**Article:**

Bauso, D. and Timmer, J. (2012) On robustness and dynamics in (un)balanced coalitional games. Automatica, 48 (10). 2592 - 2596. ISSN: 0005-1098

https://doi.org/10.1016/j.automatica.2012.06.057

# On robustness and dynamics in (un)balanced coalitional games [⋆]

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**Abstract**

In this paper we investigate robustness and dynamics for coalitional games with transferable utilities (TU games). In particular we study sequences of TU games. These sequences model dynamic situations in which the values of coalitions of players are not known beforehand, and are subject to changes over time. An allocation rule assigns a payoff to each player in each time period. This payoff is bounded by external restrictions, for example due to contractual agreements. Our main questions are: (i) under which conditions do the allocations converge to a core-element of the game, and (ii) when do the allocations converge to some specific allocation, the so-called nominal allocation? The main contribution of this paper is a design method for allocation rules that return solutions in the core or $\varepsilon$-core of the game under delayed information on the coalitions' values, and therefore the resulting allocation rule is called robust.

*Key words:* cooperative TU games, dynamics, robustness, core.

## 1 Introduction

In this paper, we study sequences of TU games in which the values of coalitions at future times are not known beforehand, and are subject to changes over time [1,9]. Such games may arise in a number of real life situations as, for instance, in joint replenishment applications [3], or communication networks [10]. In these games, an allocation rule assigns a payoff to each player in each time period. This payoff is bounded by external restrictions for example, due to contractual agreements or budget limitations.

In spirit with "approachability theory" [4,9] and "regret-based" minimization [5,7] we pose two main questions: (i) under which conditions do the allocations converge to a core-element of the game, and (ii) when do the allocations converge to some specific allocation, the so-called nominal allocation? The main contribution of this paper is a constructive way to design allocation rules for sequences of TU games that guarantee the convergence of the allocations to the core, or the $\varepsilon$-core, of the average game. Our design method also works in case the game designer has delayed information on the coalitional values, that is, he does not know the current values of coalitions. Then the payoffs are based on the cumulative excesses of the coalitions, which depend on the former coalitional values. The allocation rule has to be robust to be able to deal with the uncertain values of the coalitions.

Our approach is different from the one in [11,12] as there the values of coalitions are modeled by random variables, whose distributions are known to the players. Also, these games are static, whereas our model is dynamic since it considers values of coalitions that vary exogenously over time (cf. [6,8]).

In this paper, we extend the dynamic system theoretic framework of [3] such that it can handle external bounds on the allocations, and we also deal with unbalanced games. Our results show necessary and sufficient conditions for allocation rules to belong to the cores of the games, or to converge to specific nominal allocations.

This paper is organized as follows. In Section 2 we introduce the model. In Sections 3 and 4 we present our results for respectively the balanced and unbalanced games. Section 5 illustrates our design method by a numerical example.

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## 2 The model

In this section we introduce our model. Let $N = \{1, \dots, n\}$ be a set of players. A coalition $S$ is a nonempty subset of the player set $N$. Let the inclusion $S \subseteq N$ mean that $S$ is a coalition. Denote by $m = 2^n - 1$ the number of coalitions. A cooperative game is a pair $< N, v >$, where $v$ is the characteristic function that assigns the value $v(S)$ to coalition $S$. If all players cooperate, then the value $v(N)$ is available. An allocation vector $a \in \mathbb{R}^n$ describes how to allocate this value over the players; player $i$ receives the amount $a_i$. Such an allocation is fair if it belongs to the core $C(v)$ of the game:

$$C(v) = \{a \in \mathbb{R}^n : \sum_{i \in N} a_i = v(N); \sum_{i \in S} a_i \geq v(S), S \subseteq N\}.$$

An allocation in the core distributes the value $v(N)$ over all the players in such a way that any coalition receives at least as much as it can obtain on its own. A related concept is the so-called $\epsilon$-core [9]. The $\epsilon$-core is the set of all allocations such that the total amount received by each coalition exceeds or is equal to the value of the coalition reduced by a given tolerance $\epsilon$:

$$\begin{aligned} C_\epsilon(v) := \{a \in \mathbb{R}^n \ : & \sum_{i \in N} a_i = v(N); \\ & \sum_{i \in S} a_i \geq v(S) - \epsilon, S \subseteq N\}. \end{aligned}$$

We say that a game is $\epsilon$-balanced if and only if its $\epsilon$-core is nonempty.

