Article:

Jotz Lean, M. and Ortiz, C. (2014) Foliated groupoids and infinitesimal ideal systems. Indagationes Mathematicae, 25 (5). pp. 1019-1053. ISSN 0019-3577

https://doi.org/10.1016/j.indag.2014.07.009

FOLIATED GROUPOIDS AND INFINITESIMAL IDEAL SYSTEMS

M. JOTZ AND C. ORTIZ

Abstract. The main goal of this work is to introduce a natural notion of ideal in a Lie algebroid, the "infinitesimal ideal systems". Ideals in Lie algebras and the Bott connection associated to involutive subbundles of tangent bundles are special cases. The definition of these objects is motivated by the infinitesimal description of multiplicative distributions on Lie groupoids, which are just ideals in the Lie group case. Several examples of infinitesimal ideal systems are presented, and the quotient of a Lie algebroid by an infinitesimal ideal system is shown to be a Lie algebroid.

Contents

1. Introduction

Lie algebras and tangent bundles are the corner cases of Lie algebroids. In both cases, equivalence relations compatible with the structure are well understood.

2010 *Mathematics Subject Classification.* Primary 22A22, 53D17, 53C12; Secondary 53C05.

The first author was partially supported by Swiss NSF grant 200021-121512, by the *Dorothea Schlözer Programme* of the University of Göttingen, and by a Swiss NSF fellowship for prospective researchers (PBELP2_137534) for research conducted at UC Berkeley, the hospitality of which she is thankful for.

In the first case, the quotient of a Lie algebra by an ideal is again a Lie algebra. If TM is the tangent bundle of a manifold M and F_M is an involutive subbundle of TM, then parallel transport relative to the Bott connection associated to F_M defines an equivalence relation on TM/F_M. If the involutive subbundle is simple, i.e. if the space of leaves of the corresponding foliation on the underlying manifold is a smooth manifold, then the quotient by this equivalence relation is the tangent bundle of the leaf space.

In these two reduction processes, one quotients a Lie algebroid by a compatible equivalence relation to construct a new Lie algebroid of the same type. The usual definition of an ideal of a Lie algebroid only makes sense in Lie algebra bundles, but is useless in the case of tangent bundles of manifolds. To be more explicit, let $(q\colon A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid. An ideal in A is simply a subbundle $I \subseteq A$ over M, such that the space of sections $\Gamma(I)$ is an ideal in $\Gamma(A)$ endowed with the Lie bracket $[\cdot,\cdot]$. The first immediate consequence of this definition is the inclusion $I \subseteq \ker(\rho)$, which shows that I is totally intransitive. The main goal of this work is to present a more permissive notion of ideal in a Lie algebroid, encompassing both ideals in Lie algebras and the Bott connection associated to involutive subbundles.

We first encountered these *infinitesimal ideal systems* when we found them to correspond to multiplicative distributions on Lie groupoids. We briefly describe this problem. Let us start with a Lie group G with Lie algebra $\mathfrak{g} = T_eG$ and multiplication map $\mathsf{m} : G \times G \to G$. Then the tangent space TG of G is also a Lie group with unit $0_e \in \mathfrak{g}$ and multiplication map $T\mathsf{m} : TG \times TG \to TG$. A multiplicative distribution $S \subseteq TG$ is a distribution on G – that is, $S(g) := S \cap T_gG$ is a vector subspace of T_gG for all $g \in G$ – that is in addition a subgroup of TG. Since at each $g \in G$, $S(g)$ is a vector subspace of T_gG, the zero section of TG is contained in S. Thus, using $T_{(g,h)}\mathsf{m}(0_g, v_h) = T_hL_gv_h$ for any $g, h \in G$ and $v_h \in T_hG$, where $L_g : G \to G$ is the left translation by g, we find that the distribution S is left invariant. It follows that S is a smooth left invariant subbundle of TG defined by $S(g) = \mathfrak{s}^L(g)$, where $\mathfrak{s}$ the vector subspace $S(e) = S \cap \mathfrak{g}$ of $\mathfrak{g}$. In the same manner, S is right invariant and we find thus that $\mathfrak{s}$ is invariant under the adjoint action of G on $\mathfrak{g}$. Hence, $\mathfrak{s}$ is an ideal in $\mathfrak{g}$ and the subbundle $S \subseteq TG$ is completely integrable in the sense of Frobenius. Its leaf N through the unit element e of G is a normal subgroup of G and the leaf space G/S of S is the group G/N.

In summary, we make two observations. On the one hand, the leaf space of a multiplicative (hence integrable of constant rank) distribution on a Lie group is automatically a group. On the other hand, multiplicative distributions on a Lie group are the same as ideals in its Lie algebra.

Let G be a Lie groupoid over M. Applying the tangent functor to each of the structural maps defining $G \rightrightarrows M$, we get a Lie groupoid structure on the tangent bundle TG over TM – the **tangent groupoid**. A **multiplicative distribution** on $G \rightrightarrows M$ is an involutive subbundle $F_G \subseteq TG$ that is also a Lie subgroupoid of the tangent groupoid over a subbundle $F_M \subseteq TM$. For simplicity, the pair $(G \rightrightarrows M, F_G)$ is then said to be a **foliated groupoid**. This paper and [16] investigate the counterparts of the two observations made above in the more general situation of Lie groupoids. The paper [16] studies the leaf space of foliations associated to multiplicative involutive distributions[1] on Lie groupoids. Here, we introduce infinitesimal ideal systems as the infinitesimal counterpart of foliated groupoids, and of foliated algebroids.

Just as a Lie algebroid is the infinitesimal version of a Lie groupoid, a foliated groupoid corresponds at the infinitesimal level to a foliated algebroid. Let us be more concrete. Take a Lie algebroid $(q\colon A \to M, \rho, [\cdot\,,\cdot])$; the tangent bundle TA inherits a Lie algebroid structure over TM [24]. If G is a Lie groupoid with Lie algebroid A, then there exists

[1]Note that in the following, distributions will always be subbundles *of constant rank* of the tangent bundle.

a one-to-one correspondence between multiplicative involutive subbundles $F_G \subseteq TG$ and **morphic** involutive distributions on A, i.e. involutive subbundles $F_A \subseteq TA$ which are also Lie subalgebroids of TA [28]. It would nevertheless be rather cumbersome to infinitesimally describe a foliated group as a morphic distribution on its Lie algebra.

The ideals of the following definition infinitesimally describe foliated groupoids as the ideals in Lie algebras describe foliated groups.

Definition 1.1. *Let $(q\colon A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle, $K \subseteq A$ a subalgebroid over M such that $\rho(K) \subseteq F_M$ and ∇ a flat partial F_M-connection on A/K with the following properties:*

(1) *If $a \in \Gamma(A)$ is ∇-parallel, then $[a, b] \in \Gamma(K)$ for all $b \in \Gamma(K)$.*
(2) *If $a, b \in \Gamma(A)$ are ∇-parallel, then $[a, b]$ is also ∇-parallel.*
(3) *If $a \in \Gamma(A)$ is ∇-parallel, then $\rho(a)$ is ∇^{F_M}-parallel, where*

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M)$$

is the Bott connection associated to F_M.

The triple (F_M, K, ∇) is an infinitesimal ideal system[2] *in A.*

Note that this is an infinitesimal version of the **ideal systems**[3] in [23], which are described there to be the kernels of fibrations of Lie algebroids. Note also that infinitesimal ideal systems already appear (not under this name) in geometric quantization as the infinitesimal version of polarizations on groupoids in [13], where Eli Hawkins already finds that they correspond to foliated algebroids. Finally, let us mention that the special case where $F_M = TM$ has been studied independently in [7] in relation with a modern approach to Cartan's work on pseudogroups.

We claim that the infinitesimal ideal systems are the objects that should be considered as the ideal objects in Lie algebroids. In the second part of the paper, we describe several examples of infinitesimal ideal systems and we show that, under regularity conditions, one can take the quotient of a Lie algebroid by an ideal system to define a new (reduced) Lie algebroid. We prove that if (A, F_A) is a foliated algebroid with corresponding ideal system (F_M, K, ∇) in A, then, modulo regularity conditions, the leaf space A/F_A inherits a natural Lie algebroid structure over the leaf space M/F_M. The projections $A \to A/F_A$ and $M \to M/F_M$ form a Lie algebroid morphism. The Lie algebroid structure on the leaf space is realized as the quotient of A by the ideal system (F_M, K, ∇). In particular, infinitesimal ideal systems arise as the kernels of fibrations of Lie algebroids. We also show that if a foliated groupoid (G, F_G) – with associated foliated algebroid (A, F_A) – is such that the leaf space $G/F_G \rightrightarrows M/F_M$ is a Lie groupoid, then its Lie algebroid is isomorphic to the reduced Lie algebroid structure on $A/F_A \to M/F_M$.

