Proceedings Paper:
Bauso, D., Zhang, X. and Papachristodoulou, A. (2015) Density flow over networks: A mean-field game theoretic approach. In: Proceedings of the IEEE Conference on Decision and Control. 53rd IEEE Conference on Decision and Control, December 15-17, 2014, Los Angeles, California, USA. Institute of Electrical and Electronics Engineers , 3469 - 3474. ISBN: 978-1-4799-7746-8. ISSN: 0743-1546.

https://doi.org/10.1109/CDC.2014.7039927

Density Flow over Networks: A Mean-field Game Theoretic Approach

Dario Bauso, Xuan Zhang and Antonis Papachristodoulou

***Abstract*— A distributed routing control algorithm for dynamic networks has recently been presented in the literature. The networks were modeled using time evolution of density at network edges and the routing control algorithm allowed edge density to converge to a Wardrop equilibrium, which was characterized by an equal traffic density on all used paths. We borrow the idea and rearrange the density model to recast the problem within the framework of mean-field games. The contribution of this paper is three-fold. First, we provide a mean-field game formulation of the problem at hand. Second, we illustrate an extended state space solution approach. Third, we study the stochastic case where the density evolution is driven by a Brownian motion.**

I. INTRODUCTION

In this paper we study a routing problem over a network. The problem setup involves a population of individuals or players traversing the edges of a network in the attempt to reach a destination node starting from a source node. From a microscopic standpoint, each player jumps from one edge to an adjacent one according to a continuous-time Markov model. Players select the transition rates, which represent the control. From a macroscopic perspective, each edge is characterized by dynamics describing the time evolution of the density of players on that edge. These dynamics take the form of a classical forward Kolmogorov Ordinary Differential Equation (ODE). In the second part of the paper, we extend our analysis to the case where the Kolmogorov equation turns into a Stochastic Differential Equation (SDE) driven by a Brownian motion.

Main results. For the problem at hand we highlight three main results. First, we provide a mean-field game formulation of the problem (see Theorem 1). Second, we illustrate an extended state space solution approach (see Theorem 2). Third, we study the stochastic case where the density evolution is driven by a Brownian motion (see Theorem 3).

Related literature. The current paper finds inspiration in the distributed routing problem presented in [5], [6]. We provide a detailed analysis of a similar problem via mean-field games theory. The theory on mean-field games originated in the work of M. Y. Huang, P. E. Caines and R. Malhamé [8], [9], [10] and independently in that of J. M. Lasry and P. L. Lions [12], [13], [14], where the now standard terminology of Mean-Field Games (MFG) was introduced. In addition to this, the closely related notion of Oblivious Equilibria for large population dynamic games was introduced by G. Weintraub, C. Benkard and B. Van Roy [20] in the framework of Markov Decision Processes.

The problem we analyze in this paper has striking similarities with the optimal planning problem [1], [4], [15], [17] which in turn can be linked back to mean-field games. Essentially, in optimal planning problems the idea is to drive the density of players from a given initial configuration to a target one at a given time by an appropriate design of the optimal decisions of the agents.

Mean-field games arise in several application domains such as economics, physics, biology, and network engineering (see [1], [2], [7], [10], [16], [18]). Explicit solutions in terms of mean-field equilibria are not common unless the problem has a linear quadratic structure [3]. In this sense, a variety of solution schemes have been recently proposed based on discretization and or numerical approximations [1]. Mean-field games have precursors in *anonymous games* and *aggregative games* building upon the notion of mass interaction and can be seen as a stationary mean-field in dynamic discrete time [11]. More recently, robustness notions have been introduced in mean-field games. Robust mean-field games aim to achieve robust performance or stability in the presence of unknown disturbances when there is a large number of players. Their relationship with risk-sensitive games and risk-neutral games has been analyzed in [19].

The rest of the paper is organized as follows. In Section II we illustrate the problem and introduce the model. In Section III we present the main results of the paper. In Section IV we provide numerical examples. Finally, in Section V we draw some conclusions.

II. MODEL AND PROBLEM SET-UP

Let a graph $G = (V, E)$ be given where $V = \{1, \cdots, n\}$ is the set of vertices and $E = \{1, \cdots, m\}$ the set of edges. Let us denote by $\varepsilon^+(i)$ and $\varepsilon^-(i)$ the sets of outgoing edges from i and incoming edges to i respectively, $\forall i \in V$. We consider a "large population" of individuals or players of which each one is characterized by a time-varying state $X(t) \in E$ at time $t \in [0, T]$, where $[0, T]$ is the time horizon window. The routing policy is described by a vector-valued function $\alpha(\cdot) : \mathbb{R}_+ \to [0, 1]^m$, $t \mapsto \alpha(t)$ where $[0, 1]^m$ denotes the m-dimensional column vector whose entries are within the interval $[0, 1]$. Moreover, we have $\sum_{e\in\varepsilon^+(i)} \alpha_e = 1$ where $\forall i \in V$ and α_e is the e^{th} entry of $\alpha(t)$. In other words, $\alpha(t)$ is equivalent to $\Delta^{|\varepsilon^+(1)|} \times \cdots \times \Delta^{|\varepsilon^+(n)|}$ where $\Delta^{|\varepsilon^+(i)|}$ denotes

