/\Theta]$ does not exceed the available amount $v(t, N)$.

If we use the notation $\mathcal{V}_b^{\Theta} = \Theta \cdot \mathcal{V}_b$, the sequence of games (4)-(5), is equivalent to the sequence of games

$$\begin{array}{c} v(k),\, k = 1, 2, \dots \text{ with } v(k) \in \mathcal{V}_b^{\Theta} \text{ for each } k = 1, 2, \dots \\ \bar{v} = \lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{k=0}^{T} v(k) \end{array} \tag{7}$$

where $\bar{v} = \Theta \bar{\mathbf{v}}$. In the remainder of this paper, we will refer to the sequence of games in (7).

Now, denote by $x(k+1) \in \mathbb{R}^m$ a vector of variables describing the aggregate coalition excesses over all previous games $v(1), \dots, v(k)$ (the value $x(0)$ is the excess at time 0), i.e.,

$$x(k+1, S) = x(k, S) + \sum_{i \in S} a_i(k) - (s_S(k) + v(k, S)) \quad \text{for all } S \subseteq N, \quad (8)$$

where $a_i(k)$ is the revenue allocated to player i and $s_S(k)$ is a desired surplus for coalition S. Roughly speaking, the coalition excess at time k is the difference between the sum of the allocated revenues to the players of the coalition and the value of the coalition increased by a desired surplus for that coalition. The aggregate coalition excess $x(k+1, S)$ is the coalition excess summed over all previous games $v(1), \dots, v(k)$ and therefore represents the *state* of the system ($x(k)$ describes the history of the system). We rewrite equation (8) in the following matrix form

$$x(k+1) = x(k) + Au(k) - v(k), \quad v(k) \in \mathcal{V}_b^{\Theta}, \quad k = 1, 2, \dots, \quad (9)$$

where $u(k) = \left[\begin{smallmatrix} a(k) \\ s(k) \end{smallmatrix}\right]$, $a(k) = [a_i(k)]_{i \in N}$ and $s(k) = [s_S(k)]_{S \subset N}$. The condition $u(k) \geq 0$ is omitted for the sake of notation. Now, let the vector $\bar{u} \in \mathcal{U}(\bar{v})$ be arbitrarily chosen, where $\bar{v}$ is assigned once given the sequence of games (7). The following lemma recalls a result obtained in Bauso *et al.* (2006) [4].

Lemma 1 *(Average constraint) Let the sequence of games (7) be given. There exists an allocation rule $f : \mathbb{R}^m \longrightarrow \mathbb{R}^{n+m-1}$ such that for $u(k) = f(v(k))$,*

$$Au(k) = v(k) \quad (10)$$

$$\lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{k=0}^{T} u(k) = \bar{u} \tag{11}$$

if and only if there exists a matrix $D \in \mathbb{R}^{(n+m-1)\times m}$ *that satisfies*

$$AD = I \in \mathbb{R}^{m \times m} \tag{12}$$

$$D(v - \bar{v}) + \bar{u} \geq 0 \quad \forall v \in \mathcal{V}_b^{\Theta}. \tag{13}$$

The allocation rule is linear on $v(k)$*, that is*

$$u(k) = \bar{u} + D(v(k) - \bar{v}). \tag{14}$$

In the following we call $\lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{k=0}^{T} u(k)$ the *average allocation* (vector).

Note that condition (10) implies that $u(k) = \begin{bmatrix} a(k) \\ s(k) \end{bmatrix} \in \mathcal{U}(v(k))$ at each time k. This in turn means that $a(k)$ is an element of the core $C(v(k))$ of the game $< N, v(k) >$ obtained from freezing the coalitions' values at time k. Furthermore, it is easy to show that the above result and the following ones are still valid if the budget to allocate at each time period has a fixed size. Denoting by u^+ the maximal size of the budget, condition (13) changes to $0 \leq D(v - \bar{v}) + \bar{u} \leq u^+$, $\forall v \in \mathcal{V}_b^{\Theta}$.

