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Opinion dynamics and stubbornness through mean-field games

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Abstract— **This paper provides a mean field game theoretic interpretation of opinion dynamics and stubbornness. The model describes a crowd-seeking homogeneous population of agents, under the influence of one stubborn agent. The game takes on the form of two partial differential equations, the Hamilton-Jacobi-Bellman equation and the Kolmogorov-Fokker-Planck equation for the individual optimal response and the population evolution, respectively. For the game of interest, we establish a mean field equilibrium where all agents reach ε-consensus in a neighborhood of the stubborn agent's opinion.**

I. INTRODUCTION

Over the past few years there has been an increasing interest in the field of opinion dynamics. These describe the time evolution of the beliefs of a typically very large population of agents in response to repeated interactions among themselves over a social network (see, e.g., [6, Sect. III] and [1]). In continuous opinion dynamics, beliefs (hereafter opinions or beliefs are used interchangeably) are represented as scalars or vectors, each moving towards a convex combinations of (a subset of) other agents' current beliefs, thus modeling the attractive nature of social influence. Standard models predict, provided that the underlying social network is connected, that a consensus among the agents is achieved asymptotically in time. Exceptions are provided by bounded confidence models [11], whereby agents do not take into account the influence of other agents whose beliefs are too different from theirs, as well as models with competing stubborn agents [2] who do not change their opinions but are able to influence the ones of the rest of the population. Such stubborn agents might represent leaders, political parties or media sources attempting to influence the beliefs in the rest of the population. Scaling limits results (see, e.g., [7]) show that, if the agents' population is homogeneous, the empirical belief distribution converges, as the population size grows large, towards the solution of a certain deterministic mean-field differential equation in the space of probability measures. Such results are in the spirit of the propagation of chaos [8] in interacting particle systems.

As a matter of fact, interactions among particles is a main element in the theory of mean field games initiated by Lasry and Lions in [13], [4]. The underlying idea is that a large number of indistinguishable players (the particles) interact so that the strategies of a single player is influenced by the distribution of the other players. Such a model has been shown to be useful in several application domains such as economics, physics, biology, and network engineering (see [3], [5], [9], [10], [12], [15], [17]).

Main contribution. In this paper we propose an approach to opinion dynamics within the framework of mean-field games. We focus on a stochastic model with time-continuous beliefs comprising a homogeneous population of agents, plus one stubborn agent trying to influence them.

The main contribution of this paper is to provide a mean field game theoretic interpretation of opinion dynamics and stubbornness. The game takes the form of two partial differential equations (PDEs), which is the classical structure of a mean field game. The first PDE is the Hamilton-Jacobi-Bellman equation which has to be solved backward in time once we fix a penalty on the final deviation of the individual's opinion from the mainstream opinion (this is true for a finite horizon formulation). The second PDE is the Kolmogorov-Fokker-Planck equation which describes the density evolution of the players' opinions in response to the individuals' optimal behaviors as returned by the first PDE. [13], [16].

For the game of interest, we also establish a mean field equilibrium where all agents reach ε-consensus in a neighborhood of the stubborn agent's opinion. Mean field equilibrium strategies are shown to be state-feedback linear control policies. In addition to this, the tolerance ε appears to essentially depend on the Brownian Motion, and decrease on the number of players. We illustrate the theoretical results on numerical examples.

This paper is organized as follows. In Section II, we set up the problem and related model. In Section III, we analyze the system microscopic behavior, and study equilibria and stability. In Section IV, we provide numerical results. Finally, in Section V, we provide conclusions.

Notation We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. We let $\mathcal{B}$ be a finite-dimensional standard Brownian motion process defined on this probability space. We define $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, its natural filtration augmented by all the $\mathbb{P}-$null sets (sets of measure-zero with the respect $\mathbb{P}$). We write ∂_x and ∂^2_{xx} to stand respectively for the first and second derivatives with respect to x.

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II. Model and problem set-up

Consider a population of homogeneous agents (players), each one characterized by an opinion $X(t) \in \mathbb{R}$ at time $0 \leq t \leq T$, where $[0,T]$ is the time horizon window. The control variable is a measurable function of time $u_1(\cdot)$, defined from $[0,T]$ to $\mathbb{R}$ and establishes the rate of variation of an agent's opinion. A stubborn agent tries to disturb the agents' opinions in a way that is proportional to his advertisement efforts $u_2(\cdot)$, a measurable function from $[0,T]$ to $\mathbb{R}$, which is the control of the stubborn agent.