D. Bauso was supported by PRIN 20103S5RN3 "Robust decision making in markets and organization". D. Bauso is with DICGIM, Università di Palermo, V.le delle Scienze, 90128 Palermo, Italy and, as "Research Fellow", also with Dipartimento di Matematica, Università di Trento. D. Bauso is currently visiting professor at Department of Engineering Science, University of Oxford, UK. {dario.bauso@unipa.it}

Xuan Zhang and Antonis Papachristodoulou are with Department of Engineering Science, University of Oxford, Parks Road, Oxford, OX1 3PJ, UK. {xuan.zhang,antonis@eng.ox.ac.uk}

the simplex in $\mathbb{R}^{|\varepsilon^+(i)|}$ and $|\varepsilon^+(i)|$ is the cardinality of set $\varepsilon^+(i)$ (number of outgoing edges from i), $\forall i \in V$. Let $k \in E$ be the player's state. The state evolution of a single player is then captured by the following continuous-time Markov stochastic process:

$$\{X(t),\, t \geq 0\},\; q_{kj}(h, \phi_k, \alpha_j) = \begin{cases} \alpha_j \phi_k h & j \in Adj(k), \\ 1 - \phi_k h, & j = k, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $q_{kj}(h, \phi_k, \alpha_j)$ (q_{kj}) are the infinitesimal transition probabilities from k to j, h is the infinitesimal time interval, $\phi_k \in \mathbb{R}_+$ is the transition rate in state $k \in E$, and $Adj(k) = \{j \in E |\, j \in \varepsilon^+(i), k \in \varepsilon^-(i)\}$ represents the set of adjacent edges to k.

Denote by $\rho \in [0,1]^m$ the vector of densities on edges, which means that $\sum_{e \in E} \rho_e = 1$, ρ_e is the e^{th} entry of ρ. Let us define the flow function $f(\cdot) : [0,1]^m \to \mathbb{R}^m_+$, $\rho \mapsto f(\rho)$, which maps densities into flows for each edge. In this paper, we assume the following linear rule $f_e(\rho) = \phi_e \rho_e$, where $f_e(\rho)$ is the e^{th} entry of $f(\rho)$. The density evolution can be described by the Kolmogorov ODE given by

$$\dot{\rho}(t) = \Big(\tilde{B}^T(\alpha)\hat{B} - I\Big)\, f(\rho), \tag{2}$$

where

- the matrix-valued function $\tilde{B}(\cdot) : [0,1]^m \to [0,1]^{n \times m}$, $\alpha \mapsto \tilde{B}(\alpha)$, which relates nodes to outgoing edges, i.e., $\tilde{B}_{ij}(\alpha) = \alpha_j$ if $j \in \varepsilon^+(i)$ and $\tilde{B}_{ij}(\alpha) = 0$ otherwise. Here $[0,1]^{n\times m}$ denotes the $n \times m$-dimensional matrix whose entries are within the interval $[0,1]$, and $\tilde{B}_{ij}(\alpha)$ is the entry in the i^{th} row and j^{th} column of $\tilde{B}(\alpha)$.
- the matrix $\hat{B} \in \{0,1\}^{n\times m}$ relates nodes to incoming edges, i.e., $\hat{B}_{ij} = 1$ if $j \in \varepsilon^-(i)$ and $\hat{B}_{ij} = 0$ otherwise. Here $\{0,1\}^{n\times m}$ denotes the $n \times m$-dimensional matrix whose entries are either 0 or 1, and $\hat{B}_{ij}$ is the entry in the i^{th} row and j^{th} column of $\hat{B}$.

Equation (2) establishes that the density variation on each edge is a consequence of a discrepancy between the outgoing flow and the incoming flow on the same edge. The former is captured by the term $f(\rho)$ whereas the latter is represented by $\tilde{B}^T(\alpha)\hat{B}f(\rho)$. Then density variation depends on the difference $\tilde{B}^T(\alpha)\hat{B}f(\rho) - f(\rho)$ which gives (2). Note that $\tilde{B}^T(\alpha)$ is a column (left) stochastic matrix, i.e., $\sum_{i=1,\cdots,m}(\tilde{B}^T(\alpha))_{ij} = 1$ for all $j = 1, \cdots, n$.