This paper is organized as follows. In Section 2, we recall the definitions of the tangent Lie groupoid associated to a Lie groupoid, and of the tangent Lie algebroid defined by a Lie algebroid. We then recall some facts about flat partial connections, as well as the definition of the Bott connection associated to an involutive subbundle of the tangent of a manifold.

In Section 3, we give the definition of foliated groupoids. The first main result of this paper (Theorem 3.6) states that a foliated groupoid $(G \rightrightarrows M, F_G)$ defines an infinitesimal ideal system (F_M, K, ∇) in the Lie algebroid A of $G \rightrightarrows M$. Then, we examine how the involutivity of the multiplicative distribution is encoded by the properties of the ideal system.

[2] Infinitesimal ideal systems were called "IM-foliations" in an earlier version of this work, in analogy to the "IM-2-forms" of [5], but we find this new terminology more adequate.

[3] For simplicity, we will often just write "ideal systems" in this paper, but we will always refer to infinitesimal ideal systems.

Infinitesimal ideal systems were found in [13] to encode morphic involutive distributions on Lie algebroids. In Section 4 we summarize the approach of [8] to this result and we show how infinitesimal ideal systems are in one-to-one correspondence with foliated algebroids. The approach in [8] is an application of a result on the correspondence of morphisms of representations up to homotopy and morphisms of VB-algebroids (see [12] for the correspondence between representations up to homotopy and VB-algebroids). To avoid unnecessary technicalities in this paper, we do not introduce these objects, but describe explicitely our special situation. Yet, the reader who knows these concepts should note that one of the results in [8] is that an infinitesimal ideal system in A is equivalent to a pair of subrepresentations up to homotopy: one of the adjoint and one of the double representations up to homotopy defined by the Lie algebroid A. This shows that our proposed notion of ideal is compatible with the definition of an ideal in a Lie algebra by a subrepresentation of the adjoint representation.

We prove that the infinitesimal ideal system defined by a foliated groupoid and the infinitesimal ideal system defined by the corresponding foliated algebroid coincide (Theorem 4.9), and we conclude with the one-to-one correspondence of source-simply connected foliated groupoids with infinitesimal ideal systems on integrable Lie algebroids.

The examples of infinitesimal ideal systems in Section 5 explain why they can be seen as the ideals in Lie algebroids. We show that kernels of Dirac structures, usual ideals in Lie algebroids, Bott connections associated to involutive subbundle and kernels of transitive Lie algebroid morphisms are examples of infinitesimal ideal systems. We compare also our notion of foliated algebroids with the ones of [34], as well as the infinitesimal descriptions in both approaches.

Section 6 finally shows that the quotient of a Lie algebroid by an ideal system inherits a natural Lie algebroid structure such that the projection is a Lie algebroid morphism (Theorem 6.10).

The first appendix recalls how the Lie algebroid $A(TG) \to TM$ of the tangent groupoid $TG \rightrightarrows TM$ is isomorphic to the tangent Lie algebroid $T(A(G)) \to TM$. The second appendix proves some useful results on invariance of subbundles of a vector bundle under the flow of a vector field on it.

Notation. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sets of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $q_E\colon E \to M$, the set of (local) sections of E will be written $\Gamma(E)$. We will write $\operatorname{Dom}(\sigma)$ for the open subset of the smooth manifold M where the local section $\sigma \in \Gamma(E)$ is defined. The linear function on E associated to a section $\xi \in \Gamma(E^*)$ will be written ℓ_ξ. For any $\varepsilon \in \Gamma_M(E)$, the vector field $\varepsilon^\uparrow \in \mathfrak{X}(E)$ is defined by

$$\varepsilon^\uparrow(e_m) = \left.\frac{d}{dt}\right|_{t=0} e_m + t\varepsilon(m) \tag{1.1}$$

for all $e_m \in E$. In other words, $\varepsilon^\uparrow(\ell_\xi) = q_E^*\langle \xi, \varepsilon\rangle$ for all $\xi \in \Gamma(E^*)$, and $X(q_E^* f) = 0$ for all $f \in C^\infty(M)$. The Lie bracket of two such vector fields vanishes.

The flow of a vector field X will be written $\phi^X_\cdot$, unless specified otherwise.

Let $f : M \to N$ be a smooth map between two smooth manifolds M and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be ***f*-related** if $Tf \circ X = Y \circ f$ on $\operatorname{Dom}(X) \cap f^{-1}(\operatorname{Dom}(Y))$. We write then $X \sim_f Y$.

The pullback or restriction of a vector bundle $E \to M$ to an embedded submanifold N of M will be written $E|_N$. In the special case of the tangent and cotangent spaces of M, we will write $T_N M$ and $T^*_N M$. If $f : M \to N$ is a smooth surjective submersion, we write $T^f M$ for the kernel of $Tf : TM \to TN$.

The projection map of $TM \to M$ is finally denoted by p_M.

A **groupoid** G over the **units** M will be written $G \rightrightarrows M$. The **source** and **target** maps are denoted by $\mathsf{s}, \mathsf{t} : G \longrightarrow M$ respectively, the **unit section** $\epsilon : M \longrightarrow G$, the inversion map $\mathsf{i} : G \longrightarrow G$ and the multiplication $\mathsf{m} : G_{(2)} \longrightarrow G$, where $G_{(2)} = \{(g,h) \in G \times G \mid \mathsf{t}(h) = \mathsf{s}(g)\}$ is the set of composable groupoid pairs. A groupoid G over M is called a **Lie groupoid** if both G and M are smooth Hausdorff manifolds, the source and target maps $\mathsf{s}, \mathsf{t} : G \longrightarrow M$ are surjective submersions, and all the other structural maps are smooth. Throughout this work we only consider Lie groupoids.

The Lie algebroid of $G \rightrightarrows M$ is defined in this paper to be $AG = T^{\mathsf{s}}_M G$, with anchor $\rho_{AG} := T\mathsf{t}|_{AG}$ and bracket $[\cdot\,,\cdot]_{AG}$ defined by using right invariant vector fields.

We will write $A(\cdot)$ for the functor that sends Lie groupoids to Lie algebroids and Lie groupoid morphisms to Lie algebroid morphisms. *For simplicity,* $(AG, \rho_{AG}, [\cdot\,,\cdot]_{AG})$ *will be written* $(A, \rho, [\cdot\,,\cdot])$ *in the following.*

Acknowledgements. We would like to thank Henrique Bursztyn, Thiago Drummond, Mathieu Stiénon and Ping Xu for interesting discussions. We are especially thankful to Maria Amelia Salazar who showed us a counterexample to a former version of Theorem 3.8. Thanks to her comments, we improved considerably Section 3.3.

The first author wishes to thank Penn State University, IMPA and Universidade Federal do Paraná, Curitiba, where parts of this work have been done. The second author thanks IMPA, EPFL and ESI for the hospitality during several stages of this project.

2. Background

2.1. **Tangent and cotangent groupoids.** Let G be a Lie groupoid over M with Lie algebroid A. The tangent bundle TG has a natural Lie groupoid structure over TM, which is obtained by applying the tangent functor to each of the structure maps defining G. That is, the set of composable pairs $(TG)_{(2)}$ of this groupoid is equal to $T(G_{(2)})$ and for $(g,h) \in G_{(2)}$ and a pair $(v_g, w_h) \in (TG)_{(2)}$, the multiplication is

$$v_g \star w_h := T\mathsf{m}(v_g, w_h).$$

We refer to TG with the groupoid structure over TM as the **tangent groupoid** of G [23].

As in [22], we define **star vector fields on** G or **star sections of** TG to be vector fields $X \in \mathfrak{X}(G)$ such that there exists $\bar{X} \in \mathfrak{X}(M)$ with $X \sim_{\mathsf{s}} \bar{X}$ and $\bar{X} \sim_{\epsilon} X$, i.e. X and $\bar{X}$ are s-related and $\bar{X}$ and X are ϵ-related, i.e. X restricts to $\bar{X}$ on M. We then write $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$. In the same manner, we can define t-star sections, $X \overset{\star}{\sim}_{\mathsf{t}} \bar{X}$ with $\bar{X} \in \mathfrak{X}(M)$ and $X \in \mathfrak{X}(G)$. It is easy to see that the tangent space TG is spanned by star vector fields at each point in $G \setminus M$. Note also that the Lie bracket of two star sections of TG is again a star section.

We call a vector field $X \in \mathfrak{X}(G)$ a t**-section** if there exists $\bar{X} \in \mathfrak{X}(M)$ such that $X \sim_{\mathsf{t}} \bar{X}$.