It turns out that the opinion dynamics can be written in the form $X' = f(X, u_1, u_2) + \sigma d\mathcal{B}$ with $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ constant in X and linear in the controls:

$$\left\{ \begin{array}{l} dX(s) = (u_1(s) + u_2(s))ds + \sigma d\mathcal{B}(s), \;\; s > t \\ X(t) = x, \end{array} \right. \tag{1}$$

where $\sigma > 0$ is a weighting coefficient and $d\mathcal{B}(t)$ is a Brownian motion.

Consider a probability density function $m : \mathbb{R}\times[0,+\infty[\to \mathbb{R}$, $(x,t) \mapsto m[x](t)$, representing the percentage of agents in state x at time t, which satisfies $\int_{\mathbb{R}} m[x](t)dx = 1$ for every t. Let us also define the mean opinion at time t as $\overline{m}(t) := \int_{\mathbb{R}} xm[x](t)dx$.

The objective of the agent is to adjust his opinion based on the average opinion of all agents. This reflects a typical crowd-seeking behavior in that emulating others brings some benefits and makes an agent more comfortable and at ease.

Then, for the agents, consider a running cost $g_1 : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to [0,+\infty[$, $(x,\overline{m},u_1) \mapsto g_1(x,\overline{m},u_1)$ of the form:

$$g_1(x,\overline{m},u_1) = \frac{1}{2}\left(a_1\Big(\overline{m}-x\Big)^2 + c_1 u_1^2\right). \tag{2}$$

Also consider a final cost $\Psi_1 : \mathbb{R}\times\mathbb{R} \to [0,+\infty[$, $(\overline{m},x) \mapsto \Psi_1(\overline{m},x)$ of the form

$$\Psi_1(\overline{m},x) = \frac{1}{2}S_1\Big(\overline{m}-x\Big)^2.$$

We suppose that the stubborn agent wants to attract the agent's opinions towards his opinion itself, and that such an opinion correspond to $x = 0$. Hence, for the stubborn agent, we consider a running cost (one per every opinion evolution) $g_2 : \mathbb{R}\times\mathbb{R} \to [0,+\infty[$, $(x,u_2) \mapsto g_2(x,u_2)$ of the form:

$$g_2(x,u_2) = \frac{1}{2}\left(a_2x^2 + c_2u_2^2\right). \tag{3}$$

We also consider a final cost $\Psi_2 : \mathbb{R} \to [0,+\infty[$, of the form

$$\Psi_2(x) = \frac{1}{2}S_2x^2.$$

The parameters $a_1, c_1, S_1, a_2, c_2, S_2$ are positive and fixed.

Problem 1: The problem is the following: given an initial distribution of opinions m_0 and the corresponding mean $\overline{m}_0 : \mathbb{R} \to \mathbb{R}$, for every $(x,t) \in \mathbb{R}\times[0,T]$, minimize over all measurable control $u_1(\cdot)$ and over all measurable controls $u_2(\cdot)$, the following two cost functionals, respectively

$$\min_{u_1(.)} \mathbb{E}\left(\int_t^T g_1(X(s),\overline{m}(s),u_1(s))ds + \Psi_1(\overline{m}(T),X(T))\right),$$

$$\min_{u_2(.)} \mathbb{E}\left(\int_t^T g_2(X(s),u_2(s))ds + \Psi_2(X(T))\right),$$

where $X(\cdot)$ is the trajectory of the stochastic controlled equation starting from the single opinion x, and $\overline{m}(\cdot)$ is the evolution of the mean distribution of the opinions if every one of the agents behaves optimally.

The problem has then a differential game feature. We have a family of differential games, one per every initial opinion x, where the players are all the (homogeneous) agents with that initial opinion and the stubborn agent. The dynamics depends on both controls, and the dynamics of the opinion also depends on the mean opinion evolution (and so, inside the family, the differential games are mutually influenced).

Being a differential game, we are interested in possible Nash equilibria. A pair of controls $(u_1^*(\cdot), u_2^*(\cdot))$ is a Nash equilibrium if, denoting by v_1 (resp. v_2) the value function of the first (resp. second) player when the second (resp. first) one implements u_2^* (resp. u_1^*), then v_1 (resp. v_2) is obtained exactly when the first (resp. second) player implements u_1^* (resp. u_2^*).