Assume that the graph is acyclic, and has one source node s and one destination node d. Select a subset of paths from s to d and call it $\mathcal{P}$. Each element of $\mathcal{P}$ is an $s - d$ path $\{s, \cdots, i, \cdots, d\}$. Let the matrix $C \in \{0,1\}^{|\mathcal{P}|\times m}$ be given which relates paths to edges. Each row of C contains ones or zeros depending on what edges are included in the path. We can define the output vector-valued function $y(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{|\mathcal{P}|}$, $t \mapsto y(t)$, which represents the collective density on each path and can be expressed as $y(t) = C\rho(t)$.

In order to achieve a Wardrop equilibrium, i.e., uniform distribution over all available paths, for each player, consider a running cost $g(\cdot) : E \times [0,1]^m \to [0, +\infty[$, $(x, \rho) \mapsto g(x, \rho)$ of the form below, where $\mathcal{M}$ is the consensus manifold/Wardrop equilibria set:

$$g(x, \rho) = dist(\rho, \mathcal{M}), \tag{3}$$

$$\mathcal{M} = \{\exists p \in [0,1], y(t) = p[1]^{|\mathcal{P}|}\}, \tag{4}$$

where $dist(\rho, \mathcal{M})$ denotes the distance from the vector ρ to the manifold $\mathcal{M}$, and the $[1]^{|\mathcal{P}|}$ denotes the $|\mathcal{P}|$-dimensional vector whose entries are 1. The problem in its generic form is then the following:

Problem 1: Design a routing policy to minimize the output disagreement, i.e., each player solves the following problem:

$$\begin{cases} \inf_{\alpha(\cdot)} J(x, \alpha(\cdot), \rho[\cdot](\cdot)), \\ J(\cdot) = \mathbb{E}\Big[\int_0^T g(X(t), \rho(t))dt + g(X(T), \rho(T))\Big], \\ \{X(t), t \geq 0\} \text{ as in (1)}. \end{cases} \tag{5}$$

III. Main results

In this section we highlight three main results. First, we provide a mean-field game formulation of the problem at hand (see Theorem 1). Second, we illustrate an extended state space solution approach (see Theorem 2). Third, we study the stochastic case where the density evolution is driven by a Brownian motion (see Theorem 3).

A. Mean-field game formulation

Let us denote by $v(x,t)$ the value of the optimization problem starting from time t at state x. The first step is to show that the problem results in the following mean-field game system for the unknown scalar functions $v(x,t)$ and $\rho(t)$ when each player behaves according to (5):

Theorem 1: The mean-field system for the routing problem in Problem 1 takes on the form:

$$\begin{cases} \dot{v}(x,t) + \mathcal{H}(x, \Delta(v), t) = 0 \text{ in } E \times [0,T[, \\ \\ v(x,T) = g(x, \rho(T)), \forall x \in E, \\ \\ \dot{\rho}(t) = \Big(\tilde{B}^T(\alpha^*)\hat{B} - I\Big) f(\rho) \text{ in } [0,T[, \\ \\ \rho(0) = \rho_0,\; \rho_0 \text{ given}, \end{cases} \tag{6}$$

$\mathcal{H}(x, \Delta(v), t)$ is the Hamiltonian function given by

$$\mathcal{H}(x, \Delta(v), t) = \inf_{\alpha} \Big\{ \textstyle\sum_{z \in E} q_{xz}(v(z,t) - v(x,t)) + g(x,\rho) \Big\}. \tag{7}$$

In the expression above, $\Delta(v)$ denotes the difference of the value function computed in two successive states, q_{xz} is the transition rate given in (1). The optimal time-varying control $\alpha^*(x,t)$ is given by

$$\alpha^*(x,t) \in \arg\min_{\alpha} \Big\{ \textstyle\sum_{z \in E} q_{xz}(v(z,t) - v(x,t)) + g(x,\rho) \Big\}. \tag{8}$$

Proof: Let us start by noticing that the third and fourth equations of (6) are the forward Kolmogorov equation and the corresponding boundary condition on the initial distribution law. To prove the first equation of (6), we know that from dynamic programming it holds:

$$\dot{v}(x,t) + \inf_{\alpha} \Big\{ \sum_{z\in E} q_{xz}(v(z,t) - v(x,t)) \\ +g(x,\rho) \Big\} = 0 \text{ in } E \times [0,T[,$$

By introducing the Hamiltonian $\mathcal{H}(x, \Delta(v), t)$ given in (7), we obtain the first equation. Note that the transition rates depend on the routing policy/control α. This is then obtained as the minimizer in the computation of the Hamiltonian as expressed by (8). For the first part of the proof, it is left to notice that the second equation is the boundary condition on the terminal penalty. ■

The mean-field game system (6) appears in the form of two coupled ODEs intertwined in a forward-backward way. The first equation in (6) is the *Hamilton-Jacobi-Bellman* (HJB) equation with variable $v(x,t)$ and parametrized in $\rho(\cdot)$. Given the boundary condition on final state (second equation in (6)), and assuming a given population behavior captured by $\rho(\cdot)$, the HJB equation is solved backwards and returns the value function and best-response behavior of the players given by (8). The HJB equation is coupled with a second ODE, which is the *Fokker-Planck-Kolmogorov* (FPK) (third equation in (6)), defined on variable $\rho(\cdot)$ and parametrized in $\alpha^*(x,t)$. Given the boundary condition on initial distribution $\rho(0) = \rho_0$ (fourth equation in (6)), and assuming a given individual behavior described by α^*, the FPK equation is solved forward and returns the population behavior time evolution $\rho(t)$.