Let $\mathbf{v}(t) = [v(t, S)]_{S \subseteq N}$ be a vector of values of the coalitions at time $t$, and let $\mathcal{V}$ be a bounded polyhedron. The sequence

$$< N, \mathbf{v}(t) >, \ t = 1, 2, \dots \text{ with } \mathbf{v}(t) \in \mathcal{V} \text{ for all } t, \tag{1}$$

is a sequence of cooperative games, one for each time period. Let $\bar{\mathbf{v}}$ be the *vector of average coalitions' values*,

$$\bar{\mathbf{v}} = \lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbf{v}(t). \tag{2}$$

Notice that the coalitions' values vary according to some exogenous random process.

Let $e^S \in \mathbb{R}^n$ be the characteristic vector for coalition $S$ with $e^S_i = 1$ if $i \in S$, and $e^S_i = 0$ if $i \notin S$. Let $B \in \mathbb{R}^{m \times n}$ be the matrix whose rows are the characteristic vectors $e^S$, for all coalitions $S$. Let $I$ be the $(m-1)$-dimensional identity matrix. Define the matrix $A \in \mathbb{R}^{m \times (n+m-1)}$ by

$$A = \left[ \begin{array}{c|c} & -I \\ B & ---- \\ & 0 \dots 0 \end{array} \right]. \tag{3}$$

Denote the column vector of nonnegative surplus variables by $s = [s_1, \dots, s_{m-1}]' \geq 0$, where $\zeta'$ is the transposed of a given vector $\zeta$. Let

$$\mathcal{U}(v) = \left\{ u \in \mathbb{R}^{n+m-1} : Au = v, \ s \geq 0 \right\} \tag{4}$$

be the set of *"augmented" allocation vectors* $u := \left[ \begin{smallmatrix} a \\ s \end{smallmatrix} \right] \in \mathbb{R}^{n+m-1}$. Now, if $< N, v >$ is a balanced game then finding an allocation $a$ in the core $C(v)$ is equivalent to finding an augmented allocation vector $u$ in $\mathcal{U}(v)$.

Allocations to players are made at an integer rate $1/\Theta$, $\Theta < 1$, whereas the rate of change of the coalitional values equals one by default. Hence, we obtain a new sequence,

$$\begin{aligned} & v(k) = \mathbf{v}(t)\Theta, \quad k = \tfrac{t-1}{\Theta} + 1, \ \tfrac{t-1}{\Theta} + 2, \dots, \tfrac{t}{\Theta}, \\ & \qquad t = 1, 2, \dots. \end{aligned} \tag{5}$$

Let $\mathcal{V}^\Theta = \Theta \cdot \mathcal{V}$ and $\bar{v} = \Theta \bar{\mathbf{v}}$. Then the sequence of games (1)-(2) corresponds one to one with the sequence of games

$$\begin{aligned} & < N, v(k) >, \ k = 1, 2, \dots \text{ with } v(k) \in \mathcal{V}^\Theta \text{ for each } k \\ & \bar{v} = \lim_{T \longrightarrow \infty} \tfrac{1}{T} \textstyle\sum_{k=1}^{T} v(k). \end{aligned} \tag{6}$$

In the remainder of this paper, we always refer to this latter sequence of so-called instantaneous games.

We assume that the augmented allocation vector $u$ is bounded by the polyhedron

$$\mathcal{U} := \{u : u_{min} \leq u \leq u_{max}\}$$

with $u_{min} = \left[ \begin{smallmatrix} a_{min} \\ 0 \end{smallmatrix} \right]$ and $a_{min} \in \mathbb{R}^n$, $a_{min} \leq 0$.

The instantaneous games with the above additional bounds on allocations provide a stylized model of any situation where the allocations are subject to budget limitations, or contracts or binding agreements between the game designer and the players.