We will also need the cotangent groupoid $T^*G \rightrightarrows A^*$ in the proof of Theorem 3.8. It was shown in [6], that T^*G is a Lie groupoid over A^*. The source and target of $\alpha_g \in T^*_g G$ are defined by

$$\tilde{\mathsf{s}}(\alpha_g) \in A^*_{\mathsf{s}(g)}, \qquad \tilde{\mathsf{s}}(\alpha_g)(a) = \alpha_g(Tl_g(a - T\mathsf{t}(a))) \quad \text{for all} \quad a \in A_{\mathsf{s}(g)}$$

and

$$\tilde{\mathsf{t}}(\alpha_g) \in A^*_{\mathsf{t}(g)}, \qquad \tilde{\mathsf{t}}(\alpha_g)(b) = \alpha_g(Tr_g(b)) \quad \text{for all} \quad b \in A_{\mathsf{t}(g)}.$$

A one-form $\eta \in \Omega^1(G)$ is a t-section of T^*G if $\tilde{\mathsf{t}} \circ \eta = \bar{\eta} \circ \mathsf{t}$ for some $\bar{\eta} \in \Gamma(A^*)$.

2.2. **The tangent Lie algebroid $TA \to TM$.** Consider a vector bundle $q_A\colon A \to M$. Then the tangent space TA of A has two vector bundle structures. First, the usual vector bundle structure of the tangent space, $p_A\colon TA \to A$ and second the vector bundle structure $Tq_A\colon TA \to TM$, with the addition defined as follows. If x_{a_m} and $x_{a'_m}$ are such that $Tq_A(x_{a_m}) = Tq_A(x_{a'_m}) =: x_m \in TM$, then there exist curves $c, c'\colon (-\varepsilon, \varepsilon) \to A$ such that $\dot{c}(0) = x_{a_m}$, $\dot{c}'(0) = x_{a'_m}$ and $q_A \circ c = q_A \circ c'$. The sum $x_{a_m} +_{Tq_A} x_{a'_m}$ is then defined by

$$x_{a_m} +_{Tq_A} x_{a'_m} = \left.\frac{d}{dt}\right|_{t=0} c(t) +_{q_A} c'(t) \in T_{a_m + a'_m}A.$$

We get a double vector bundle

$$\begin{tikzcd} TA \arrow[r, "p_A"] \arrow[d, "Tq_A"'] & A \arrow[d, "q_A"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd},$$

that is, the structure maps of each vector bundle structure are vector bundle morphisms relative to the other structure [23]. Note that a subbundle H of TA over A is said to be **linear** if it is closed under the addition of TA as a vector bundle over TM.

Assume now that $q_A : A \longrightarrow M$ has a Lie algebroid structure with anchor map $\rho : A \longrightarrow TM$ and Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$. Then there is a Lie algebroid structure on TA over TM. In order to describe it explicitly, we recall first that there exists a **canonical involution**

$$\begin{tikzcd} TTM \arrow[r, "J_M"] \arrow[d, "p_{TM}"'] & TTM \arrow[d, "Tp_M"] \\ TM \arrow[r, "\mathrm{Id}_{TM}"'] & TM \end{tikzcd} \tag{2.2}$$

which is given as follows [23, 33]. Elements $(\xi; v, x; m) \in TTM$, that is, with $p_{TM}(\xi) = v \in T_mM$ and $Tp_M(\xi) = x \in T_mM$, are considered as second derivatives

$$\xi = \frac{\partial^2\sigma}{\partial t \partial u}(0,0),$$

where $\sigma : \mathbb{R}^2 \to M$ is a smooth square of elements of M[4]. The canonical involution $J_M : TTM \to TTM$ is defined by

$$J_M(\xi) := \frac{\partial^2\sigma}{\partial u \partial t}(0,0).$$

We can apply the tangent functor to the anchor map $\rho : A \longrightarrow TM$, and then compose with the canonical involution to obtain a bundle map $\rho_{TA} : TA \longrightarrow TTM$ defined by

$$\rho_{TA} = J_M \circ T\rho.$$

This defines the tangent anchor map. In order to define the tangent Lie bracket, we observe that every section $a \in \Gamma_M(A)$ induces two types of sections of $TA \longrightarrow TM$. The first type of section is simply $Ta : TM \longrightarrow TA$, and the second type of section are the **core** sections $a^\dagger : TM \longrightarrow TA$, which are defined by

$$a^\dagger(v_m) = T_m0^A(v_m) +_{p_A} \left.\frac{d}{dt}\right|_{t=0} ta(m) \tag{2.3}$$

[4]The notation means that σ is first differentiated with respect to u, yielding a curve $v(t) = \frac{\partial\sigma}{\partial u}(t,0)$ in TM with $\left.\frac{d}{dt}\right|_{t=0} v(t) = \xi$. Thus, $v = \frac{\partial\sigma}{\partial u}(0,0) = p_{TM}(\xi)$ and $x = \frac{\partial\sigma}{\partial t}(0,0) = Tp_M(\xi)$.

where $0^A : M \longrightarrow A$ denotes the zero section. As observed in [25], sections of the form Ta and $a^\dagger$ generate the module of sections $\Gamma_{TM}(TA)$. Therefore, the tangent Lie bracket $[\cdot\,,\cdot]_{TA}$ is completely determined by

$$[Ta, Tb]_{TA} = T[a,b], \quad \left[Ta, b^\dagger\right]_{TA} = [a,b]^\dagger, \quad \left[a^\dagger, b^\dagger\right]_{TA} = 0$$

for all $a, b \in \Gamma(A)$, the extension to general sections is done using the Leibniz rule with respect to the tangent anchor ρ_{TA}.

2.3. **Flat partial connections.**

Definition 2.1. *([2]) Let M be a smooth manifold and $F \subseteq TM$ a smooth involutive vector subbundle of the tangent bundle. Let $E \to M$ be a vector bundle over M. An F-**partial connection** is a map $\nabla : \Gamma(F) \times \Gamma(E) \to \Gamma(E)$, written $\nabla(X, e) =: \nabla_X e$ for $X \in \Gamma(F)$ and $e \in \Gamma(E)$, such that:*

1. *∇ is tensorial in the F-argument,*
2. *∇ is $\mathbb{R}$-linear in the E-argument,*
3. *∇ satisfies the Leibniz rule*
$$\nabla_X(fe) = X(f)e + f\nabla_X e$$
for all $X \in \Gamma(F)$, $e \in \Gamma(E)$, $f \in C^\infty(M)$.

The connection is flat if its curvature tensor vanishes.

Example 2.2 (The Bott connection)**.** Let M be a smooth manifold and $F \subseteq TM$ an involutive subbundle. The **Bott connection**

$$\nabla^F : \Gamma(F) \times \Gamma(TM/F) \to \Gamma(TM/F)$$

defined by

$$\nabla_X \bar{Y} = \overline{[X, Y]},$$

where $\bar{Y} \in \Gamma(TM/F)$ is the projection of $Y \in \mathfrak{X}(M)$, is a flat F-partial connection on $TM/F \to M$.

The class $\bar{Y} \in \Gamma(TM/F)$ of a vector field is ∇^F-parallel if and only if $[Y, \Gamma(F)] \subseteq \Gamma(F)$. Since F is involutive, this does not depend on the representative of $\bar{Y}$. We say by abuse of notation that Y is ∇^F-parallel.

The following proposition can be easily shown by using the fact that the parallel transport defined by a flat connection does not depend on the chosen path in simply connected sets (see [16], [18] for similar statements).

Proposition 2.3. *Let $E \to M$ be a smooth vector bundle of rank k, $F \subseteq TM$ an involutive subbundle and ∇ a flat partial F-connection on E. Then there exists for each point $m \in M$ a frame of local ∇-parallel sections $e_1, \ldots, e_k \in \Gamma(E)$ defined on an open neighborhood U of m in M.*

We will also use the following lemma, which is easy to prove.

Lemma 2.4. *Let $E \to M$ be a smooth vector bundle of rank k, $F \subseteq TM$ an involutive subbundle and ∇ a partial F-connection on E.*

1. *Assume that $f \in C^\infty(M)$ is F-invariant, i.e. $X(f) = 0$ for all $X \in \Gamma(F)$. Then $f \cdot e$ is ∇-parallel for any ∇-parallel section $e \in \Gamma(E)$.*
2. *Assume that the foliation defined by F on M is simple, i.e. the leaf space has a smooth manifold structure such that the quotient $\pi : M \to M/F$ is a smooth surjective submersion. Then $X \in \mathfrak{X}(M)$ is ∇^F-parallel if and only if there exists $\bar{X} \in \mathfrak{X}(M/F)$ such that $X \sim_\pi \bar{X}$.*

3. Foliated groupoids

3.1. **Definition and properties.**

Definition 3.1. *Let $G \rightrightarrows M$ be a Lie groupoid. A subbundle $F_G \subseteq TG$ is* ***multiplicative*** *if it is a subgroupoid of $TG \rightrightarrows TM$ over $F_G \cap TM =: F_M$. We also say that F_G is a* ***multiplicative involutive distribution*** *on G. To simplify the terminology, we that that the pair $(G \rightrightarrows M, F_G)$ is a* ***foliated groupoid****.*

Remark 3.2. Multiplicative subbundles were introduced in [32] (see also [13]) as follows. A subbundle $F_G \subseteq TG$ is multiplicative if for all composable $g, h \in G$ and $u \in F_G(g \star h)$, there exist $v \in F_G(g)$, $w \in F_G(h)$ such that $u = v \star w$. It is easy to check that a multiplicative distribution in the sense of Definition 3.1 is multiplicative in the sense of [32], but the converse is not necessarily true, unless for instance if the Lie groupoid is a Lie group (see [15]). The case of involutive wide subgroupoids of $TG \rightrightarrows TM$ has also been studied in [1].