B. State space extension

Our solution approach to (6) involves expanding the state space including ρ as an additional state variable. Then, we look for a new value function $V(x,\rho,t)$ (note that we have abused the notation V and its meaning should be clear from the context) which depends not only on x but rather also on the density vector ρ. With the above reasoning in mind the mean-field system of the problem at hand can be rewritten as follows.

Lemma 1: The mean-field system for the routing problem in Problem 1 in extended form appears as:

$$\begin{cases} \partial_t V(x,\rho,t) + \tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V, t) = 0 \\ \text{in } E \times [0,1]^m \times [0,T[, \\ \\ V(x,\rho,T) = g(x,\rho(T)), \forall (x,\rho) \in E \times [0,1]^m, \end{cases} \tag{9}$$

where

$$\tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V, t) = \inf_{\alpha} \Big\{ \sum_{z\in E} q_{xz}(V(z,\rho,t) \\ -V(x,\rho,t)) + \partial_\rho V(x,\rho,t)^T \Big[\Big(\tilde{B}^T(\alpha)\hat{B} \\ -I \Big) f(\rho) \Big] + g(x,\rho) \Big\}, \tag{10}$$

and the optimal time-varying control $\alpha^*(x,\rho,t)$ is given by

$$\alpha^*(x,\rho,t) \in \arg\min_{\alpha} \Big\{ \sum_{z\in E} q_{xz}(V(z,\rho,t) \\ -V(x,\rho,t)) + \partial_\rho V(x,\rho,t) \Big[\Big(\tilde{B}^T(\alpha)\hat{B} \\ -I \Big) f(\rho) \Big] + g(x,\rho) \Big\}. \tag{11}$$

Proof: From dynamic programming we obtain

$$\partial_t V(x,\rho,t) + \inf_{\alpha} \Big\{ \sum_{z\in E} q_{xz}(V(z,\rho,t) \\ -V(x,\rho,t)) + \partial_\rho V(x,\rho,t)^T \Big[\Big(\tilde{B}^T(\alpha)\hat{B} - I \Big) f(\rho) \Big] \\ +g(x,\rho) \Big\} = 0 \text{ in } E \times [0,1]^m \times [0,T[.$$

By introducing the Hamiltonian $\tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V, t)$ given in (10), the first equation is proven. To prove (11), observe that the optimal control is the minimizer in the computation of the extended Hamiltonian. It remains to notice that the second equation in (9) is the boundary condition on the terminal penalty. ■

Assumption 1: (**Attainability condition**) The value of the projected game, $val[\lambda]$, is negative for every $\lambda \in \mathbb{R}^m$, i.e.,

$$val[\lambda] = \inf_{\alpha} \Big\{ \lambda^T \Big[q_{x\cdot} + \dot{\rho} \Big] \Big\} \\ = \inf_{\alpha} \{ \sum_{z\in E} (q_{xz} + \dot{\rho}_z)\lambda_z \} < 0, \forall \lambda \in \mathbb{R}^m, \tag{12}$$

where $q_{x\cdot} = [q_{xz}]_{z\in E} \in \mathbb{R}^m$.

This assumption ensures that for a given feasible target manifold, there always exists a routing policy $\alpha(t)$ that drives the edge density ρ towards the manifold (λ can be viewed as the vector connecting the current density projection point on the target manifold and the current density point, with the direction pointing out from the target manifold). We can then establish the following result.

Theorem 2: Let Assumption 1 hold true. Then the mean-field game for the routing problem is given by

$$\begin{cases} \partial_t V(x,\rho,t) + val[\partial_\rho V(x,\rho,t)] + g(x,\rho) = 0 \\ \text{in } E \times [0,1]^m \times [0,T[, \\ \\ V(x,\rho,T) = g(x,\rho(T)), \forall (x,\rho) \in E \times [0,1]^m. \end{cases} \tag{13}$$

Furthermore, the optimal control is:

$$\alpha^*(x,\rho,t) = \\ \arg\min_{\alpha} \Big\{ \partial_\rho V(x,\rho,t)^T \Big[\Big(\tilde{B}^T(\alpha)\hat{B} - I \Big) f(\rho) \Big] \Big\}. \tag{14}$$