Define $x(k+1) \in \mathbb{R}^m$ as the state variable of the system at stage $k+1$, with $x(1)$ the excess at time 1 (take $x(1) = 0$ for sake of simplicity). This vector of variables describes the cumulative excesses of the coalitions over the games $v(1), \dots, v(k)$,

$$\begin{aligned} x(k+1) &= x(k) + Au(k) - v(k), \\ & v(k) \in \mathcal{V}^{\Theta},\ u(k) \in \mathcal{U},\ k = 1, 2, \dots. \end{aligned} \tag{7}$$

Here we write $u(k) = \begin{bmatrix} a(k) \\ s(k) \end{bmatrix}$, $a(k) = [a_i(k)]_{i \in N}$ with $a_i(k)$ the revenue allocated to player $i$, and $s(k) = [s_S(k)]_{S \subset N}$. Note that we can interpret $u(k)$ as the *control variable* as it reflects the revenues that the game designer chooses to allocate to the players at stage $k$. If at stage $k$ the game designer knows $x(k)$ and $v(k-1)$, so the information on the game values is delayed by one period, then she will design allocation rules that depend on the state $x(k)$.

## 3 Balanced games

We start by analysing allocation rules for balanced games. The lemma below provides necessary and sufficient conditions for sequences of games to be balanced in terms of the sets $\mathcal{V}^{\Theta}$ and $\mathcal{U}$.

**Lemma 1** *All the games in the sequence (6) are balanced if and only if*

$$\mathcal{V}^{\Theta} \subseteq A\mathcal{U}. \tag{8}$$

*Then there exists an augmented allocation rule $u(k)$ that depends on $v(k)$ such that*

$$a(k) \in C(v(k)),\ \forall k. \tag{9}$$

PROOF. From the definitions it follows that balancedness is equivalent to (8). Next, we prove (9).

(Sufficiency) If (8) is true, then there exists a vector $u(k) \in \{u \in \mathcal{U} :\ Au = v(k)\} \subseteq \mathcal{U}(v(k))$ such that $u(k)$ depends on $v(k)$. Thus $a(k) \in C(v(k))$ by (4).

(Necessity) If (8) is false, i.e., $\mathcal{V}^{\Theta} \not\subseteq A\mathcal{U}$, then there exists a vertex $v^{(r)}$ of $\mathcal{V}^{\Theta}$ such that $Au \neq v^{(r)}$ for all $u \in \mathcal{U}$; $U \cap U(v^{(r)}) = \emptyset$. At each time $k$ where $v(k) = v^{(r)}$ there exists no $a(k) \in C(v(k))$; the core $C(v(k))$ is empty. □

We consider situations where the game designer knows $x(k)$ and $v(k-1)$ at time $k$; there is a one-period delay in the information of the game values. Given a function $f(k)$, denote by $\bar{f}$ the long term average of a given function $f(k)$, i.e., $\bar{f} = \lim_{k \longrightarrow \infty} \frac{1}{k} \sum_{j=1}^{k} f(j)$ and $\bar{f}^k$ the average up to time $k$. The game $< N, \bar{v}^k >$ is called the *average game* (up to time $k$).

If $v(k)$ is not known at time $k$ then the core $C(v(k))$ is also not known, and its elements cannot be used for allocations. In this case, allocations outside the core may be approximately close to the core of the average game according to a certain tolerance $\epsilon$. Average games and $\epsilon$-balancedness are related as follows.

**Theorem 2** *Take $\epsilon = \max_{v \in \mathcal{V}^{\Theta}} \|v\|$, the infinity norm of the vector $v$, and consider a tolerance $\epsilon(k) := \frac{\epsilon}{k}$. Assume $0 \in \mathcal{V}^{\Theta}$. There exists a time $\tilde{k}$ such that all average games $< N, \bar{v}^k >$ are $\epsilon(k)$-balanced for all $k \geq \tilde{k}$ if and only if*

$$\mathcal{V}^{\Theta} \subseteq int\{A\mathcal{U}\}. \tag{10}$$

*Furthermore, there exists an allocation rule $u(k)$ as a function of $x(k)$ such that*

$$\bar{a}^k \in C_{\epsilon(k)}(\bar{v}^k),\ \forall k \geq \tilde{k}. \tag{11}$$

PROOF.