Definition 3.3. *Let F_G be a multiplicative distribution on $G \rightrightarrows M$. The subbundle*

$$K = F_G \cap A = \{v \in F_G \mid p_G(v) \in M \text{ and } T\mathsf{s}(v) = 0\}$$

is called the ***core*** *of $(G \rightrightarrows M, F_G)$.*

A multiplicative subbundle $F_G \subseteq TG$ determines a $\mathcal{VB}$-groupoid

$$\begin{tikzcd} F_G \arrow[r, "p_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & G \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ F_M \arrow[r, "p_M"'] & M \end{tikzcd}$$

with **core** K. In particular, we have the following lemma [16].

Lemma 3.4. *Let $G \rightrightarrows M$ be a Lie groupoid and $F_G \subseteq TG$ a multiplicative subbundle. Then the intersection $F_M := F_G \cap TM$ has constant rank on M. Since it is the set of units of F_G seen as a subgroupoid of TG, the pair $F_G \rightrightarrows F_M$ is a Lie groupoid.*

The bundle $F_G|_M$ splits as $F_G|_M = F_M \oplus K$, where $K := F_G \cap A$. We have

$$(F_G \cap T^{\mathsf{s}}G)(g) = K(\mathsf{t}(g)) \star 0_g = T_{\mathsf{t}(g)} r_g \left(K(\mathsf{t}(g))\right)$$

for all $g \in G$.

In the same manner, if $F^{\mathsf{t}} := (F_G \cap T^{\mathsf{t}}G)|_M$, we have $(F_G \cap T^{\mathsf{t}}G)(g) = 0_g \star F^{\mathsf{t}}(\mathsf{s}(g))$ for all $g \in G$. As a consequence, the intersections $F_G \cap T^{\mathsf{t}}G$ and $F_G \cap T^{\mathsf{s}}G$ have constant rank on G.

3.2. **The connection associated to a foliated groupoid.**

Our first result on multiplicative distributions is easy to prove, by considering right-invariant and s-sections of F_G (see also [17, 14]).

Proposition 3.5. *Let F_G be a multiplicative involutive distribution on a Lie groupoid $G \rightrightarrows M$. Then K is a subalgebroid of A and F_M is an involutive subbundle of TM.*

The main goal of this subsection is the construction of a partial F_M-connection on A/K induced by the Bott F_G-connection on TG/F_G. We will see later how the quadruple (A, F_M, K, ∇) contains the whole information about the foliated groupoid.

We write $\bar{a}$ for the class in A/K of $a \in \Gamma(A)$.

Theorem 3.6. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid. Then there is a partial F_M-connection on A/K*

$$\nabla : \Gamma(F_M) \times \Gamma(A/K) \to \Gamma(A/K)\,. \tag{3.4}$$

defined by

$$a \in \Gamma(A) \text{ is } \nabla\text{-parallel if and only if } a^r \in \mathfrak{X}(G) \text{ is } \nabla^{F_G}\text{-parallel.}$$

The triple (F_M, K, ∇) *is an infinitesimal ideal system in* A.

For the proof of this theorem, we need the following result, which can be shown with the same techniques as its general counterpart on Dirac groupoids in [17, 14].

Lemma 3.7. *Let* $(G \rightrightarrows M, F_G)$ *be a Lie groupoid endowed with a multiplicative subbundle* $F_G \subseteq TG$, X *a* t*-section of* F_G*, i.e.* t*-related to some* $\bar{X} \in \Gamma(F_M)$*, and consider* $a \in \Gamma(A)$*. Then the derivative* $\pounds_{a^r} X$ *can be written as a sum*

$$\pounds_{a^r} X = Z_{a,X} + b^r_{a,X}$$

with $b_{a,X} \in \Gamma(A)$*, and* $Z_{a,X}$ *a* t*-section of* F_G*. In addition, if* $X \sim_{\mathsf{t}} 0$*, then* $\pounds_{a^r} X \in \Gamma(F_G \cap T^{\mathsf{t}} G)$*. In particular, its restriction to* M *is a section of* F^{t} *and* $b_{a,X}$ *is a section of* K*.*

Assume now that F_G is involutive and define

$$\nabla : \Gamma(F_M) \times \Gamma(A/K) \to \Gamma(A/K)$$

by

$$\nabla_{\bar{X}} \bar{a} = -\overline{b_{a,X}},$$

with $b_{a,X}$ as in Lemma 3.7, for any choice of t-section $X \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and any choice of representative $a \in \Gamma(A)$ for $\bar{a}$. We will show that this is a well-defined partial F_M-connection and complete the proof of Theorem 3.6.

Proof of Theorem 3.6. Choose $X, X' \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and $X' \sim_{\mathsf{t}} \bar{X}$. Then $Y := X - X' \sim_{\mathsf{t}} 0$ and, by Lemma 3.7, we find $b_{a,Y} \in \Gamma(K)$ for any $a \in \Gamma(A)$, i.e. $\overline{b_{a,X}} = \overline{b_{a,X'}}$.

Choose now $a \in \Gamma(K)$ and $X \in \Gamma(F_G)$, $X \sim_{\mathsf{t}} \bar{X} \in \Gamma(F_M)$. Then we have $a^r \in \Gamma(F_G)$ and since F_G is involutive, $\pounds_{a^r} X \in \Gamma(F_G)$. Again, since $Z_{a,X} \in \Gamma(F_G)$, we find $b_{a,X} \in \Gamma(K)$. This shows that ∇ is well-defined.

By definition, if $a \in \Gamma(A)$ is such that $\nabla_{\bar{X}} \bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$, then we have $\pounds_{a^r} X = Z_{a,X} + b^r_{a,X} \in \Gamma(F_G)$ for all t-sections $X \in \Gamma(F_G)$. Since $\Gamma(F_G)$ is spanned as a $C^\infty(G)$-module by its t-sections, we get

$$[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G).$$

Conversely, $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$ implies immediately $\nabla_{\bar{X}} \bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$. This proves the second claim of the theorem.

We check that ∇ is a flat partial F_M-connection. Choose $a \in \Gamma(A)$, $\bar{X} \in \Gamma(F_M)$, $X \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and $f \in C^\infty(M)$. Then we have $\mathsf{t}^* f \cdot X \sim_{\mathsf{t}} f\bar{X}$ and

$$\pounds_{a^r}(\mathsf{t}^* f \cdot X) = \mathsf{t}^*(\rho(a)(f)) \cdot X + \mathsf{t}^* f \cdot \pounds_{a^r} X.$$

In particular, we find

$$\begin{aligned} \overline{b_{a,\mathsf{t}^* f \cdot X}} &= \overline{(1 - T\mathsf{s})\left(\mathsf{t}^*(\rho(a)(f)) \cdot X + \mathsf{t}^* f \cdot \pounds_{a^r} X\right)|_M} \\ &= \overline{\rho(a)(f) \cdot (1 - T\mathsf{s}) X|_M + f \cdot (1 - T\mathsf{s})(\pounds_{a^r} X)|_M}. \end{aligned}$$

Since $(T\mathsf{s} - 1)X|_M \in \Gamma(K)$, this leads to $\overline{b_{a,\mathsf{t}^* f \cdot X}} = f \cdot \overline{b_{a,X}}$ and hence $\nabla_{f\bar{X}} \bar{a} = -f \cdot \overline{b_{a,X}} = f \cdot \nabla_{\bar{X}} \bar{a}$.

Since $(fa)^r = \mathsf{t}^* f \cdot a^r$, we have in the same manner

$$\pounds_{(fa)^r} X = -\pounds_X(\mathsf{t}^* f \cdot a^r) = -\mathsf{t}^*(\bar{X}(f)) \cdot a^r + \mathsf{t}^* f \cdot \pounds_{a^r} X,$$

which leads to $\nabla_{\bar{X}}(f \cdot \bar{a}) = \bar{X}(f) \cdot \bar{a} + f \cdot \nabla_{\bar{X}} \bar{a}$.