Proof: From (12) we have

$$val[\partial_\rho V(x,\rho,t)] = \inf_{\alpha} \Big\{ \partial_\rho V(x,\rho,t)^T \Big[q_{x\cdot} + \dot{\rho} \Big] \Big\} \\ = \inf_{\alpha} \Big\{ \partial_\rho V(x,\rho,t)^T \Big[q_{x\cdot} + \Big(\tilde{B}^T(\alpha)\hat{B} - I \Big) f(\rho) \Big] \Big\} \\ = \tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V, t) - g(x,\rho).$$

Invoking Lemma 1, and the first equation in (9), we obtain the first equation in (13). The second equation in (13) is again the boundary condition on the terminal penalty. It remain to notice that the optimal control is the minimizer in the computation of the extended Hamiltonian and thus is obtained from (14). ■

C. Stochastic case

In this section, we analyze the case where the density evolves according to a stochastic differential equation driven by a Brownian motion. The Kolmogorov equation is then replaced by a geometric Brownian motion dynamics as illustrated below:

$$d\rho(t) = \Big(\tilde{B}^T(\alpha)\hat{B} - I \Big) f(\rho)dt + \sigma dist(\rho, \mathcal{M}) d\mathcal{B}(t). \tag{15}$$

Extending the state space as in the earlier case, and introducing the extended Hamiltonian for the stochastic case as

$$\begin{array}{l}\tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V,t)=\inf_\alpha \Big\{ \sum_{z\in E} q_{xz}(V(z,\rho,t)\\ -V(x,\rho,t))+\partial_\rho V(x,\rho,t)^T\Big[\Big(\tilde{B}^T(\alpha)\hat{B}\\ -I\Big)f(\rho)\Big]+g(x,\rho)\Big\},\end{array} \tag{16}$$

the mean-field system turns into the system of equations below in the value function $V(x,\rho,t)$ in $E\times[0,1]^m\times[0,T[$:

$$\left\{\begin{array}{l}\partial_t V(x,\rho,t)+\tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V,t)\\ +\frac{\sigma^2}{2}dist^2(\rho,\mathcal{M})Tr\Big(\partial^2_{\rho\rho}V(x,\rho,t)\Big)=0\\ \text{in } E\times[0,1]^m\times[0,T[,\\ \\ V(x,\rho,T)=g(x,\rho(T)),\forall(x,\rho)\in E\times[0,1]^m,\end{array}\right. \tag{17}$$

where the optimal time-varying control $\alpha^*(x,\rho,t)$ is obtained as

$$\begin{array}{l}\alpha^*(x,\rho,t)\in\arg\min_\alpha \Big\{ \sum_{z\in E} q_{xz}(V(z,\rho,t)\\ -V(x,\rho,t))+\partial_\rho V(x,\rho,t)^T\Big[\Big(\tilde{B}^T(\alpha)\hat{B}\\ -I\Big)f(\rho)\Big]+g(x,\rho)\Big\}.\end{array} \tag{18}$$

Assumption 2: (**Expected attainability condition**) The expected value of the projected game, $val[\lambda]$, is negative for every $\lambda\in\mathbb{R}^m$, i.e.,

$$\begin{array}{l}expval[\lambda]\\ =\inf_\alpha \mathbb{E}\Big\{\lambda^T\Big[q_{x\cdot}+\Big(\tilde{B}^T(\alpha)\hat{B}-I\Big)f(\rho)\Big]\Big\}\\ =\inf_\alpha \mathbb{E}\{\sum_{z\in E}(q_{xz}+\dot{\rho}_z)\lambda_z\}<0,\forall\lambda\in\mathbb{R}^m.\end{array} \tag{19}$$

We can then establish the following result.

Theorem 3: Let Assumpion 2 hold true. Then, the mean-field game for the routing problem is given by

$$\left\{\begin{array}{l}\partial_t V(x,\rho,t)+val[\partial_\rho V(x,\rho,t)]+g(x,\rho)\\ +\frac{\sigma^2}{2}dist^2(\rho,\mathcal{M})Tr\Big(\partial^2_{\rho\rho}V(x,\rho,t)\Big)=0\\ \text{in } E\times[0,1]^m\times[0,T[,\\ \\ V(x,\rho,T)=g(x,\rho(T)),\forall(x,\rho)\in E\times[0,1]^m.\end{array}\right. \tag{20}$$

Furthermore, the optimal control is:

$$\begin{array}{l}\alpha^*(x,\rho,t)=\\ \arg\min_\alpha \Big\{\partial_\rho V(x,\rho,t)^T\Big[\Big(\tilde{B}^T(\alpha)\hat{B}-I\Big)f(\rho)\Big]\Big\}.\end{array} \tag{21}$$