Observe that the linear allocation rule (14) requires perfect knowledge of the coalition values at each sample time. Differently, in this paper the coalitions' values $v(k)$ are unknown and revenues are allocated at time k based on the aggregate coalition excesses $x(k)$. Any allocation rule based on the state $x(k)$ will be referred to as a *feedback rule*. We are interested in finding dynamic allocation rules that keep the excess vector bounded within a pre-specified threshold while satisfying the condition that if the average coalitions's value is $\bar{v}$ then the average allocation is $\bar{u}$. For this we need the following definition, see Bauso *et al.* (2006) [4]. For $\xi \in \mathbb{R}^m$, let ξ_i denote the ith component of ξ, and define

$$|\xi| = \max_i |\xi_i|.$$

Let $\mathbb{Z}$ denote the set of integers, and $\mathbb{Z}^+$ the set of nonnegative integers. Let $f = \{f(0), f(1), f(2), \ldots\}$ be any bounded one-sided sequence in $\mathbb{R}^m$, and define

$$\|f(k)\| = \sup_{k \in \mathbb{Z}^+} |f(k)|.$$

Our dynamic allocation rule is defined as follows.

Definition 3 *Given $\epsilon > 0$ and a reference value x_{ref} for system (9), an* ϵ-stabilizing allocation rule *is a feedback rule for which there exists a continuous positive function $\phi(k)$, monotonically decreasing and converging to 0 as $k \longrightarrow \infty$ such that for all $x(0)$, the following condition holds true*

$$\|x(k) - x_{ref}\| \leq \max\{\|x(0)\|\phi(k), \epsilon\}.$$

For the sake of simplicity, take $x_{\text{ref}} = 0$. Then the above condition implies that $x(k)$ does not deviate more than ϵ from 0 in the long run. For any $x(0)$ with $\|x(0)\| \leq \epsilon$ the condition simply requires that $\|x(k)\| \leq \epsilon$ for all k. With this in mind, our problem of interest can be stated as follows.

Problem 1 For the sequence of games (7), find an ϵ-stabilizing allocation rule such that its average allocation equals $\bar{u}$, i.e., $\lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{k=0}^{T} u(k) = \bar{u}$.

Note that the requirement $\lim_{T\longrightarrow\infty} \frac{1}{T}\sum_{k=0}^{T} u(k) = \bar{u}$ simply represents a constraint on the coalitions' excess in the long run.

Also, observe that the ϵ-stabilization of the excess vector $x(k)$ means that at each time k the excess $x(k)$ does not exceed a pre-defined threshold ϵ of the game $< N, v(k) >$. Using the definition of the ϵ-core from Lehrer (2002) [12], the above problem corresponds to finding an allocation rule that at each time k returns a vector in the ϵ-core of the one-shot game $< N, v(k) >$.

Remark 1 We can refrain from the assumption that the average vector of coalitions' values is known a-priori and formulate the above problem under the milder assumption that $\bar{v}$ is simply averaged on-line over past coalitions' values. Later on we show that the allocation rule depends on matrix D as in (12)-(13). Hence, if the value $\bar{v}$ is averaged on-line, it becomes time varying and since it appears in the computation of D in (13), the latter matrix must be updated iteratively.

2.3 Motivating example

Consider a single-period one-warehouse multi-retailer inventory system (see, e.g., Hartman *et al.* (2000) [9], Meca *et al.* (2003) [13], and Meca *et al.* (2004) [14]). Figure 1 displays a warehouse W serving three retailers R_1, R_2 and R_3. Each retailer faces a demand bounded by a minimum and a maximum value. For instance R_1 faces a demand d_1 in the interval $[d_1^-, d_1^+] =$

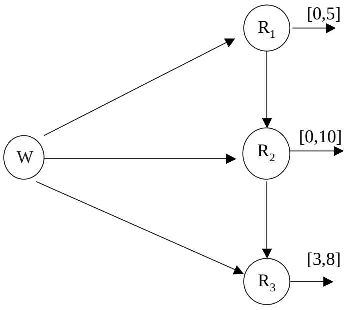

Fig. 1 Example of one warehouse W and three retailers R_1, R_2 and R_3. Retailer R_1 faces a demand in the interval $[0, 5]$, R_2 in the interval $[0, 10]$, and R_3 in the interval $[3, 8]$.

$[0, 5]$, R_2 faces a demand d_2 in the interval $[d_2^-, d_2^+] = [0, 10]$, and R_3 faces a demand d_3 in the interval $[d_3^-, d_3^+] = [3, 8]$.