The problem then results in the following mean field game system for the unknown scalar functions $v_1(x,t)$, $v_2(x,t)$, and $m(x,t)$

$$\left\{ \begin{array}{l} -\partial_t v_1(x,t) - \dfrac{\sigma^2}{2}\partial^2_{xx}v_1(x,t) + \{-f(x,u_1^*,u_2^*)\partial_x v_1(x,t) \\ -g_1(x,m(t),u_1)\} = 0 \text{ in } \mathbb{R}\times[0,T[, \\ v_1(x,T) = \Psi_1(x) \text{ in } \mathbb{R}, \\ -\partial_t v_2(x,t) - \dfrac{\sigma^2}{2}\partial^2_{xx}v_2(x,t) + \{-f(x,u_1^*,u_2^*)\partial_x v_2(x,t) \\ -g_2(x,m(t),u_2(x,t))\} = 0 \text{ in } \mathbb{R}\times[0,T[, \\ v_2(x,T) = \Psi_2(x) \text{ in } \mathbb{R}, \\ \partial_t m(x,t) + \partial_x(m(x,t)f(X,u_1^*,u_2^*)) \\ -\frac{\sigma^2}{2}\partial^2_{xx}m(x,t) = 0, \text{ in } \mathbb{R}\times[0,T[, \\ m(x,0) = m_0(x) \text{ in } \mathbb{R}. \end{array} \right.$$

Now, it is reasonable, to consider v_1 and v_2 have a quadratic form:

$$v_1(x,t) = \frac{q_1(t)}{2}(x-\overline{m}(t))^2, \quad v_2(x,t) = \frac{q_2(t)}{2}x^2,$$

and so, searching the optimal time-varying state-feedback controls as

$$\left\{ \begin{array}{l} u_1^*(x,t) = \text{argmax}_{u_1\in\mathbb{R}}\{-f(x,u_1,u_2^*)\partial_x v_1(x,t) \\ \quad -g_1(x,m(t),u_1)\}, \\ \\ u_2^*(x,t) = \text{argmax}_{u_2\in\mathbb{R}}\{-f(x,u_1^*,u_2)\partial_x v_2(x,t) \\ \quad -g_2(x,m(t),u_2)\}, \end{array} \right. \tag{4}$$

we get

$$u_1^*(x,t) = -\frac{q_1(t)}{c_1}(x-\overline{m}(t)), \quad u_2^*(x,t) = -\frac{q_1(t)}{c_2}x.$$

Here, $q_1(\cdot)$ and $q_2(\cdot)$ are solutions of the corresponding Riccati equations.

III. MICROSCOPIC MODEL

Let a finite set of players $\{1,\dots,n\}$ be given and let $Y_i(t)$ for all $i=1,\dots,n$ be the corresponding state. Let us collect all states in a state vector $Y(t)=[Y_1(t),\dots,Y_n(t)]^T$. Given the optimal controls $u_1^*(x,t)$ and $u_2^*(x,t)$ as computed above the evolution of the state vector is captured by the Stochastic Differential Equation (SDE)

$$dY(t)=\left[\frac{q_1}{c_1}(\overline{m}\mathbf{1}-Y(t))-\frac{q_2}{c_2}Y(t)\right]dt+\sigma d\mathcal{B}(t). \quad (5)$$

For future purposes it is convenient to rewrite (5) making use of a stochastic matrix. To do this, let us introduce $W=-\frac{q_1}{c_1}L+I$, where L is the Laplacian of a fully connected network. Observe that

$$W=W^T \quad W\mathbf{1}=\mathbf{1}. \quad (6)$$

Then, we can rewrite (5) as

$$dY(t)=\left[(W-I)\,Y(t)-\frac{q_2}{c_2}Y(t)\right]dt+\sigma d\mathcal{B}(t) \quad (7)$$

The above equation is useful as it allows us to analyze the evolution of the stochastic properties of the mean opinion $\overline{m}(t)$. Indeed, observe that $\overline{m}(t)$ is a stochastic process with first-order moment satisfying

$$\mathbb{E}d\overline{m}(t)=-\frac{q_2}{c_2}\mathbb{E}\overline{m}(t)dt.$$

Then, we can infer that the first-order moment converges exponentially to zero according to

$$\mathbb{E}\overline{m}(t)=e^{-\frac{q_2}{c_2}t}\mathbb{E}\overline{m}(t)$$

and therefore $\mathbb{E}\overline{m}(t)\to 0$.

Definition 1 (asymptotic ε-stability): We say that a stochastic process $\xi(t)$, $t\geq 0$ is *asymptotically ε-stable* if $\lim_{t\to\infty}\sup\|\xi\|\leq\varepsilon$.

The next theorem establishes that the mean opinion converges to a neighborhood of zero almost surely.