Proof: Let us observe that from (19) we have

$$\begin{array}{l}expval[\partial_\rho V(x,\rho,t)]=\inf_\alpha \mathbb{E}\Big\{\partial_\rho V(x,\rho,t)^T\Big[q_{x\cdot}+\dot{\rho}\Big]\Big\}\\ =\inf_\alpha \mathbb{E}\Big\{\partial_\rho V(x,\rho,t)^T\Big[q_{x\cdot}+\Big(\tilde{B}^T(\alpha)\hat{B}-I\Big)f(\rho)\Big]\Big\}\\ =\tilde{\mathcal{H}}(x,\rho,\Delta(v),\partial_\rho V,t)-g(x,\rho).\end{array}$$

From the above equation and invoking the first equation in (17), we obtain the first equation in (20). The second equation in (20) represents the boundary condition on the terminal penalty. To conclude our proof we notice that the optimal control is the minimizer in the computation of the extended Hamiltonian and thus is obtained from (21). ■

IV. Numerical example

Consider the following example, consisting of 4 vertices and 5 edges, as shown in Fig. 1 (vertex 'S' stands for the source and vertex 'D' stands for the destination, edge e is marked with f_e, the incoming flow f_0 is equal to the outgoing flow $f_6=f_4+f_5$).

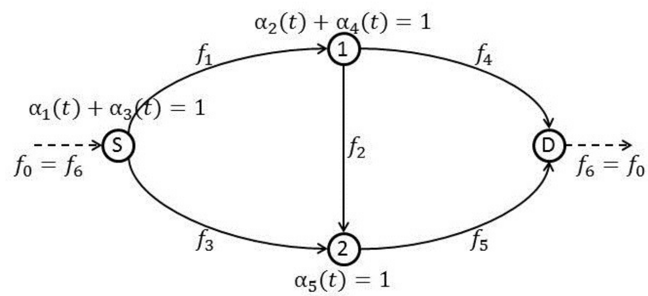

Fig. 1: Network system.

The matrices introduced in the sections above are

$$\tilde{B}^T(\alpha)=\begin{bmatrix}\alpha_1&0&0\\0&\alpha_2&0\\\alpha_3&0&0\\0&\alpha_4&0\\0&0&\alpha_5\end{bmatrix}$$

$$\tilde{B}=\begin{bmatrix}1&0&1&0&0\\0&1&0&1&0\\0&0&0&0&1\end{bmatrix}$$

The density evolution expressed by (2) takes on the form, where we use $f_e(\rho_e(t))=\phi\rho_e(t)$:

$$\left\{\begin{array}{lcl}\dot{\rho}_1(t)&=&\alpha_1(t)(\phi\rho_4(t)+\phi\rho_5(t))-\phi\rho_1(t)\\ \dot{\rho}_2(t)&=&\alpha_2(t)\phi\rho_1(t)-\phi\rho_2(t)\\ \dot{\rho}_3(t)&=&\alpha_3(t)(\phi\rho_4(t)+\phi\rho_5(t))-\phi\rho_3(t)\\ \dot{\rho}_4(t)&=&\alpha_4(t)\phi\rho_1(t)-\phi\rho_4(t)\\ \dot{\rho}_5(t)&=&\alpha_5(t)(\phi\rho_2(t)+\phi\rho_3(t))-\phi\rho_5(t)\end{array}\right. \tag{22}$$

and

$$\left\{\begin{array}{lcl}\alpha_1(t)+\alpha_3(t)&=&1\\ \alpha_2(t)+\alpha_4(t)&=&1\\ \alpha_5(t)&=&1\end{array}\right. \tag{23}$$

Let us consider the paths $\{1,4\}$, $\{1,2,5\}$ and $\{3,5\}$. In other words, $\mathcal{P}=\Big\{\{1,4\},\{1,2,5\},\{3,5\}\Big\}$ which corresponds to defining an output

$$\begin{bmatrix}y_1(t)\\y_2(t)\\y_3(t)\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&0&1&0\\1&1&0&0&1\\0&0&1&0&1\end{bmatrix}}_{C}\begin{bmatrix}\rho_1(t)\\\rho_2(t)\\\rho_3(t)\\\rho_4(t)\\\rho_5(t)\end{bmatrix}$$

Deterministic case. We first consider the deterministic case. Table I shows the parameters of the overall system. According to Theorem 2, we have the following Algorithm to solve the distributed routing problem. The simulations are carried out with MATLAB on an Inter(R) Xeon(R) CPU E31245 at 3.30GHz and 8 GB of RAM, and the results are

Parameter	Value	Variable	Initial Value
ϕ	0.8	$\rho(t)$	$(0.3, 0.5, 0.2, 0, 0)$
Time step h	0.01	$\alpha(t)$	$(0.6, 0.5, 0.4, 0.5, 1)$
Time span T	20		

TABLE I: Parameters of the overall system.