After demands d_i are realized, each retailer R_i must choose whether to fulfill the demand or not. The retailers do not hold any private inventory. Therefore, if they wish to fulfill their demands, they must reorder goods at the central warehouse. The retailers may share the total transportation cost $K = 7$. Before demands are realized, the warehouse holder decides how to allocate the transportation costs among the retailers. This decision is only based on the knowledge of the minimum demand d_i^- and maximum demand d_i^+.

The corresponding cost game has a set of three players $N = \{1, 2, 3\}$, namely the three retailers. If player i plays alone, the cost of reordering coincides with the full transportation cost (since a single truck serves him only) whereas the cost of not reordering is the cost of unfulfilled demand, that is, lost demand. Assume the latter cost is one unit per unit of unfulfilled demand. Then the cost associated to the retailers R_1 and R_2 are respectively,

$$c(\{1\}) \in \left[\min\{d_1^-, K\}, \min\{d_1^+, K\}\right] = [0, 5]$$
$$c(\{2\}) \in \left[\min\{d_2^-, K\}, \min\{d_2^+, K\}\right] = [0, 7].$$

If two players form a coalition they are forced to select a joint decision ("both reorder" or "both do not reorder"). The cost of reordering for the coalition also equals the total transportation cost that, this time, must be shared between the two players. The cost of not reordering is the sum of the unfulfilled demands of both players. For instance, the cost of coalition $S = \{1, 2\}$ is $c(\{1, 2\}) \in \left[\min\{(d_1^- + d_2^-), K\}, \min\{(d_1^+ + d_2^+), K\}\right] = [0, 7]$.

For a generic n-player game, we have for all coalitions $S \subseteq N$

$$\min(K, \sum_{i \in S} d_i^-) \leq c(S) \leq \min(K, \sum_{i \in S} d_i^+). \tag{15}$$

We can compute the cost savings $v(S)$ of a coalition S as the difference between the sum of the costs of the coalitions of the individual players in S and the cost of the coalition itself, namely,

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S). \tag{16}$$

Given the upper and lower bounds for $c(S)$ in (15), the value $v(S)$ is bounded as follows:

$$\sum_{i \in S} \min(K, d_i^-) - \min(K, \sum_{i \in S} d_i^-) \leq v(S) \leq \sum_{i \in S} \min(K, d_i^+) - \min(K, \sum_{i \in S} d_i^+).$$

For example, the cost savings of coalition $S = \{1, 2\}$ are $v(\{1, 2\}) = c(\{1\}) + c(\{2\}) - c(\{1, 2\}) \in [0, 5]$.

It turns out that the cost savings, or value, of each coalition are bounded by a minimum and a maximum value, namely, $v_{min}(S) \leq v(S) \leq v_{max}(S)$ with fixed bounds $v_{min}(S)$ and $v_{max}(S)$. Hence, in the light of Definition 1, an equivalent description of the joint replenishment application is obtained by replacing the family of cost games by the family of cost-savings games $< N, \mathcal{V} >$ with

$$\mathcal{V} = \{v \in \mathbb{R}^m : v_{min}(S) \leq v(S) \leq v_{max}(S), \text{for all } S \subseteq N\} \qquad (17)$$

and $v(S)$ as in (16). For the sake of brevity we omit the proof that each game in the polyhedron (17) corresponds to a balanced game. We conclude that the joint replenishment problem can be described by the family of balanced games $< N, \mathcal{V} >$. To see the dynamic aspect of the application, consider a situation where the discussed scenario occurs repeatedly in time, i.e., at each time (day, week) $k = 0, 1, \ldots$, the warehouse manager allocates the costs and demands are realized.

3 Dynamic Allocation Rule

The dynamic allocation rule that we propose as a solution to Problem 1 depends on an augmented state variable, to be defined below. Such a state variable models the excess level of each coalition combined with the deviation of the instantaneous allocation from the pre-defined average allocation of each coalition. With the given augmented state variable Problem 1 reduces to finding an ϵ-stabilizing allocation rule for the augmented dynamic system. Actually, as it will be clearer later on, ϵ-stabilizing the augmented system implies both ϵ-stabilizing the excess vector and meeting the average constraints.