(Sufficiency) Assume first that $\tilde{k}$ exists (we prove its existence below), and take for it the first time instant where $-x(\tilde{k}) \in \mathcal{V}^{\Theta}$. We show that $-x(\tilde{k}+1) \in \mathcal{V}^{\Theta}$ for some $u(\tilde{k}) \in \mathcal{U}$. By (10), there exists an allocation rule $u(\tilde{k}) \in \mathcal{U}$ such that $Au(\tilde{k}) = -x(\tilde{k})$. Then $x(\tilde{k}+1) = -v(\tilde{k})$ by (7) and also $-x(\tilde{k}+1) \in \mathcal{V}^{\Theta}$. We can repeat the same argument inductively for $\tilde{k}+2$ and so on. This proves that $-x(k) \in \mathcal{V}^{\Theta}$ for all $k \geq \tilde{k}$.

Now, we prove the existence of $\tilde{k}$. Consider any time instant $k$ such that $-x(k) \notin \mathcal{V}^{\Theta}$. Define a new variable $w(k) = x(k-1) + Au(k-1)$ so that $w(k) - x(k) = v(k-1) \in \mathcal{V}^{\Theta}$. Now, choose $u(k) := u_1(k) + u_2(k) \in \mathcal{U}$ with $u_1(k)$ satisfying

$$Au_1(k) = w(k) - x(k) = v(k-1) \in \mathcal{V}^{\Theta}.$$

Using the above equality, the dynamics for $w(k)$ turn out to be

$$\begin{aligned} w(k+1) &= x(k) + Au(k) \\ &= x(k) + Au_1(k) + Au_2(k) \\ &= w(k) + Au_2(k). \end{aligned}$$

Now select $u_2(k)$ such that the norm of $w(k+1)$ is minimized, i.e.,

$$u_2(k) = \arg\min_{u \in \mathcal{U}} \|w(k) + Au\|. \tag{12}$$

If $u_2$ is chosen such that $Au_2$ is in the opposite direction of $w$, then the formulation of $w(k+1)$ implies that the norm of $w$ is reduced. By (10) and $0 \in \mathcal{V}^{\Theta}$ we have $0 \in int\{A\mathcal{U}\}$, and so

$$\|w(k+1)\| - \|w(k)\| \leq \beta < 0$$

for some $\beta$, until $\|w(\tilde{k})\| = 0$ for a large enough $\tilde{k}$. This in turn implies that $-x(k)$, whose dynamics can be rewritten as $-x(k) = -w(k) + v(k-1)$, is ultimately bounded in $\mathcal{V}^{\Theta}$.

Hence we have shown that there exists a time $\tilde{k}$ such that $-x(k) \in \mathcal{V}^{\Theta}$, and $\|x(k)\| \leq \epsilon$ as well, for all $k \geq \tilde{k}$. Now, using $\epsilon(k) = \epsilon/k$, the latter inequality implies $-\epsilon(k) \leq \sum_{i \in S} \bar{a}_i^k - \bar{s}_S^k - \bar{v}_S^k \leq \epsilon(k)$ for all $S \subset N$. Therefore $\sum_{i \in S} \bar{a}_i^k \geq \bar{v}_S^k + \bar{s}_S^k - \epsilon(k) \geq \bar{v}_S^k - \epsilon(k)$. This proves that the average games $< N, \bar{v}^k >$ are $\epsilon(k)$-balanced for all $k \geq \tilde{k}$ and also that (11) holds.

(Necessity) Assume that (10) is false. Then there certainly exists a vertex $v^{(r)}$ of the polytopic set $\mathcal{V}^{\Theta}$ such that either there does not exist $u_1(k) \in \mathcal{U}$ for which $Au_1(k) = v^{(r)}$ or, if it exists, it is such that $Au_1(k) \in \partial A\mathcal{U}$. In the first case we necessarily have $u_2(k) = 0$, while in the second case (12) shows that $u_2(k)$ belongs to the null space of $A$, i.e., $Au_2(k) = 0$. Thus if $v(k) = v^{(r)}$ for all $k > 0$ then we cannot keep $x(k)$ in $\mathcal{V}^{\Theta}$ for any $k$ large enough. Then the average game $< N, \bar{v}^k >$ is not $\epsilon(k)$-balanced for some $k > 0$, and condition (11) is not true. $\square$