Choose $\bar{X}, \bar{Y} \in \Gamma(F_M)$ and $X, Y \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and $Y \sim_{\mathsf{t}} \bar{Y}$. Then we have $[X, Y] \sim_{\mathsf{t}} [\bar{X}, \bar{Y}]$ and $[X, Y] \in \Gamma(F_G)$ since F_G is involutive. For any $a \in \Gamma(A)$, we have by the Jacobi-identity:

$$\begin{aligned}
\pounds_{a^r}[X,Y] &= [\pounds_{a^r}X, Y] - [\pounds_{a^r}Y, X] \\
&= \left[Z_{a,X} + b^r_{a,X}, Y\right] - \left[Z_{a,Y} + b^r_{a,Y}, X\right] \\
&= [Z_{a,X}, Y] - [Z_{a,Y}, X] + \pounds_{b^r_{a,X}} Y - \pounds_{b^r_{a,Y}} X \\
&= [Z_{a,X}, Y] - [Z_{a,Y}, X] + Z_{b_{a,X},Y} + {b_{b_{a,X},Y}}^r - Z_{b_{a,Y},X} - {b_{b_{a,Y},X}}^r.
\end{aligned}$$

Since $[Z_{a,X}, Y] - [Z_{a,Y}, X] + Z_{b_{a,X},Y} - Z_{b_{a,Y},X}$ is a t-section of F_G, we find that

$$\nabla_{[\bar{X},\bar{Y}]}\bar{a} = \overline{b_{b_{a,Y},X}} - \overline{b_{b_{a,X},Y}} = \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{a} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{a}$$

which shows the flatness of ∇.

Choose now $a \in \Gamma(A)$ such that $\nabla_{\bar{X}}\bar{a} = 0 \in \Gamma(A/K)$ for all $\bar{X} \in \Gamma(F_M)$. If $b \in \Gamma(K)$, then $b^r \in \Gamma(F_G)$, $\rho(b) \in \Gamma(F_M)$ and $b^r \sim_{\mathsf{t}} \rho(b)$. This leads to

$$\overline{[b,a]} = \nabla_{\rho(b)}\bar{a} = 0 \in \Gamma(A/K)$$

and hence $[a, b] \in \Gamma(K)$. This shows 2. For each $\bar{X} \in \Gamma(F_M)$, there exists $X \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$. Since $[a^r, X] \in \Gamma(F_G)$, $a^r \sim_{\mathsf{t}} \rho(a)$ and $T\mathsf{t}(F_G) = F_M$, we find $[\rho(a), \bar{X}] \in \Gamma(F_M)$, which proves 4.

To show 3., choose two sections $a, b \in \Gamma(A)$ such that $\bar{a}$ and $\bar{b}$ are ∇-parallel. We have then for any t-section $X \sim_{\mathsf{t}} \bar{X}$ of F_G:

$$\begin{aligned}
\pounds_{[a,b]^r} X &= \pounds_{a^r}(Z_{b,X} + b^r_{b,X}) - \pounds_{b^r}(Z_{a,X} + b^r_{a,X}) \\
&= \pounds_{a^r}(Z_{b,X}) + [a, b_{b,X}]^r - \pounds_{b^r}(Z_{a,X}) - [b, b_{a,X}]^r.
\end{aligned}$$

Since $\bar{a}$ and $\bar{b}$ are ∇-parallel, this yields $\nabla_{\bar{X}}\overline{[a,b]} = -\overline{[a, b_{b,X}]} + \overline{[b, b_{a,X}]}$. Since $\bar{a}$ and $\bar{b}$ are parallel, we have $b_{b,X}, b_{a,X} \in \Gamma(K)$ and 3. follows using 2. □

3.3. **Involutivity of a multiplicative subbundle of TG.** It is natural to ask here how exactly the involutivity of F_G is encoded in the data (F_M, K, ∇). For an arbitrary (not necessarily involutive) multiplicative subbundle $F_G \subseteq TG$, we can consider the map

$$\tilde{\nabla} : \Gamma(F_M) \times \Gamma(A) \to \Gamma(A/K),$$
$$\tilde{\nabla}_{\bar{X}} a = -\overline{b_{a,X}}$$

which is well-defined by the proof of Theorem 3.6.

Theorem 3.8. *Let (G, F_G) be a source-connected Lie groupoid endowed with a multiplicative subbundle. Then F_G is involutive if and only if the following holds:*

1. *$F_M \subseteq TM$ is involutive,*
2. *$\tilde{\nabla}$ vanishes on sections of K,*
3. *the induced map $\nabla : \Gamma(F_M) \times \Gamma(A/K) \to \Gamma(A/K)$ is a flat partial F_M-connection on A/K.*

The proof of this theorem is a simplified version of the proof of the general criterion for the integrability property of multiplicative Dirac structures (see [14]).

Proof. We have already shown in Proposition 3.5 and Theorem 3.6 that the involutivity of F_G implies (1), (2) and (3).

For the converse implication, note that the t-star sections of F_G span F_G outside of the set of units M. Hence, it is sufficient to show involutivity on t-star sections and right-invariant sections of F_G. Choose first two right-invariant sections a^r, b^r of F_G, i.e. with $a, b \in \Gamma(K)$. We then have $\rho(b) \in \Gamma(F_M)$, $b^r \sim_{\mathsf{t}} \rho(b)$ and, since $\tilde{\nabla}_{\rho(b)} a = 0$ by (2), we find that $[a^r, b^r] \in \Gamma(F_G)$. In the same manner, by the definition of $\tilde{\nabla}$ and Lemma 3.7, Condition

(2) implies that the bracket of a right-invariant section of F_G and a t-star section is always a section of F_G.

We have thus only to show that the bracket of two t-star sections of F_G is again a section of F_G. Let K° be the annihilator of K in A^* and consider the dual F_M-connection on $(A/K)^* \simeq K^\circ \subseteq A^*$, i.e. the (by (3)) flat connection

$$\nabla^* : \Gamma(F_M) \times \Gamma(K^\circ) \to \Gamma(K^\circ)$$

given by

$$\left(\nabla^*_{\bar{X}}\alpha\right)(\bar{a}) = \bar{X}(\alpha(\bar{a})) - \alpha\left(\nabla_{\bar{X}}\bar{a}\right)$$

for all $\bar{X} \in \Gamma(F_M)$, $\alpha \in \Gamma(K^\circ)$ and $a \in \Gamma(A)$.

Choose a t-star section $X \in \Gamma(F_G)$, $X \sim_{\mathsf{t}} \bar{X}$, $X|_M = \bar{X}$ and a $\tilde{\mathsf{t}}$-section $\eta \in \Gamma(F_G^\circ)$, $\eta \sim_{\tilde{\mathsf{t}}} \bar{\eta}$. Then, for any section a of A, we have

$$\begin{aligned}
(\pounds_X\eta)(a^r) &= X\left(\eta(a^r)\right) + \eta\left(\pounds_{a^r}X\right)\\
&= \mathsf{t}^*\left(\bar{X}(\bar{\eta}(\bar{a}))\right) + \eta\left(Z_{a,X} + b^r_{a,X}\right) \qquad \text{by Lemma 3.7}\\
&= \mathsf{t}^*\left(\bar{X}(\bar{\eta}(\bar{a})) - \bar{\eta}\left(\nabla_{\bar{X}}\bar{a}\right)\right) \qquad \text{since } \eta \in \Gamma(F_G^\circ) \text{ and } Z_{a,X} \in \Gamma(F_G)\\
&= \mathsf{t}^*(\nabla^*_{\bar{X}}\bar{\eta}(\bar{a})).
\end{aligned}$$

This shows that $\pounds_X\eta \sim_{\tilde{\mathsf{t}}} \nabla^*_{\bar{X}}\bar{\eta} \in \Gamma(K^\circ)$. Note that we have not shown yet that $\pounds_X\eta$ is a section of F_G°. Choose a second t-star section Y of F_G, $Y \sim_{\mathsf{t}} \bar{Y}$ and $Y|_M = \bar{Y}$. An easy computation using $\eta(X) = \eta(Y) = 0$ yields

$$-2\mathbf{d}\left(\eta([X,Y])\right) = \pounds_X\pounds_Y\eta - \pounds_Y\pounds_X\eta - \pounds_{[X,Y]}\eta.$$