Given two matrices A and D as in (12) and (13), from a standard property of linear algebra, see also the appendix, we can find two matrices C and F that "square" A and D and satisfy

$$\begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I. \tag{18}$$

Now, consider the augmented system

$$\begin{aligned} x(k+1) &= x(k) + Au(k) - v(k), \\ y(k+1) &= y(k) + Cu(k), \end{aligned} \tag{19}$$

where $v(k)$ is as in (6). The additional dynamic variable $y(k)$ keeps track of the deviation between the instantaneous and the average allocation of each player. Define the augmented state variable $z \in \mathbb{R}^{n+m-1}$ as

$$z(k) = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}, \quad \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} z(k).$$

This variable satisfies the equation

$$\begin{aligned} z(k+1) &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k+1) \\ y(k+1) \end{bmatrix} \\ &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} u(k) - \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} v(k) \\ 0 \end{bmatrix} \\ &= z(k) + u(k) - Dv(k). \end{aligned} \tag{20}$$

This indicates that the allocation rule $u(k) = -z(k)$, which is linear in z, solves our problem.

Theorem 1 *Consider system (20) with $v(k)$ as in (6). The allocation rule in feedback form*

$$u(k) = -z(k) \tag{21}$$

satisfies

$$\|z(k)\| \leq \|Dv(k)\|. \tag{22}$$

Further, if the average coalitions' value is $\bar{v}$ then the average allocation vector is $\bar{u}$.

Proof To prove (22), we substitute the allocation rule (21) in the dynamics of system (20). This results in $z(k+1) = -Dv(k)$ for all k, which implies (22). For the rest of the proof, by summing (20) for different $k = 1, 2, \ldots$, we obtain

$$\frac{1}{T}\sum_{k=0}^{T-1} u(k) - \frac{1}{T}\sum_{k=0}^{T-1} Dv(k) = \frac{z(T) - z(0)}{T} \to 0$$

as $T \to \infty$, since the numerator is a finite quantity whereas the denominator tends to infinity. Therefore $\bar{u} = D\bar{v}$, which concludes the proof. □

For fixed ϵ we wish to find the maximum time period Θ^* such that $\|Dv(k)\| \leq \epsilon$. Trivially, such a value is $\Theta^* = \frac{\epsilon}{\delta}$ where $\delta = \max_{v \in \mathcal{V}_b} |Dv|$. Then we have the following corollary.

Corollary 1 *Consider system (20) with $v(k)$ as in (6). For any ϵ and corresponding Θ^*, if $\Theta \leq \min\{\Theta^*, 1\}$, then the allocation rule in feedback form*

$$u(k) = -z(k), \tag{23}$$

is ϵ-stabilizing.

Proof It is easy to show that

$$\|z(k)\| \leq \|D\mathbf{v}(t)\Theta\| \leq \|D\mathbf{v}(t)\Theta^*\| \leq \max_{v\in\mathcal{V}_b} |Dv\Theta^*| \leq \epsilon.$$

□

A side effect of $\|z\| \leq \epsilon$ is that also $\|u\| \leq \epsilon$ as $u = -z$. This means that the smaller ϵ the smaller the maximum allocation (in magnitude). Also, observe that the above results can be extended to the case where $\bar{v}$ is averaged on-line (see Remark 1), with the difference that matrix D must be updated iteratively according to (12)-(13).

4 Other algorithms in the literature

The idea of ϵ-stabilizing or shrinking the excess vector can be found, from a different perspective, also in the algorithms proposed by Cesco (1998) [7], Lehrer (2002) [12] and Sengupta *et al.* (1996) [16]. Though we have developed our algorithm independently from the ones cited above, a-posteriori we recognize that all of them propose an allocation rule that uses a measure of the extra benefit that a coalition has received up to the current time by re-distributing the budget among the players. Budget distribution occurs iteratively until the allocation process converges to an element in the core or in the ϵ-core if the game is not balanced (in this last case the core is empty). However, unlike the game in this paper, the games dealt with in [7, 12, 16] are games with complete information in the sense that the values of the coalitions are known and time invariant. Therefore these dynamic processes refer to allocations of payments to the players and not to the variation of the coalitions' values. Owing to the fact that, in this paper, the values of the coalitions vary unknowingly, we have been able to guarantee the convergence to the core only in the long run. Furthermore, if we look at the game $< N, v(k) >$, ϵ-stabilization implies that the vector of payments belongs to the ϵ-core of the game at each time $k = 0, 1, \ldots$.

A last comment regards the computational complexity of the algorithm. In this sense, it must be noted that to compute the matrix D, upon which the allocation rule is based, the number of constraints of type (10) to consider grows exponentially on the number of players n. We refer the reader to Bauso *et al.* [4], Section 5, for a procedure based on constraints generation that returns the matrix D in polynomial time.