Theorem 1: There exists $\varepsilon>0$ such that the mean opinion is ε-stable w.p.1,

$$\lim_{t\to\infty}\|\overline{m}(t)\|\leq\varepsilon, \text{ w.p.1.}$$

Furthermore, the smallest ε for which the above holds is

$$\varepsilon=\sqrt{\frac{1}{2}\frac{c_2}{q_2}\frac{\sigma^2}{n^2}}. \quad (8)$$

Proof: This proof is within the framework of Lyapunov stochastic stability theory (cfg. [14], system (64.51)). We start by observing that

$$\begin{array}{rcl} d\overline{m}(t) & = & \frac{1}{n}\mathbf{1}^T dY(t) \\ & = & \frac{1}{n}\mathbf{1}^T\left[(W-I)Y(t)-\frac{q_2}{c_2}Y(t)\right]dt+\frac{1}{n}\mathbf{1}^T\sigma d\mathcal{B}(t) \\ & = & -\frac{1}{n}\mathbf{1}^T\frac{q_2}{c_2}Y(t)dt+\frac{1}{n}\mathbf{1}^T\sigma d\mathcal{B}(t) \\ & = & -\frac{q_2}{c_2}\overline{m}(t)dt+\frac{1}{n}\mathbf{1}^T\sigma d\mathcal{B}(t). \end{array} \quad (9)$$

The above SDE is linear and the corresponding stochastic process can be studied in the framework of stochastic stability theory [14]. To do this, consider the infinitesimal generator

$$\mathcal{L}=\frac{1}{2}\sigma^2\frac{d^2}{d\overline{m}^2}-\frac{q_2}{c_2}\overline{m}(t)\frac{d}{d\overline{m}(t)}. \quad (10)$$

The above equation is obtained from

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\left(d\overline{m}^2\frac{d^2}{d\overline{m}^2}\right)+\mathbb{E}\left(d\overline{m}\frac{d}{d\overline{m}}\right)\\ &=\frac{1}{2}\left[\mathbb{E}\left((-\frac{q_2}{c_2})^2\overline{m}(t)^2dt^2\right)+\mathbb{E}\left(\sigma^2 d\mathcal{B}(t)^2\right)\right.\\ &\quad\left.+\mathbb{E}\left(2(-\frac{q_2}{c_2})\overline{m}(t)dt\sigma d\mathcal{B}(t)\right)\right]\frac{d^2}{d\overline{m}(t)^2}\\ &+\left[\mathbb{E}\left((-\frac{q_2}{c_2})\overline{m}(t)dt\right)+\mathbb{E}\left(\sigma d\mathcal{B}(t)\right)\right]\frac{d}{d\overline{m}(t)}. \end{aligned}$$

Now, recalling that for a Brownian motion it holds $\mathbb{E}d\mathcal{B}(t)=0$ and $\mathbb{E}d\mathcal{B}(t)^2\to 0$ and ignoring the second-order terms (in dt^2 or $dtd\mathcal{B}(t)$) we obtain (10).

Consider a candidate Lyapunov function $V(\overline{m})=\frac{1}{2}\overline{m}(t)^2$. The idea is to show that there exists a finite scalar κ and a neighborhood of zero of size κ, denoted by $\mathcal{N}_\kappa=\{\overline{m}\in\mathbb{R}|\,V(\overline{m})\leq\kappa\}$, such that the stochastic derivative of $V(\overline{m})$ is negative, i.e.

$$\begin{array}{c} \mathcal{L}V(\overline{m}(t)):=\lim_{dt\to 0}\frac{\mathbb{E}V(\overline{m}(t+dt))-V(\overline{m}(t))}{dt}<0, \\ \text{for all } \overline{m}(t)\notin\mathcal{N}_\kappa. \end{array} \quad (11)$$

Given the above set, we also need to show that once $\overline{m}(t)\in\mathcal{N}_\kappa$ then $\overline{m}(t)+d\overline{m}(t)\in\mathcal{N}_\kappa$. Actually, the theory establishes that if the former condition holds true, which is $\mathcal{L}V(\overline{m}(t))<0$, then $V(\overline{m}(t))$ is a supermartingale whenever $\overline{m}(t)$ is not in $\mathcal{N}_\kappa$ and therefore by the martingale convergence theorem $\exists\, t$ such that $V(\overline{m}(t))\leq\kappa$. Combining this with the property $\overline{m}(t)+d\overline{m}(t)\in\mathcal{N}_\kappa$ for every $\overline{m}(t)\in\mathcal{N}_\kappa$ we obtain $\lim_{t\to\infty}V(\overline{m}(t))\leq\kappa$ w.p.1 (almost surely), which in turn implies $\lim_{t\to\infty}\|\overline{m}(t)\|\leq\varepsilon=\sqrt{2\kappa}$ w.p.1.