Under certain conditions we can design augmented allocation rules such that averaging the allocations over the long run results in a desired value, called the *nominal allocation*. Let $a_{nom} \in \mathbb{R}_+^N$ be an *a priori* given allocation vector, referred to as the *nominal allocation*. Let $Ext\{\mathcal{V}^{\Theta}\} = \{1, 2, \ldots, b\}$ be the set of indices of all vertices of $\mathcal{V}^{\Theta}$. Denote by $v^{(r)}$, $r \in Ext\{\mathcal{V}^{\Theta}\}$, a vertex of the polytopic set $\mathcal{V}^{\Theta}$. Assume that $\bar{v} = v_{nom}$, the vector of known *nominal coalitions' values*. Consider a matrix $D \in \mathbb{R}^{(n+m-1)\times m}$ subject to the conditions:

$$AD = I \in \mathbb{R}^{m \times m}, \text{ and} \tag{13}$$

$$u_{min} \leq D(v^{(r)} - v_{nom}) + u_{nom} \leq u_{max}, \quad r \in Ext\{\mathcal{V}^{\Theta}\}, \tag{14}$$

where $u_{nom} = \begin{bmatrix} a_{nom} \\ s_{nom} \end{bmatrix} \in \mathcal{U}$ is such that $Au_{nom} = v_{nom}$. We investigate under which conditions

$$\bar{a} = a_{nom}. \tag{15}$$

**Theorem 3** *Assume that condition (10) is satisfied, and the average coalitions' values $\bar{v}$ are equal to a fixed $v_{nom} \in \mathcal{V}^{\Theta}$. Furthermore, consider $u_{nom} = \begin{bmatrix} a_{nom} \\ s_{nom} \end{bmatrix} \in \mathcal{U}$ such that $Au_{nom} = v_{nom}$. There exists an allocation rule $u(k)$, as a function of $x(k)$, such that (11) with $\epsilon = \max_{v \in \mathcal{V}^{\Theta}} \|v\|$ and (15) hold if and only if there exists a matrix $D \in \mathbb{R}^{(n+m-1)\times m}$ that satisfies (13) and (15).*

PROOF. (Sufficiency) Assume $D$ satisfies (13) and (15). Using a standard property of linear algebra, we can find matrices $C \in \mathbb{R}^{(n-1)\times(n+m-1)}$ and $F \in \mathbb{R}^{(n+m-1)\times(n-1)}$ such that

$$\begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I, \tag{16}$$

with $I$ the $(n+m-1)$-dimensional identity matrix. Using $C$ and $y(k) \in \mathbb{R}^{n-1}$ we construct the augmented dynamic system

$$\begin{aligned} x(k+1) &= x(k) + Au(k) - v(k) \\ y(k+1) &= y(k) + Cu(k). \end{aligned} \tag{17}$$

Also, we use matrix $F$ to define a new variable $z(k) \in \mathbb{R}^{n+m-1}$ as expressed below:

$$z(k) = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} z(k).$$

Using (16) and (17), this variable evolves according to the following dynamic equation:

$$\begin{aligned} z(k+1) &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k+1) \\ y(k+1) \end{bmatrix} \\ &= z(k) + u(k) - Dv(k). \end{aligned} \tag{18}$$

It is useful to write the above dynamics componentwise. Then

$$z_i(k+1) = z_i(k) + u_i(k) - D_i v(k), \tag{19}$$

where $D_i$ is the $i$th row of $D$ and $u_{i,\min} \leq u_i(k) \leq u_{i,\max}$. A possible allocation rule is

$$u_i(k) = \begin{cases} u_{i,min} & \text{if } z_i(k) > -u_{i,\min} \\ u_{i,max} & \text{if } z_i(k) < -u_{i,\max} \\ -z_i(k) & \text{if } -u_{i,\max} \leq z_i(k) \leq -u_{i,\min} \end{cases}. \tag{20}$$

First, we show that there exists a time $\tilde{k}$ such that $-u_{i,\max} \leq z_i(\tilde{k}) \leq -u_{i,\min}$. Consider $z_i(k) < -u_{i,\max}$. Using (20) in the dynamics (19) and again because of (15), we obtain $z_i(k+1) - z_i(k) = u_{i,\max} - D_i v(k) > 0$. This holds until $-u_{i,max} \leq z_i(\tilde{k})$ for $\tilde{k}$ large enough. Further, the proof for $z_i(k) > -u_{i,\min}$ is along similar lines.