5 The Shapley value as a linear allocation rule

In this section we study the Shapley value as a special linear allocation rule of the form (14). In particular, we show that there is a matrix Φ that satisfies (12).

The *Shapley value* ϕ, introduced in Shapley (1953) [17], is defined by $\phi = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma$ where $\Pi(N)$ is the set of all permutations of N and m^σ is the marginal vector corresponding to the permutation $\sigma : N \to N$. A marginal vector m^σ corresponds to a situation in which the players enter a room one by one in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$ and where each player receives the marginal contribution he creates upon entering. Hence, m^σ is the vector in $\mathbb{R}^n$ with elements

$$
\begin{aligned}
m^\sigma_{\sigma(1)} &= v(\{\sigma(1)\}), \\
m^\sigma_{\sigma(2)} &= v(\{\sigma(1), \sigma(2)\}) - v(\{\sigma(1)\}), \\
&\vdots \\
m^\sigma_{\sigma(k)} &= v(\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(k-1)\}).
\end{aligned}
$$

Theorem 2 *The* Shapley value ϕ *is linear in* v*, i.e.,* $\phi = Lv$*, where the matrix* $L \in \mathbb{R}^{n\times m}$ *is defined by*

$$
L_{ij} = \frac{1}{n!} \cdot \begin{cases} -\mu!(n-(\mu+1))! & \text{if } i \notin S \\ (\mu-1)!(n-\mu)! & \text{if } i \in S. \end{cases} \tag{24}
$$

if column j *corresponds to coalition* S *with* $\mu = |S|$*.*

Proof The proof follows immediately from the definition of the Shapley value in Shapley (1953) [17]. □

To emphasize the dependence of ϕ on v we henceforth write $\phi(v)$ instead of ϕ. Let $s(\phi(v))$ be the vector of surplus variables when revenues are allocated according to the Shapley value $\phi(v)$. The idea is now to express $s(\phi(v))$ linearly in v.

Theorem 3 *The vector of surplus variables is linear in* v*, i.e.,*

$$
s(\phi(v)) = Qv, \tag{25}
$$

where $Q \in \mathbb{R}^{(m-1)\times m}$ *has row* i *associated to a surplus variable (a coalition* $S \subset N$*), column* j *associated to a coalition* $M \subseteq N$*, and generic* ij*th* *element*

$$Q_{ij} = \begin{cases} \sum_{p\in S} L_{pj} & \text{if } i \neq j \\ \sum_{p\in S} L_{pj} - 1 & \text{if } i = j. \end{cases} \tag{26}$$

Proof First, consider the coalition containing just player 1 and let $L_{i\bullet}$ be the generic ith row of L. The associated surplus variable is

$$s_1(\phi(v)) = \phi_1 - v(\{1\}) = L_{1\bullet}v - v(\{1\}) = (L_{11}-1)v(\{1\}) + L_{12}v(\{2\}) + \ldots + L_{1m}v(N).$$

The latter equation yields $Q_{1\bullet} = [(L_{11}-1)\, L_{12} \ldots L_{1m}]$, which is in accordance with (26).

If we repeat the same reasoning for a generic coalition $M \subset N$, the surplus variable is

$$s_M(\phi(v)) = \sum_{i\in M} \phi_i - v(M) = \sum_{i\in M} L_{i\bullet}v - v(M).$$

Remind j is the column associated to coalition M. Then, the latter equation yields $Q_{jk} = \sum_{i\in M} L_{ik}$ if $k \neq j$ and $Q_{jj} = \sum_{i\in M} L_{ij} - 1$ which is in accordance with (26). $\square$

Using the fact that $\phi(v)$ and $s(\phi(v))$ are linear in v, we define the allocation vector associated to the Shapley value by $u(\phi(v)) = [\phi(v)' \quad s(\phi(v))']'$.

Corollary 2 *There exists a matrix $\Phi \in \mathbb{R}^{(n+m-1)\times m}$, defined by $\Phi = [L' \quad Q']'$ such that $u(\phi(v)) = \Phi v$. Furthermore Φ is a right inverse of A, i.e., $A\Phi = I$.*

Proof From the Theorems 2 and 3 we conclude $[\phi(v)' \quad s(\phi(v))']' = [L' \quad Q']'\, v$. This finishes the proof of the first part.