Hence, we get for $a \in \Gamma(A)$:

$$\begin{aligned}
-2\cdot a^r(\eta([X,Y])) &= \left(\pounds_X\pounds_Y\eta - \pounds_Y\pounds_X\eta - \pounds_{[X,Y]}\eta\right)(a^r)\\
&= X(\pounds_Y\eta(a^r)) - \pounds_Y\eta([X,a^r]) - Y(\pounds_X\eta(a^r)) - \pounds_X\eta([Y,a^r])\\
&\quad - [X,Y](\eta(a^r)) + \eta([[X,Y],a^r])\\
&= \mathsf{t}^*\bar{X}(\nabla^*_{\bar{Y}}\bar{\eta}(\bar{a})) + (\pounds_Y\eta)(Z_{a,X} + b^r_{a,X})\\
&\quad - \mathsf{t}^*\bar{Y}(\nabla^*_{\bar{X}}\bar{\eta}(\bar{a})) - (\pounds_X\eta)(Z_{a,Y} + b^r_{a,Y})\\
&\quad - \mathsf{t}^*[\bar{X},\bar{Y}](\bar{\eta}(\bar{a})) - \eta\left([\pounds_{a^r}X, Y] + [X, \pounds_{a^r}Y]\right)\\
&= \mathsf{t}^*\bar{X}(\nabla^*_{\bar{Y}}\bar{\eta}(\bar{a})) + \pounds_Y\eta(Z_{a,X}) - \mathsf{t}^*(\nabla^*_{\bar{Y}}\bar{\eta})(\nabla_{\bar{X}}\bar{a})\\
&\quad - \mathsf{t}^*\bar{Y}(\nabla^*_{\bar{X}}\bar{\eta}(\bar{a})) - \pounds_X\eta(Z_{a,Y}) + \mathsf{t}^*(\nabla^*_{\bar{X}}\bar{\eta})(\nabla_{\bar{Y}}\bar{a})\\
&\quad - \mathsf{t}^*[\bar{X},\bar{Y}](\bar{\eta}(\bar{a})) - \eta\left([Z_{a,X} + b^r_{a,X}, Y] + [X, Z_{a,Y} + b^r_{a,Y}]\right)\\
&= \mathsf{t}^*\left(\left(\nabla^*_{\bar{X}}\nabla^*_{\bar{Y}}\bar{\eta} - \nabla^*_{\bar{Y}}\nabla^*_{\bar{X}}\bar{\eta}\right)(\bar{a}) - [\bar{X},\bar{Y}](\bar{\eta}(\bar{a}))\right)\\
&\quad + Y(\eta(Z_{a,X})) - X(\eta(Z_{a,Y})) - \eta\left(Z_{b_{a,X},Y} + b^r_{b_{a,X},Y} - Z_{b_{a,Y},X} - b^r_{b_{a,Y},X}\right)\\
&\overset{(3)}{=} \mathsf{t}^*\left(\bar{\eta}(-\nabla_{[\bar{X},\bar{Y}]}\bar{a} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{a} + \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{a})\right) \overset{(3)}{=} 0.
\end{aligned}$$

Hence, $a^r(\eta([X,Y])) = 0$ for all $a \in \Gamma(A)$ and since G is source-connected, this implies that $\eta([X,Y])(g) = \eta([X,Y])(\mathsf{s}(g))$ for all $g \in G$. But since for $m \in M$, we have

$$[X,Y](m) = [\bar{X},\bar{Y}](m)$$

and F_M is a subalgebroid of TM by (1), we find that $[X,Y](\mathsf{s}(g)) \in F_G(\mathsf{s}(g))$ for all $g \in G$ and hence $\eta([X,Y])(g) = \eta([X,Y])(\mathsf{s}(g)) = 0$. Since η was a $\tilde{\mathsf{t}}$-section of F_G° and $\tilde{\mathsf{t}}$-sections of F_G° span F_G° on G, we have shown that $[X,Y] \in \Gamma(F_G)$ and the proof is complete. □

Remark 3.9. (1) We have seen in this proof that Condition (2) implies the fact that K is a subalgebroid of A.

(2) The same result has been shown independently in [7], using Lie groupoid and Lie algebroid cocycles, in the special case where $F_M = TM$, i.e. where F_G is a wide subgroupoid of TG.

Example 3.10. Assume that G is a Lie group (hence with $M = \{e\}$) with Lie algebra $\mathfrak{g}$. Let F_G be a multiplicative distribution. In this case, the core $K =: \mathfrak{k}$ is the fiber of F_G over the identity and $F_M = 0$. As a consequence, any partial F_M-connection on $\mathfrak{g}/\mathfrak{k}$ is trivial. We check that all the conditions in Theorem 3.6 are automatically satisfied.

First of all, any element ξ of $\mathfrak{g}$ is ∇-parallel. This implies that

$$[\xi^r, \Gamma(F_G)] \subseteq \Gamma(F_G) \qquad \text{for all} \qquad \xi \in \mathfrak{g},$$

i.e. F_G is left-invariant, in agreement with [27, 15, 16].

1), 3) and 4) are trivially satisfied and 2) is exactly the fact that $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. This recovers the results proved in [27, 15, 16]. Note also that since all the conditions in Theorem 3.8 are trivially satisfied and a multiplicative distribution on a Lie group is hence always involutive.

Example 3.11. Let $G \rightrightarrows M$ be a Lie groupoid with a smooth, free and proper action of a Lie group H by Lie groupoid automorphisms. Let $\mathcal{V}_G$ be the vertical space of the action, i.e. the smooth subbundle of TG that is generated by the infinitesimal vector fields ξ_G, for all $\xi \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. The involutive subbundle $\mathcal{V}_G$ is easily seen to be multiplicative (see for instance [16]).

The action restricts to a free and proper action of H on M, and it is easy to check that $\mathcal{V}_G \cap TM = \mathcal{V}_M$ is the vertical vector space of the action of H on M. Furthermore, $\mathcal{V}_G \cap T^{\mathsf{s}}G = \mathcal{V}_G \cap T^{\mathsf{t}}G = 0^{TG}$ and we get $K = 0^A$.

The infinitesimal vector fields (ξ_G, ξ_M), $\xi \in \mathfrak{h}$, are multiplicative (in the sense of [25] for instance). We get hence from [25] that the Lie bracket $[a^r, \xi_G]$ is right-invariant for any $\xi \in \mathfrak{h}$ and $a \in \Gamma(A)$. We obtain a map (see also [25])

$$\begin{array}{ccc} \mathfrak{h} \times \Gamma(A) & \to & \Gamma(A) \\ (\xi, a) & \mapsto & [\xi_G, a^r]|_M \end{array},$$

and we recover the connection

$$\nabla : \quad \Gamma(\mathcal{V}_M) \times \Gamma(A) \quad \to \quad \Gamma(A)$$

defined by $\nabla_{\xi_M} a = [\xi_G, a^r]|_M$ for all $\xi \in \mathfrak{h}$ and $a \in \Gamma(A)$. This connection is obviously flat and satisfies all the conditions in Theorem 3.6.

Example 3.12. Let $(G \rightrightarrows M, J_G)$ be a complex Lie groupoid, i.e. a Lie groupoid endowed with a complex structure J_G that is multiplicative in the sense that the map

$$\begin{tikzcd} TG \arrow[r, "J_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & TG \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] \\ TM \arrow[r, "J_M"'] & TM \end{tikzcd}$$

is a Lie groupoid morphism over some map J_M. Since ${J_G}^2 = -\mathrm{Id}_{TG}$, we conclude that ${J_M}^2 = -\mathrm{Id}_{TM}$ and the Nijenhuis condition for J_M is easy to prove using s-related vector fields. The map J_G restricts also to a map j_A on the core A, i.e. a fiberwise complex structure that satisfies also a Nijenhuis condition. (This can be seen by noting that the Nijenhuis tensor of J_G restricts to right-invariant vector fields.)

The subbundles $T^{1,0}G = E_i$ and $T^{0,1}G = E_{-i}$ of $TG \otimes \mathbb{C}$ are multiplicative and involutive with bases $T^{1,0}M$ and $T^{0,1}M$ and cores $A^{1,0}$ and $A^{0,1}$. The quotient $(A \otimes \mathbb{C})/A^{1,0}$ is isomorphic as a vector bundle to $A^{0,1}$ and a straightforward computation shows that the connection that we get from the multiplicative involutive complex distribution $T^{1,0}G$ is exactly the connection $\nabla : \Gamma(T^{1,0}M) \times \Gamma(A^{0,1}) \to \Gamma(A^{0,1})$ as in Lemma 4.7 of [20].

Since the parallel sections of the connection in this Lemma are exactly the holomorphic sections of $A^{0,1}$, one can reconstruct the map $J_A : TA \to TA$ defined by $J_A = \sigma^{-1} \circ A(J_G) \circ \sigma$ [5] as in [21] by requiring that $J_A(Ta) = Ta \circ J_M$ for all parallel sections $a \in \Gamma(A)$, and $J_A(\hat{b}) = \widehat{j_A(b)} \circ J_M$ for all sections $b \in \Gamma(A)$.

By Lemma 4.7 in [20] and the integration results in [21], the complex structure J_G is hence equivalent to the datum (J_M, j_A, ∇) with its properties. This is in agreement with the results that we will prove in the next section.

4. Foliated algebroids

In this section we study Lie algebroids equipped with distributions compatible with both Lie algebroid structures $TA \to TM$ and $TA \to A$ on TA. This is the first step towards an infinitesimal description of foliated groupoids.

4.1. **Definition and properties.**

Definition 4.1. *Let $A \to M$ be a Lie algebroid. A subbundle $F_A \subseteq TA$ is called* ***morphic*** *if it is a Lie subalgebroid of $TA \to TM$ over some subbundle $F_M \subseteq TM$.*

If F_A is involutive and morphic, then the pair (A, F_A) is a ***foliated Lie algebroid****.*

Since foliated algebroids have already been shown in [13, 8] to correspond to infinitesimal ideal systems, we only summarize here the approach in [8], see also [19]. (Note that another approach could be found in a former version of this paper.)

4.1.1. *Connections on a vector bundle A, linear splittings of TA and the Lie bracket on $\mathfrak{X}(A)$.* We recall here the relation between a connection on a vector bundle A and the Lie bracket of vector fields on A.