Next, take without loss of generality $u_{nom} = v_{nom} = 0$. Because of (15), note that the dynamics (19) and (20) imply $-u_{i,\max} \leq z_i(\tilde{k}+1) = -D_i v(\tilde{k}) \leq -u_{i,\min}$. Repeating the same argument forward in time results in $-u_{i,\max} \leq z_i(k+1) = -D_i v(k) \leq -u_{i,\min}$ for all $k \geq \tilde{k}$. This proves that for $\tilde{k}$ large enough

$\|z(k)\| \leq \max_{\tau \geq \tilde{k}} \|Dv(\tau)\|$ for all $k \geq \tilde{k}$. This condition also implies that $\|x(k)\| \leq \max_{k \geq \tilde{k}} \|Az(k)\| \leq \max_{v \in \mathcal{V}^\Theta} \|ADv\| = \max_{v \in \mathcal{V}^\Theta} \|v\| = \max_{j=1,\dots,b} \|v^{(j)}\| = \epsilon$ for all $k \geq \tilde{k}$, which proves condition (11).

To prove (15), we use (18) to obtain $\frac{1}{T}\sum_{k=1}^{T} u(k) - \frac{1}{T}\sum_{k=1}^{T} Dv(k) = (z(T+1) - z(1))/T$. This converges to 0 as $T \to \infty$, since the numerator is a finite quantity whereas the denominator tends to infinity. Therefore, $\bar{u} = D\bar{v}$, and so $\bar{a} = a_{nom}$.

(Necessity) We show that if (13) and (15) do not hold then (11) does not hold as well. Actually, if (13) and (15) are not true then invoking results in [2] Section 3 there exists no allocation rule $u(k)$ such that (9) and (15) hold. This implies that (8) and therefore (10) are not verified. Invoking Theorem 2, we also have that (11) does not hold. This concludes our proof. □

## 4 Unbalanced games

In this section, we consider allocation rules for sequences of games that are, in general, not balanced. This is the case, for instance, when condition (8) in Lemma 1 does not hold.

**Assumption 1** *The following condition is satisfied:*

$$\mathcal{V}^\Theta \not\subseteq A\mathcal{U}. \tag{21}$$

We also assume that the expected coalitions' values are not correlated with the state, and coincide at each time with the long term average. This is reasonable since the coalitions' values are independent of the past allocations and, differently from [6,8], vary according to some exogenous random process.

**Assumption 2** *The vector of coalitions' values $v(k)$ satisfies $E[v(k)] = \bar{v}$ and $E[x(k)'v(k)] = 0$.*

We translate the origin of the $u$ and $v$ spaces without loss of generality.

**Assumption 3** *For ease of calculations set $u_{nom} = v_{nom} = 0$ and assume that $\bar{v} = v_{nom} = 0 \in int\{A\mathcal{U}\}$.*

Define the *distance* between a point $x \in \mathbb{R}^n$ and a set $S$ in $\mathbb{R}^n$ as $d(x, S) = \min_{y \in S} \|x - y\|$, and define the function $V(x) = x^T x/2$. Our main result on unbalanced games is stated below.

**Theorem 4** *Under Assumptions 2 and 3, there exists an allocation rule $u(k)$ such that $d(\bar{a}^k, C(\bar{v}^k)) \to 0$, for $k \to \infty$ with probability one. Furthermore, such an allocation rule satisfies (15). A possible augmented allocation rule is*

$$u(k) = \arg\min_{u \in \mathcal{U}} V(x(k) + Au). \tag{22}$$

PROOF. The first part of the theorem, which establishes $d(\bar{a}^k, C(\bar{v}^k)) \to 0$ for $k \to \infty$, is proved if we show that $x(k)$ tends to zero with probability one. Let $u(k)$ be defined as in (22), and recall that $0 \in int\{A\mathcal{U}\}$. For $x \neq 0$ the new variable $w(k) = x(k-1) + Au(k-1)$ satisfies the condition