To see that $\mathcal{L}V(\overline{m}(t))<0$ is true, observe that from (9) we have

$$\mathcal{L}V(\overline{m})=-\frac{q_2}{c_2}\overline{m}^2+\frac{1}{2}\frac{\sigma^2}{n^2}.$$

Now, consider the level sets $\mathcal{N}_\kappa=\{\overline{m}(t)\in\mathbb{R}|\,V(\overline{m}(t))\leq\kappa\}$ and observe that there always exists a $\hat{\kappa}$ big enough and finite such that for every $\overline{m}(t)\notin\mathcal{N}_{\hat{\kappa}}$, i.e., $\frac{1}{2}\overline{m}(t)^2>\hat{\kappa}$, we have $\frac{q_2}{c_2}\overline{m}^2>\frac{1}{2}\frac{\sigma^2}{n^2}$. The latter implies $\mathcal{L}V(\overline{m}(t))<0$ for all $\overline{m}(t)\notin\mathcal{N}_{\hat{\kappa}}$, which proves that every level set $\mathcal{N}_\kappa$ where $\kappa\geq\hat{\kappa}$ is contractive.

In other words, for every $\overline{m}(t)\in\partial\mathcal{N}_{\hat{\kappa}}$, $\overline{m}(t+dt)\in\mathcal{N}_{\hat{\kappa}}$. The same reasoning proves that every level set $\mathcal{N}_\kappa$ where $\kappa\geq\hat{\kappa}$ is contractive. Thus, we can conclude that for every $\kappa\geq\hat{\kappa}$ there exists an $\varepsilon=\sqrt{2\kappa}$ for which the level set $\{\overline{m}\in\mathbb{R}|\,\|\overline{m}\|\leq\varepsilon\}$ is contractive. A value for $\hat{k}$ is

$$\left\{\begin{array}{l} \hat{k}:=\min k \\ \{\overline{m}|\,V(\overline{m})\leq k\}\supset\left\{\overline{m}|\,\frac{q_2}{c_2}\overline{m}^2>\frac{1}{2}\frac{\sigma^2}{n^2}\right\}. \end{array}\right. \quad (12)$$

which returns

$$\hat{k} = \frac{1}{4}\frac{c_2}{q_2}\frac{\sigma^2}{n^2}.$$

It is apparent that $\mathcal{N}_{\hat{\kappa}}$ is also control invariant, which means that for every $\overline{m} \in \mathcal{N}_{\hat{\kappa}}$, $\overline{m} + d\overline{m} \in \mathcal{N}_{\hat{\kappa}}$.

To prove (8), let us substitute $\hat{k} = \frac{1}{4}\frac{c_2}{q_2}\frac{\sigma^2}{n^2}$ into $\varepsilon = \sqrt{2\kappa}$ and then we obtain $\varepsilon = \sqrt{\frac{1}{2}\frac{c_2}{q_2}\frac{\sigma^2}{n^2}}$. ■

A direct consequence of the above result is that the bounding set $\mathcal{N}_{\hat{\kappa}}$ shrinks for increasing number of players and collapses asymptotically to the origin for n tending to infinity.

Corollary 1: For $n \to \infty$ the mean opinion converges asymptotically to zero,

$$\lim_{t\to\infty} \overline{m}(t) = 0.$$

Now, our aim is to analyze convergence of the agents to their average. To this purpose, define the averaging matrix $\mathcal{M} = \frac{1}{n}\mathbf{1} \otimes \mathbf{1}$. Then for a given vector $Y(t)$ we have $\mathcal{M}Y(t) = (\frac{1}{n}\mathbf{1} \otimes \mathbf{1})Y(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^T Y(t) = \overline{m}(t)\mathbf{1}$. In other words $\mathcal{M}Y(t)$ is the vector all of whose components are the average of the entries of $Y(t)$. The averaging matrix is useful to introduce the error vector $e(t)$ describing the deviations of the components of $Y(t)$ from their average. For the error vector we can write the expression below, which relates $e(t)$ to $Y(t)$:

$$\begin{aligned} e(t) &= Y(t) - \frac{1}{n}\mathbf{1} \otimes \mathbf{1}^T Y(t) \\ &= Y(t) - \overline{m}(t)\mathbf{1} \\ &= (I - \mathcal{M})Y(t). \end{aligned}$$

The next result establishes that the error vector converges to zero which implies that all opinions converge to the mean opinion.