Let $q_A\colon A \to M$ be a vector bundle. A linear vector field on A is a derivation of $C^\infty(A)$ that sends linear functions to linear functions and pullbacks to pullbacks. More explicitely, $X \in \mathfrak{X}(A)$ is linear over $\bar{X} \in \mathfrak{X}(M)$ if for all $\xi \in \Gamma(A^*)$, $X(\ell_\xi) = \ell_{D_X^*\xi}$ with $D_X^*\xi \in \Gamma(A^*)$ and for all $f \in C^\infty(M)$, $X(q_A^*f) = q_A^*(\bar{X}(f))$. Hence, a vector field X which is q_A-related to $\bar{X} \in \mathfrak{X}(M)$ defines a derivation $D_X^* : \Gamma(A^*) \to \Gamma(A^*)$ over $\bar{X}$. The dual derivation $D_X : \Gamma(A) \to \Gamma(A)$ is then defined by $\langle D_X a, \xi\rangle = \bar{X}\langle a, \xi\rangle - \langle a, D_X^*\xi\rangle$ and describes the Lie bracket of X with core vector fields:

$$[X, a^\uparrow] = (D_X a)^\uparrow \tag{4.5}$$

for all $a \in \Gamma(A)$. The Lie bracket $[X, Y]$ of two linear vector fields X and $Y \in \mathfrak{X}(A)$ over $\bar{X}$ and $\bar{Y} \in \mathfrak{X}(M)$ is again linear over $[\bar{X}, \bar{Y}]$ and the derivation $D_{[X,Y]}$ is equal to the commutator of the derivations D_X and D_Y.

Let $\nabla\colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ be a connection. For each $X \in \mathfrak{X}(M)$, ∇_X is a derivation of $\Gamma(A)$ and we have a corresponding linear vector field $\widetilde{\nabla_X}$ over X. The set of all vector fields on A defined in this manner spans a linear subbundle H_∇ of $p_A\colon TA \to A$ that is in direct sum with the vertical space $V := T^{q_A}A = \{v_{a_m} \in TA \mid T_{a_m}q_A v_{a_m} = 0\}$:

$$TA \cong V \oplus H_\nabla \to A.$$

For all functions $\varphi \in C^\infty(M)$ and sections $\xi \in \Gamma(A^*)$, we have

$$\widetilde{\nabla_X}(\ell_\xi) = \ell_{\nabla_X^*\xi}, \qquad \widetilde{\nabla_X}(q_A^*\varphi) = q_A^*(X(\varphi)), \qquad b^\uparrow(\ell_\xi) = q_A^*\langle \xi, b\rangle, \qquad b^\uparrow(q_A^*\varphi) = 0. \tag{4.6}$$

Conversely, consider a linear splitting $TA \cong V \oplus H$ of $TA \to A$. Then, since $H \cong TA/V$ is isomorphic to the pullback $q_A^!TM \to A$, and by the linearity of H, we find for each vector

[5] To avoid confusions, we write in this example $\sigma : TA \to A(TG)$ for the canonical flip map.

field $X \in \mathfrak{X}(M)$ a unique linear vector field $\tilde{X} \in \Gamma(H)$ such that $\tilde{X} \sim_{q_A} X$. One can then *define* a connection $\nabla^H : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ by setting

$$\nabla^H_X = D_X$$

for all $X \in \mathfrak{X}(M)$.

This shows the correspondence of the two definitions of a connection; the first as the map

$$\nabla : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A),$$

the second as a linear splitting

$$TA \cong V \oplus H \to A.$$

Given ∇ or H_∇, it is easy to see using the equalities in (4.6) that

$$\begin{aligned} \left[\tilde{X}, \tilde{Y}\right] &= \widetilde{[X,Y]} - R_\nabla(X,Y)^\uparrow, \\ \left[\tilde{X}, a^\uparrow\right] &= (\nabla_X a)^\uparrow, \\ \left[a^\uparrow, b^\uparrow\right] &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and $a, b \in \Gamma(A)$. Here, $R_\nabla(X,Y)^\uparrow \in \mathfrak{X}(A)$ is defined by $R_\nabla(X,Y)^\uparrow(a_m) = (R_\nabla(X,Y)a_m)^\uparrow$ for all $a_m \in A$. That is, the Lie bracket of vector fields on A can be completely described in terms of the connection.

4.1.2. *The Lie bracket on sections of $TA \to TM$.* Consider now a connection ∇ on a Lie algebroid A. Then TA splits over TM as $H_\nabla \oplus \ker(p_A)$ and we can define sections $\tilde{a} \in \Gamma_{TM}(H_\nabla)$, for $a \in \Gamma(A)$, by

$$\tilde{a}(v_m) = T_m a v_m - \left.\frac{d}{dt}\right|_{t=0} a_m + t\nabla_{v_m} a$$

for all $v_m \in TM$. Recall that for $a \in \Gamma(A)$, we also have the core section $a^\dagger$ of $\ker(p_A) \to TM$:

$$a^\dagger(v_m) = T_m 0^A v_m + \left.\frac{d}{dt}\right|_{t=0} ta(m).$$

The vector bundle $TA \to TM$ is spanned by the sections $\tilde{a}$ and $a^\dagger$ for all $a \in \Gamma(A)$ and the Lie algebroid structure on $TA \to TM$ can be described as follows:

(1) $[\tilde{a}, \tilde{b}] = \widetilde{[a,b]} - R^{\rm bas}_\Delta(a,b)^\dagger$,
(2) $[\tilde{a}, b^\dagger] = (\nabla^{\rm bas}_a b)^\dagger$,
(3) $[a^\dagger, b^\dagger] = 0$,
(4) $\rho_{TA}(\tilde{a}) = \widehat{\nabla^{\rm bas}_a} \in \mathfrak{X}(TM)$,
(5) $\rho_{TA}(b^\dagger) = (\rho(b))^\uparrow \in \mathfrak{X}(TM)$,

where $\nabla^{\rm bas} : \Gamma(A) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $\nabla^{\rm bas} : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ are given by

$$\nabla^{\rm bas}_a X = \rho(\nabla_X a) + [\rho(a), X]$$

for all $a \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$ and

$$\nabla^{\rm bas}_a b = \nabla_{\rho(b)} a + [a, b]$$

for all $b \in \Gamma(A)$ [9, 10], and $R^{\rm bas}_\nabla \in \Omega^2(A, \mathrm{Hom}(A, TM))$ by

$$R^{\rm bas}_\nabla(a,b)(X) = -\nabla_X[a,b] + [\nabla_X a, b] + [a, \nabla_X b] + \nabla_{\nabla^{\rm bas}_b X} a - \nabla_{\nabla^{\rm bas}_a X} b.$$

4.1.3. *Double subbundles of the double vector bundle* $(TA;TM,A;M)$. Given a double subbundle $(F_A;F_M,A;M)$ of $(TA;TM,A;M)$, we can always choose a connection ∇ that is *adapted* to F_A, i.e. such that F_A splitts as

$$(F_A\cap H_\nabla)\oplus(F_A\cap V)$$

as a vector bundle over A, and as

$$(F_A\cap H_\nabla)\oplus(F_A\cap \ker(p_A))$$

as a vector bundle over F_M. The restriction to the zero section of the intersection $F_A\cap V$ can be identified with a subbundle K of A and is called the **core** of F_A. For any $\xi\in F_A\cap V$ with $p_A(\xi)=a_m$, there exists $k_m\in K$ such that $\xi=\frac{d}{dt}\big|_{t=0}a_m+tb_m$. Conversely, for any $a_m\in A$ and $k_m\in K$, we have $\frac{d}{dt}\big|_{t=0}a_m+tb_m\in F_A\cap V$.

Note that $F_A\cap H_\nabla\to A$ is then spanned by the sections $\widetilde{X}$ for all $X\in\Gamma(F_M)$, $F_A\cap H_\nabla\to F_M$ by the sections $\tilde{a}|_{F_M}$ for all $a\in\Gamma(A)$, $F_A\cap V\to A$ by the sections $a^\uparrow$ for all $a\in\Gamma(K)$ and $F_A\cap\ker(p_A)\to F_M$ by $a^\dagger|_{F_M}$ for all $a\in\Gamma(K)$.

This leads to the following two propositions.