Algorithm

Input: Set of parameters as in Table I.
Output: Density $\rho(t)$, policy $\alpha(t)$ and $dist(\rho(t), \mathcal{M})$

1 : **Initialize:** Set of initial values as in Table I.
2 : **for** time $t = 0, h, 2h, \dots, T - h$ **do**
3 : compute projected point of $\rho(t)$ on $\mathcal{M}$
4 : compute the optimal control $\alpha^*(t)$ using Theorem 2, and the distance $dist(\rho(t), \mathcal{M})$
5 : set $\beta(0) = \alpha(t)$
  for $k = 0, 1, \dots, 100$ **do**
    compute
    $\beta(k+1) = \beta(k) + \frac{h}{100}(\alpha^*(t) - \beta(k))$
  end for
  set $\alpha(t) = (\beta_1(101), \beta_2(101), 1 - \beta_1(101), 1 - \beta_2(101), 1)$
6 : compute $\rho(t + h)$
7 : **end for**
8 : **STOP**

illustrated in Figures 2-4. The run time of the simulation is around 25 seconds. Since $\sum_e \dot{\rho}_e(t) = 0$ (i.e., conservation law holds), $\sum_e \rho_e(t) = \sum_e \rho_e(0) = 1$ always holds, which is shown in Fig. 2. When achieving consensus, $\rho_2(t) = 0$ holds, indicating that all players choose either leaving the source vertex through edge 1 and returning it through edge 4, or leaving through edge 3 and going back through edge 5. Moreover, the players choose these two routes almost equiprobably, i.e., $\alpha_1 \approx \alpha_3 \approx 0.5$, as illustrated in Fig. 3. The distance from the consensus manifold converges to zero, as illustrated in Fig. 4. Note that in order to avoid chirping in $\alpha(t)$, we have introduced lowpass dynamics $\dot{\beta}(t) = \alpha^*(t) - \beta(t)$ (the relevant transfer function is $\beta(s) = \frac{1}{s+1}\alpha^*(s)$ which is actually a lowpass filter for $\alpha(t)$), corresponding to Step 5 in the Algorithm.

Stochastic case. We now consider the stochastic case. In this case, the dynamics of the network (22) change to

$$\begin{cases} \dot{\rho}_1(t) &= \alpha_1(t)(\phi\rho_4(t) + \phi\rho_5(t)) - \phi\rho_1(t) + w_1(t) \\ \dot{\rho}_2(t) &= \alpha_2(t)\phi\rho_1(t) - \phi\rho_2(t) + w_2(t) \\ \dot{\rho}_3(t) &= \alpha_3(t)(\phi\rho_4(t) + \phi\rho_5(t)) - \phi\rho_3(t) + w_3(t) \\ \dot{\rho}_4(t) &= \alpha_4(t)\phi\rho_1(t) - \phi\rho_4(t) + w_4(t) \\ \dot{\rho}_5(t) &= \alpha_5(t)(\phi\rho_2(t) + \phi\rho_3(t)) - \phi\rho_5(t) + w_5(t) \end{cases} \tag{24}$$

where $w_e(t)$ represents the Gaussian noise whose mean is 0 and variance is $\frac{1}{2}\sigma^2 dist^2(\rho(t), \mathcal{M})$. The above algorithm can still solve the distributed routing problem. We continue to use the parameters in Table I, set $\sigma = 1$, run 50 different Monte Carlo trajectories, and compute the average value of

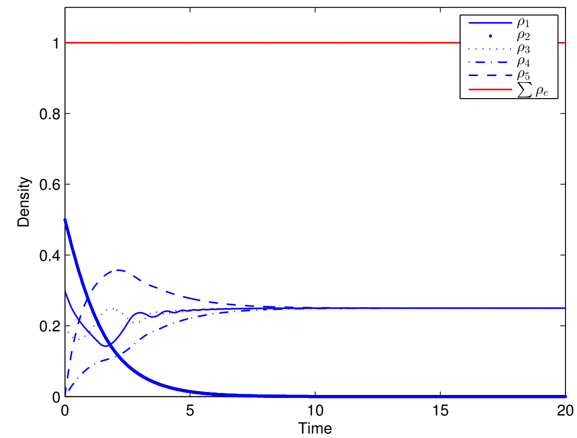

Fig. 2: Simulation results of the deterministic case: density.