$$V(w(k+1)) = V(x(k)+Au(k)) \leq V(x(k)+A0) = V(x(k))$$

by definition of $u(k)$, and because $0 \in \mathcal{U}$. Further, $w(k+1) = x(k) + Au(k) = x(k+1) + v(k)$ by definition of $w(k)$ and (7). This implies $x(k+1) = w(k+1) - v(k)$. Applying the triangle inequality results in

$$V(x(k+1)) \leq V(w(k+1))+V(v(k)) \leq V(x(k))+V(v(k)).$$

If we take expectations we obtain $E(V(x(k+1))) \leq E(V(x(k)))+E(V(v(k)))$. Recall that $\lim_{\Theta\to 0} E(V(v(k))) = 0$ as $v(k) \in \mathcal{V}^\Theta$ and $\mathcal{V}^\Theta = \Theta\mathcal{V}$. Then, taking the limit we have $\lim_{\Theta\to 0} E(V(x(k+1))) \leq \lim_{\Theta\to 0}(E(V(x(k))) + E(V(v(k)))) = \lim_{\Theta\to 0} E(V(x(k)))$. This last inequality implies that $x(k)$ tends to zero with probability one (and $x(k)$ is said to be stochastically stable).

The proposed rule does imply stochastic stability but it does not necessarily satisfy (15). To enforce (15) we use (19) which we rewrite as $z_i(k+1) = z_i(k)+u_i(k)-\delta_i(k)$, where $\delta_i(k) = D_i v(k)$, and $D_i$ is the $i$th row of any matrix $D$ satisfying (13) but not necessarily (15). Note that $E[\delta_i] = D_i E[v(k)] = 0$. If we consider the function $V(z_i(k)) = z_i(k)^2/2$, and slightly modify (22) to

$$u_i(k) = \arg\min_{u_{i,\min} \leq \mu \leq u_{i,\max}} V(z_i(k) + \mu),$$

then we see that $E(V(z_i(k+1))) \leq E(V(z_i(k))) + E(V(\delta_i(k)))$. Taking limits results in $\lim_{\Theta\to 0} E(V(z_i(k+1))) \leq \lim_{\Theta\to 0} E(V(z_i(k))) + \underbrace{\lim_{\Theta\to 0} E(V(\delta_i(k)))}_{=0} \leq \lim_{\Theta\to 0} E(V(z_i(k)))$ which means that the $z_i(k)$ subsystem is stable with probability one. Then, $\frac{1}{T}[z_i(T) - z_i(0)] \to 0$, which implies $\frac{1}{T}\sum_{k=1}^{T} [u_i(k) - \delta_i(k)] \to 0$ with probability one. This proves (15). □

## 5 Numerical example.

Consider three players and the following coalitions' values (c.f. multi retailer system in [3]): $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$, $v(\{1,2\}) \in [0,5]$, $v(\{1,3\}) \in [0,5]$, $v(\{2,3\}) \in [0,7]$, $v(N) = 12$. Let the nominal coalitions' values and the nominal average allocation vector be

$$v_{nom} = [0, 0, 0, 2, 3, 4, 12]^T, \; u_{nom} = [4, 5, 3, 4, 5, 3, 7, 4, 4]^T.$$

Note that $Au_{nom} = v_{nom}$. We translate the origin of the $u$-$v$ space to $u_{nom}$-$v_{nom}$. First, we calculate $D$ by using the linear program in [2, Section 5.2]. Then we compute matrices $C$ and $F$ that square $B$ and $D$, as described in the proof of Theorem 3, using the method explained in detail in the appendix of [3]. For the maximum sample time we get $\Theta^* > 0.1$ and choose $\Theta = 0.1$.