Theorem 2: For each $\pi > 0$ there exists an $\varepsilon(\pi) > 0$ such that

$$\mathbb{P}(\|e(t)\|_\infty \leq \varepsilon(\pi)) > 1 - \pi. \tag{13}$$

Proof: The time evolution of the error vector is represented by the SDE

$$\begin{aligned} de(t) &= (I - \mathcal{M})\left[(W - I)Y(t) - \frac{q_2}{c_2}Y(t)\right]dt \\ &\quad + (I - \mathcal{M})\sigma d\mathcal{B}(t) \\ &= (W - \mathcal{M})(I - \mathcal{M})Y(t)dt - e(t)dt \\ &\quad - (I - \mathcal{M})\frac{q_2}{c_2}Y(t)dt + (I - \mathcal{M})\sigma d\mathcal{B}(t) \\ &= \underbrace{\left(W - \mathcal{M} - I - \frac{q_2}{c_2}I\right)}_{A} e(t)dt + (I - \mathcal{M})\sigma d\mathcal{B}(t). \end{aligned}$$

The above SDE is linear and the corresponding stochastic process can be studied in the framework of stochastic stability theory [14]. To do this, consider the infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2(I - \mathcal{M})^T(I - \mathcal{M})\frac{d^2}{de(t)^2} + Ae(t)\frac{d}{de(t)}. \tag{14}$$

Now, recalling that for a Brownian motion it holds $\mathbb{E}d\mathcal{B}(t) = 0$ and $\mathbb{E}d\mathcal{B}(t)^2 \to 0$ and ignoring the second-order terms (in dt^2 or $dtd\mathcal{B}(t)$) we obtain (14).

Observe that from (6) we have that $\|W - \mathcal{M}\| < 1$ which in turn implies that A is negative definite, i.e., $\xi^T A \xi < 0$ for all $\xi \in \mathbb{R}^n$. We use this fact to study the infinitesimal generator of the Lyapunov function $V(e) = \frac{1}{2}e^T e$.

We aim to prove that there exists a finite scalar κ and a neighborhood of zero of size κ, denoted by $\mathcal{N}_\kappa = \{e \in \mathbb{R}^n | \, V(e) \leq \kappa\}$, such that $\mathcal{L}V(e(t)) < 0$ for all $e(t) \notin \mathcal{N}_\kappa$, where $\mathcal{L}$ is the infinitesimal generator of the process $e(t)$.

Again we recall that if the former condition holds true, which is $\mathcal{L}V(e(t)) < 0$, then $V(e(t))$ is a supermartingale whenever $e(t)$ is not in $\mathcal{N}_\kappa$ and therefore by the martingale convergence theorem $\exists\, t$ such that $V(e(t)) \leq \kappa$.

Let us first consider the SDE for the error vector $de(t) = Ae(t)dt + (I - \mathcal{M})\sigma d\mathcal{B}(t)$ and rewrite $(I - \mathcal{M})\sigma d\mathcal{B}(t) = \sum b_i d\mathcal{B}_i(t)$ where

$$b_i = \sigma \begin{bmatrix} -\frac{1}{n} \\ \vdots \\ 1 - \frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{bmatrix} \quad i\text{th row}. \tag{15}$$

Then, for the infinitesimal generator of the Lyapunov function it holds

$$\begin{aligned} \mathcal{L}V(e) &= e(t)^T Ae(t) + \frac{1}{2}\sum_{i=1}^{n}\Sigma_{ii} \\ &= e(t)^T Ae(t) + \frac{1}{2}n\sigma^2[(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2], \end{aligned}$$

where $\Sigma = \sum_{k=1}^{n} b_k b_k^T \in \mathbb{R}^{n\times n}$ whose elements in the principal diagonal are $\Sigma_{ii} = \sigma^2[(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2]$.

Now, consider the level sets $\mathcal{N}_\kappa = \{e(t) \in \mathbb{R}^n | \, V(e(t)) \leq \kappa\}$ and observe that there always exists a $\hat{\kappa}$ big enough and finite such that for every $e(t) \notin \mathcal{N}_{\hat{\kappa}}$, i.e., $\frac{1}{2}e(t)^T e(t) > \hat{\kappa}$, we have $e(t)^T Ae(t) + \frac{1}{2}n\sigma^2[(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2] < 0$. The latter means $\mathcal{L}V(e(t)) < 0$ for all $e(t) \notin \mathcal{N}_{\hat{\kappa}}$, which proves that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive.

In other words, for every $e(t) \in \partial\mathcal{N}_{\hat{\kappa}}$, $e(t + dt) \in \mathcal{N}_{\hat{\kappa}}$. The same reasoning proves that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive. Thus, we can conclude that for every $\kappa \geq \hat{\kappa}$ there exists an $\varepsilon = \sqrt{2\kappa}$ for which the level set $\{\overline{m} \in \mathbb{R} | \, \|\overline{m}\| \leq \varepsilon\}$ is contractive. A value for $\hat{k}$ can be obtained solving the optimization problem

$$\left\{ \begin{array}{l} \hat{k} := \min k \\ \{e | \, V(e) \leq k\} \supset \{e | \, e(t)^T Ae(t) \\ \qquad + \frac{1}{2}n\sigma^2[(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2] < 0\}. \end{array} \right. \tag{16}$$