To prove that $A\Phi = I$, it suffices to show that $A_{i\bullet}\Phi_{\bullet j} = 1$ if $i = j$ and zero otherwise. Observe that row i of A, denoted by $A_{i\bullet} \in \mathbb{R}^{1\times(n+m-1)}$, is associated to a coalition $M \subseteq N$, whereas column j of Φ, denoted by $\Phi_{\bullet j} \in \mathbb{R}^{(n+m-1)\times 1}$, is associated to a coalition $S \subseteq N$. Hence, the condition $i = j$ is equivalent to $M = S$.

Now consider once again the row vector $A_{i\bullet}$. The first n elements of this vector correspond to players $p = 1, \ldots, n$ and the last $m-1$ elements correspond to all coalitions $R \subset N$ (recall the structure of A as described in (2)). Now the structure of row $A_{i\bullet}$ may be formulated as:

$$A_{i\bullet} = [\ldots \underbrace{1}_{\forall p \in M} \ldots\ldots \underbrace{0}_{\forall p \notin M} \ldots\ldots \underbrace{-1}_{R=M} \ldots\ldots \underbrace{0}_{\forall R \neq M} \ldots]. \tag{27}$$

Analogously, the first n elements of $\Phi_{\bullet j}$ correspond to players $p = 1 \ldots n$, and the last $m-1$ elements correspond to all coalitions $R \subset N$ (see (24) and (26)).

Concluding, if $i = j$, or $M = S$, then $A_{i\bullet}\Phi_{\bullet j} = \sum_{p\in S} L_{pj} - (\sum_{p\in S} L_{pj} - 1) = 1$. On the other hand, if $i \neq j$, or $M \neq S$, then $A_{i\bullet}\Phi_{\bullet j} = \frac{1}{n!}[\sum_{p\in M} L_{pj} - \sum_{p\in M} L_{pj}] = 0$. $\square$

6 Conclusions

Inspired by a joint replenishment application, we studied a dynamic cooperative game where at each point in time the value of each coalition of players is unknown and fluctuates within a bounded polyhedron. Under the assumption that the average value of each coalition in the long run is known with certainty, we have presented a constructive method to find “robust” allocation rules, i.e., allocation rules that are close to an excess vector and guarantee a certain average allocation vector.

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A Computation of the matrices C and F

In this appendix we show how to compute the matrices C and F given the matrices A and D, as mentioned in Section 3. To simplify notation let $n+m-1=r$. Note that the following conditions follow from (18): $AD = I$, $AF = 0$, $CD = 0$, and $CF = I$. First, we rewrite the matrices A, C, D and F as follows.

- $A = [\ A_0\ A_1\ ]$ where A_0 is a $m \times (r-m)$ matrix and A_1 is an $m \times m$ non singular matrix.
- $C = [\ C_0\ C_1\ ]$ where C_0 is a $(r-m) \times (r-m)$ matrix and C_1 is an $(r-m) \times m$ matrix.
- $D = [D_0' \quad D_1']'$ where D_0 is an $(r-m) \times m$ matrix and D_1 is an $m \times m$ non singular matrix.
- $F = [F_0' \quad F_1']'$ where F_0 is a $(r-m) \times (r-m)$ matrix and F_1 is an $m \times (r-m)$ matrix.

Next, we derive the following relations:

- from $AF = 0$, we obtain $A_1F_1 = -A_0F_0$. So $F_1 = -A_1^{-1}A_0F_0$;
- from $CD = 0$, we obtain $C_1D_1 = -C_0D_0$. Thus $C_1 = -C_0D_0D_1^{-1}$;
- from $CF = I$, we obtain $C_0F_0 = I - C_1F_1 = I + C_0D_0D_1^{-1}F_1 = I - C_0D_0D_1^{-1}A_1^{-1}A_0F_0$.

Imposing, e.g., $C_0 = I$ we have $F_0 = (I + D_0 D_1^{-1} A_1^{-1} A_0)^{-1}$. Consequently,

$$C = [\ I \mid -D_0 D_1^{-1}\ ]$$

$$F = \begin{bmatrix} (I + D_0 D_1^{-1} A_1^{-1} A_0)^{-1} \\ -A_1^{-1} A_0 (I + D_0 D_1^{-1} A_1^{-1} A_0)^{-1} \end{bmatrix}.$$