In a first set of simulations, we consider the bounding polyhedron $\mathcal{U} := \{u \in \mathbb{R}^9 : -10^{-1} \cdot u_{nom} \le u \le 10^{-1} \cdot 5 \cdot \mathbf{1})\}$ where $\mathbf{1}$ is the 9-dimensional vector of ones. Note that after translation of the origin to $(u_{nom}, v_{nom})$ the surplus variables may take negative values. Condition (10) holds and the resulting games in the sequence are balanced. To see why (10) is true note that

$$\mathcal{V}^\Theta = 10^{-1} \cdot$$
$$\{v \in \mathbb{R}^7 : v(\{1\}) = v(\{2\}) = v(\{3\}) = v(N) = 0,$$
$$v(\{1,2\}) \in [-2, 3], \; v(\{1,3\}) \in [-3, 2], \; v(\{2,3\}) \in [-4, 3]\}.$$

Also observe that there exists an augmented allocation vector of type $\hat{u} = 10^{-1} \cdot [0, 0, 0, 0, 0, 0, \times, \times, \times]^T$ that satisfies $v = A\hat{u} \in int\{A\mathcal{U}\}$ for any $v \in \mathcal{V}^\Theta$. The symbols $\times$ refer to the surplus values of coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ that turn out to belong to intervals $[-0.3, 0.2]$ $[-0.2, 0.3]$ and $[-0.3, 0.4]$ whichever $v \in \mathcal{V}^\Theta$ and as such the condition $\hat{u} \in \mathcal{U}$ holds.

Now, we implement the dynamic allocation rule (20) to simulate the evolution of the system as displayed in Figure 1, left. In the simulation, coalition $\{1, 2\}$ takes on values from the repeated deterministic sequence $[-2, 3, -2, 3, -2, 3, -2, 3, -2, -2]$. Likewise for coalition $\{1, 3\}$ and sequence $[-3, 2, -3, 2, -3, 2, -3, 2, 2, 2]$ and coalition $\{2, 3\}$ and sequence $[-4, 3, -4, 3, -4, 3, 3]$. Note that after translation of the origin to $v_{nom}$ we must have $\bar{v} = 0$ and therefore we need to consider sequences with zero mean.

Fig. 1 top left, illustrates the time plot of $x(.)$. The variable is $\epsilon$-stabilized with $\epsilon < 0.4$ in accordance to (11). Fig. 1 bottom left shows the time plot of $\bar{u}^k - u_{nom}$, where $\bar{u}^k$ is the average of $u(k)$ up to time $k$. All plots tend to zero which means that the average $\bar{u}^k$ tends to $u_{nom}$.

In a second set of simulations, we consider the bounding polyhedron $\mathcal{U} := \{u \in \mathbb{R}^9 : -10^{-1} \cdot u_{nom} \le u \le 10^{-1} \cdot 2 \cdot \mathbf{1}\}$. Condition (10) no longer holds (the bounds in $\mathcal{U}$ are too tight) and the resulting games in the sequence are not balanced. Fig. 1 top right displays the time plot of $x(.)$. Peaks illustrate that the variable is not $\epsilon$-stabilized

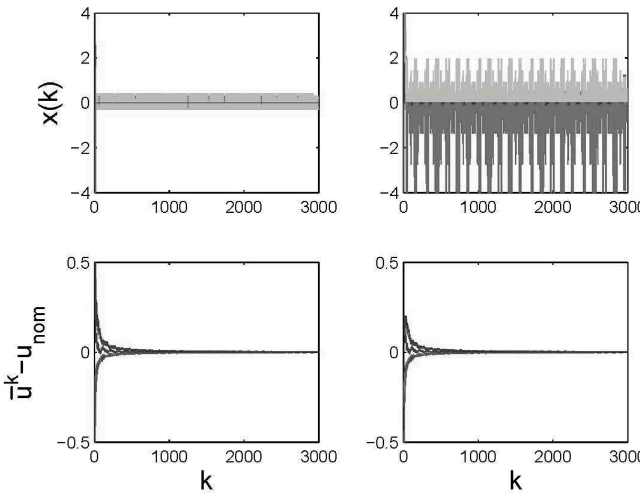


Fig. 1. Time plot of $x(k)$ (top) and $\bar{u}^k - u_{nom}$ (bottom) for the balanced case (left) and unbalanced case (right).

with $\epsilon < 0.4$, in accordance with the results in Section 4. Finally, Fig. 1 bottom right shows the time plot of $\bar{u}^k - u_{nom}$, where $\bar{u}^k$ is the average of $u(k)$ up to time $k$. All plots tend to zero which means that the average $\bar{u}(k)$ tends to $u_{nom}$.

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