■

IV. NUMERICAL STUDIES

Numerical studies include three main sets of simulations as summarized in Figg. 1-3. The first set highlights the relation between the system response and the coefficient of attraction q_1 among the opinions: the mean distribution

n	x_{min}	x_{max}	c_1	c_2	T	$\overline{m}_0$	$std(m_0)$
10^3	0	1	1	1	10	0.8	0.05

TABLE I
CONSTANT SIMULATION PARAMETERS.

	q_1	q_2	σ
I	$\{1, 2, 3\}$	1.5	0.001
II	1	$\{1.5, 2.5, 3\}$	0.001
III	1	1.5	$\{0.001, 0.01, 0.05\}$

TABLE II
VARYING SIMULATION PARAMETERS WITH DIFFERENT SIMULATION SETS.

$\overline{m}(t)$ fluctuates while decreasing and the standard deviation $std(m(t))$ decays gradually to zero. The second set emphasizes how the system evolves in response to a higher coefficient of attraction q_2 to zero, which corresponds to increasing the stubborn agent's attraction force: both the mean distribution $\overline{m}(t)$ and the standard deviation $std(m(t))$ decrease monotonically, similarly to the evolutions shown in the first set of simulations. The third set simulates the system under various effects of the Brownian motion: the mean distribution $\overline{m}(t)$ first increases, and then decreases linearly and the standard deviation $std(m(t))$ first increases until it hits a peak, and then fluctuates.

Simulations of numerical examples have been done using the algorithm below and the following parameters, also shown in Tables I-II. The number of agents is set to $n = 10^3$. The set of states is a discretization of the interval $[0, 1[$ with step size $dx = 10^{-4}$, i.e. $\mathcal{X} = \{x_{min}, x_{min} + 0.001, \ldots, x_{max}\}$. The horizon length is $T = 10$, large enough to show convergence of the population plots. As regards the initial distribution, we assume m_0 to be gaussian with mean $\bar{m}_0 = 0.8$ and standard deviation $std(m_0) = 0.05$. Parameter σ is set to a value between 0.001 and 0.05.

First set of simulations. The first set of simulations highlights how the coefficient that regulates the aggregation forces among the opinions, q_1, is a factor in reducing the sparsity of the opinions, which is measured by the standard deviation $std(m(t))$. From top-left to bottom-left, Figure 1 shows the distribution evolution $m(t)$ vs. the state $x(t)$ at different times. The initial distribution is modeled as a gaussian with mean $\overline{m}_0 = 0.8$ and standard deviation $std(m_0) = 0.05$. Parameter q_1 varies from $q_1 = 1$ (top), $q_1 = 2$ (middle) and $q_1 = 3$ (bottom). The graphics on the right display the time plot $\overline{m}(t)$ (solid line and y-axis labels on the left) and the evolution of the standard deviation $std(m(t))$ (dashed line and y-axis labels on the right). It is worth noting that the standard deviation tends to zero faster and faster as long as the attraction among the opinions grows: as it can be seen from the graphics, the distributions at a same time instant get sharper with higher values of q_1.

Algorithm

Input: Set of parameters as in Tables I-II
Output: Distribution function $m(t)$, mean $\overline{m}(t)$ and standard deviation $std(m(t))$.
Initialize. Generate x_0 from Gaussian distribution with mean $\bar{m}_0$ and standard deviation $std(m_0)$,
for time $t = 0, 1, \ldots, T-1$ **do**
  if $t > 0$, **then** compute $m(t)$, $\overline{m}(t)$, and $std(m_t)$,
  end if
  for player $i = 1, 2, \ldots, n$ **do**
    compute $X(t+1)$ by executing (1),
  end for
end for
STOP

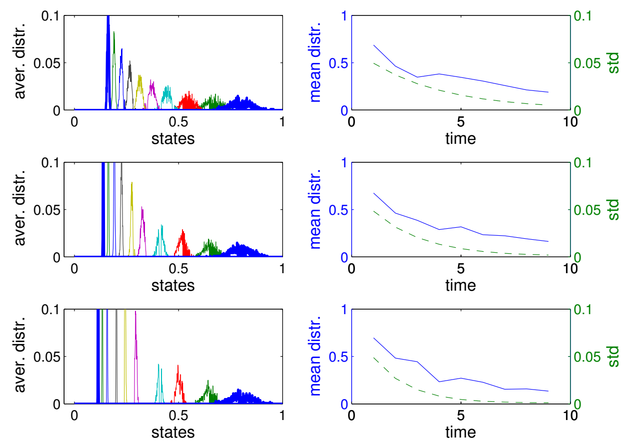

Fig. 1. First set of simulations: the sparsity of the distribution evolution becomes smaller as the coefficient q_1 grows. The standard deviation $std(m(t))$ goes to zero faster, from top to bottom.