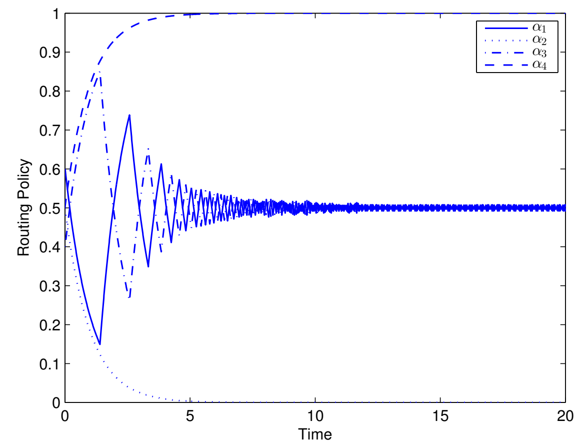

Fig. 3: Simulation results of the deterministic case: routing policy ($\alpha_5(t) = 1$ holds all the time).

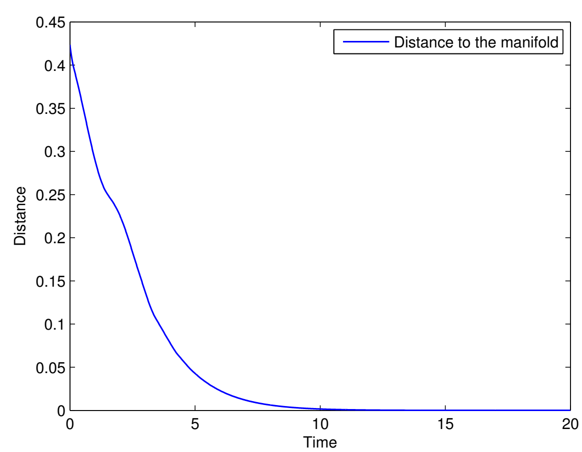

Fig. 4: Simulation results of the deterministic case: distance to the consensus manifold.

these trajectories (see Figures 5-7). We can see that the

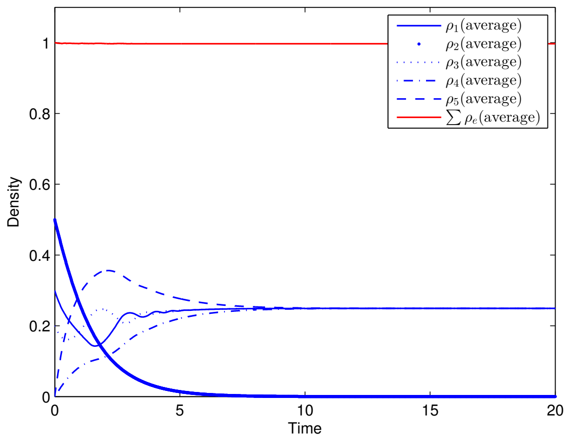

Fig. 5: Simulation results of the stochastic case: average density.

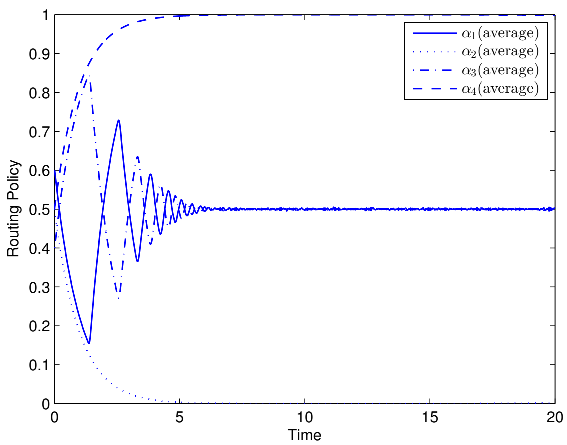

Fig. 6: Simulation results of the stochastic case: average routing policy ($\alpha_5(t) = 1$ holds all the time).

average trajectories are almost the same as those in the deterministic case. Moreover, the average trajectory of $\alpha(t)$ is now much more smooth. The sampled average distance from the consensus manifold converges to zero.

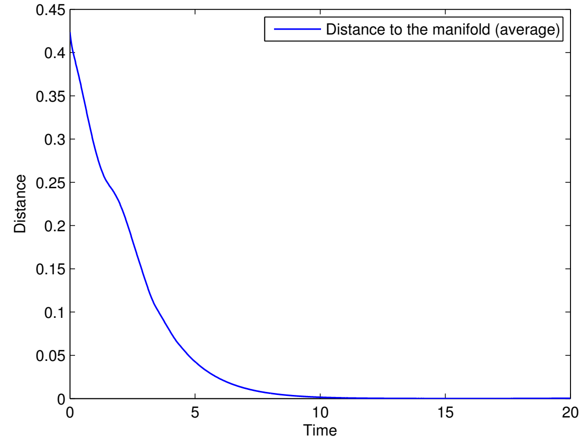

Fig. 7: Simulation results of the stochastic case: average distance to the consensus manifold.

V. Conclusions and Future Directions

We have provided a mean-field game formulation of a distributed routing problem. The problem intersects recent research on optimal planning and transportation. Future research will address the presence of adversarial disturbances in the spirit of H_∞ optimal control.

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