Second set of simulations. The second set of simulation shows the connection between the coefficient q_2, which describes the attracting force exhibited by the stubborn agent, and the convergence speed of the distribution toward zero. The graphics on the left show this effect. In particular, the graphics plot the distribution evolution $m(t)$ with respect to the state $x(t)$ at different times. The initial distribution is the same as in the first set of simulations, with identical mean and standard deviation, while q_2 varies from $q_2 = 1.5$ (top), $q_2 = 2.5$ (middle) and $q_2 = 3$ (bottom). The opinions approach zero with a speed that increases with q_3. The graphics on the right display the time plot $\bar{m}(t)$ (solid line and y-axis labels on the left) and the evolution of the standard deviation $std(m_t)$ (dashed line and y-axis labels on the right), pointing out that the mean tends to zero faster with higher values of the coefficient.

Third set of simulations. The last set of simulations shows

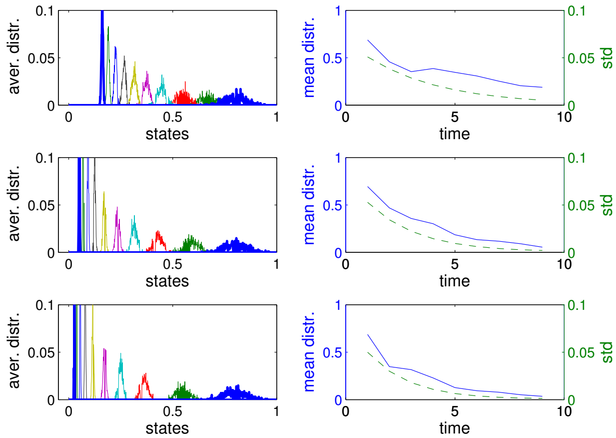

Fig. 2. Second set of simulations: the distribution evolution goes faster toward zero as q_2 grows. The mean $\bar{m}(t)$ goes to zero faster, from top to bottom.

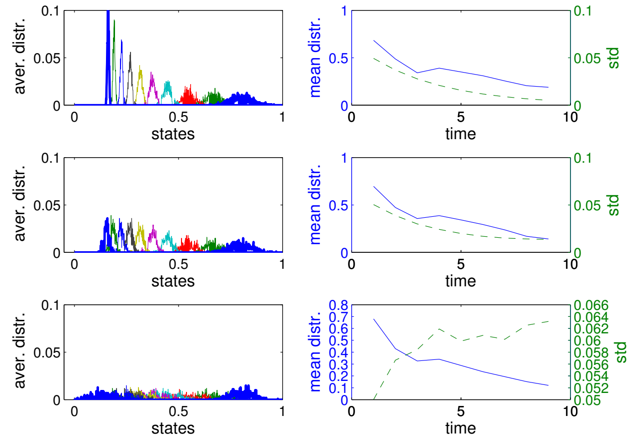

Fig. 3. Third set of simulations: effects of Brownian motion on the coefficients q_1 and q_2. Smaller coefficient values and a higher σ prevent the opinion from reaching consensus. The convergence of standard deviation $std(m(t))$ is guaranteed only by increasing the aforementioned coefficients.

the effects of the Brownian motion. The initial conditions are identical to the ones of the previous simulations, and the only varying parameter is σ from $\sigma = 0.001$ (top), $\sigma = 0.01$ (middle) and $\sigma = 0.05$ (bottom). The graphics on the left show this effect, by plotting the distribution evolution $m(t)$ as function of the state $x(t)$ at different times. With higher values of σ the evolution reacts in two ways: for σ moderately small (second graphics from top to bottom on the left column), opinions manage to gather around a mean, thus letting the standard deviation slowly decrease to zero; when σ becomes bigger, the attraction among the opinions and the force in zero are too weak to let them gather (for $q_1 = 1$ and $q_2 = 1.5$). The only way to compensate this is to increase those two coefficients. The graphics on the right display the time plot $\overline{m}(t)$ (solid line and y-axis labeling on the left) and the evolution of the standard deviation $std(m(t))$ (dashed line and y-axis labeling on the right). In the last graph, the only way to incentivize the convergence of the standard deviation $std(m(t))$ to zero is by increasing the coefficients q_1 and q_2.

V. CONCLUSIONS

We have studied a scenario where all agents reach ε-consensus almost surely in a neighborhood of the stubborn agent's opinion. We have shown that such a scenario is mean field equilibrium for the game of interest. Future research will address local interactions among the players and provide a game theoretic understanding for the formation of clusters.

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