Proposition 4.2. *The subbundle $F_A\to A$ of $TA\to A$ is involutive if and only if*
1. *F_M is involutive,*
2. *$\nabla_X a\in\Gamma(K)$ for all $X\in\Gamma(F_M)$ and $a\in\Gamma(K)$,*
3. *and the induced connection $\bar\nabla:\Gamma(F_M)\times\Gamma(A/K)\to\Gamma(A/K)$ is flat.*

Proposition 4.3. *The subbundle $F_A\to F_M$ of $TA\to TM$ is a subalgebroid if and only if*
1. *$\rho(K)\subseteq F_M$,*
2. *$\nabla_a^{\rm bas}X\in\Gamma(F_M)$ for all $a\in\Gamma(A)$ and $X\in\Gamma(F_M)$,*
3. *$\nabla_a^{\rm bas}b\in\Gamma(K)$ for all $a\in\Gamma(A)$ and $b\in\Gamma(K)$,*
4. *and $R_\nabla^{\rm bas}(a,b)(X)\in\Gamma(K)$ for all $a,b\in\Gamma(A)$ and $X\in\Gamma(K)$.*

Corollary 4.4. *The subbundle F_A is involutive and a Lie algebroid over F_M if and only if for any adapted connection ∇, $\nabla_X a\in\Gamma(K)$ for all $X\in\Gamma(F_M)$ and $a\in\Gamma(K)$ and the induced connection $\bar\nabla:\Gamma(F_M)\times\Gamma(A/K)\to\Gamma(A/K)$ defines an infinitesimal ideal system $(F_M,K,\bar\nabla)$ in A.*

Since for any two connections ∇ and ∇' that are adapted to F_A, the difference $\nabla-\nabla'$ satisfies

$$(\nabla-\nabla')_{v_m}a\in K(m)$$

for all $v_m\in F_M$ and $a\in\Gamma(A)$, the induced connection $\bar\nabla$ does not depend on the choice of the adapted connection ∇ and we have the following theorem.

Theorem 4.5. *Let A be a Lie algebroid. Morphic involutive distributions on A are in one-to-one correspondence with infinitesimal ideal systems in A.*

Remark 4.6. Let (A,F_A) be a foliated algebroid and (F_M,K,∇) the corresponding ideal system in A. Let X be a linear section of F_A over $\bar X\in\Gamma(F_M)$. Then

$$\nabla_{\bar X}\bar a=0\quad\Leftrightarrow\quad D_Xa\in\Gamma(K)$$

for $a\in\Gamma(A)$.

Example 4.7. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Assume that H acts on a Lie algebroid $A\longrightarrow M$ in a free and proper manner, by Lie algebroid automorphisms. That is, for all $h\in H$, the diffeomorphism Φ_h is a Lie algebroid morphism over $\phi_h:M\to M$. Consider the vertical spaces $\mathcal{V}_A$, $\mathcal{V}_M$ defined as follows

$$\mathcal{V}_A(a)=\{\xi_A(a)\mid\xi\in\mathfrak{h}\},\qquad \mathcal{V}_M(m)=\{\xi_M(m)\mid\xi\in\mathfrak{h}\},$$

for $a\in A$ and $m\in M$. We check that $\mathcal{V}_A$ inherits a Lie algebroid structure over $\mathcal{V}_M$ making the pair $(A,\mathcal{V}_A)$ into a foliated algebroid with core zero. Choose $\xi_A(a_m)\in\mathcal{V}_A(a_m)$ for some

$\xi \in \mathfrak{g}$, then $T_{a_m}q_A\xi_A(a_m) = \xi_M(m) \in \mathcal{V}_M(m)$. Hence, if $T_{a_m}q_A\xi_A(a_m) = 0$, then $\xi = 0$ since the action is supposed to be free and we find $K = 0$. Notice that, since the action is by algebroid automorphisms, each infinitesimal generator ξ_A is in fact a morphic vector field covering ξ_M.

If $\rho(a_m) = \dot{c}(0)$, a simple computation, using $\rho \circ \Phi_{\exp(t\xi)} = T\phi_{\exp(t\xi)} \circ \rho$ for all $t \in \mathbb{R}$, yields

$$\rho_{TA}(\xi_A(a_m)) = \left.\frac{d}{ds}\right|_{s=0} \xi_M(c(s)) \in T_{\xi_M(m)}\mathcal{V}_M.$$

If $a \in \Gamma(A)$ is such that $T_m a(\xi_M(m)) \in \mathcal{V}_A(a_m)$ for some $\xi \in \mathfrak{g}$ and $m \in M$, then there exists $\eta \in \mathfrak{g}$ such that $T_m a(\xi_M(m)) = \eta_A(a(m))$. But applying $T_{a_m}q_A$ to both sides of this equality yields then $\xi_M(m) = \eta_M(m)$, which leads to $\xi = \eta$, since the action is free, and hence $T_m a(\xi_M(m)) = \xi_A(a(m))$.

Here, the induced partial $\mathcal{V}_M$-connection ∇ is defined on A by

$$[\xi_A, a^\uparrow] = (\nabla_{\xi_M} a)^\uparrow$$

for any $a \in \Gamma(A)$. If $a \in \Gamma(A)$ is ∇-parallel, then we find $[\xi_A, a^\uparrow] = 0$ for all $\xi \in \mathfrak{g}$ and hence the flows commute, which leads to

$$\Phi_{\exp(t\xi)}(b_m) + s \cdot a\left(\phi_{\exp(t\xi)}(m)\right) = \Phi_{\exp(t\xi)}(b_m) + s \cdot \Phi_{\exp(t\xi)}(a(m))$$

for all $b_m \in A$ and $s, t \in \mathbb{R}$ and hence to

$$a\left(\phi_{\exp(t\xi)}(m)\right) = \Phi_{\exp(t\xi)}(a(m)).$$

Since H is assumed to be connected, this yields $a \circ \phi_h = \Phi_h \circ a$ for all $h \in H$.

Because the action is by Lie algebroid morphisms, we find then that the Lie algebroid bracket of ∇-parallel sections $a, b \in \Gamma(A)$ is again ∇-parallel. Since the core sections of $\mathcal{V}_A$ are all trivial, this shows that the Lie bracket of $TA \to TM$ restricts to $\mathcal{V}_A \to \mathcal{V}_M$.

4.2. **The Lie algebroid of a multiplicative involutive distribution.** The following construction can be found in [28] in the more general setting of multiplicative Dirac structures.

Let F_G be a multiplicative subbundle of TG with space of units $F_M \subseteq TM$. Since $F_G \subseteq TG$ is a Lie subgroupoid, we can apply the Lie functor, leading to a Lie subalgebroid $A(F_G) \subseteq A(TG)$ over $F_M \subseteq TM$.

As we have seen in Subsection 2.2, the canonical involution $J_G : TTG \longrightarrow TTG$ restricts to an isomorphism of double vector bundles $j_G : TA \longrightarrow A(TG)$ inducing the identity map on both the side bundles and the core. Since $j_G : TA \longrightarrow A(TG)$ is an isomorphism of Lie algebroids over TM, we conclude that

$$F_A := j_G^{-1}(A(F_G)) \subseteq TA$$

is a Lie algebroid over $F_M \subseteq TM$. Since

$$\begin{tikzcd} F_G \arrow[r, "p_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & G \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ F_M \arrow[r, "p_M"'] & M \end{tikzcd}$$

is a $\mathcal{VB}$-subgroupoid of

$$\begin{tikzcd} TG \arrow[r, "p_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & G \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd} ,$$

the Lie algebroid

$$\begin{tikzcd} A(F_G) \arrow[r, "A(p_G)"] \arrow[d] & A \arrow[d] \\ F_M \arrow[r, "p_M"'] & M \end{tikzcd}$$

is a $\mathcal{VB}$-subalgebroid of

$$\begin{tikzcd} A(TG) \arrow[r, "A(p_G)"] \arrow[d] & A \arrow[d] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd}$$

([3]), and $F_A \to A$ is also a subbundle of $TA \to A$.

The main theorem of this subsection is the following.

Theorem 4.8. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid with core K. Then $(A, F_A = j_G^{-1}(A(F_G)))$ is a foliated algebroid with core K.*

Conversely, let (A, F_A) be a foliated Lie algebroid. Assume that A integrates to a source simply connected Lie groupoid $G \rightrightarrows M$. Then there is a unique multiplicative distribution F_G on G such that $F_A = j_G^{-1}(A(F_G))$.

We will use a result of [3], which states that a $\mathcal{VB}$-algebroid

$$\begin{tikzcd} E \arrow[r, "q_E^h"] \arrow[d, "q_E^v"'] & A \arrow[d, "q_A"] \\ B \arrow[r, "q_B"'] & M \end{tikzcd}$$

integrates to a $\mathcal{VB}$-groupoid

$$\begin{tikzcd} G(E) \arrow[r, "q_{G(E)}"] \arrow[d, shift left] \arrow[d, shift right] & G(A) \arrow[d, shift left] \arrow[d, shift right] \\ B \arrow[r, "q_B"'] & M \end{tikzcd} .$$

Furthermore, if $E' \hookrightarrow E$, $B' \hookrightarrow B$ is a $\mathcal{VB}$-subalgebroid with the same horizontal base A,

$$\begin{tikzcd} E' \arrow[r, "q_E^h"] \arrow[d, "q_E^v"'] & A \arrow[d, "q_A"] \\ B' \arrow[r, "q_{B'}"'] & M \end{tikzcd} ,$$

then $E' \to B'$ integrates to an embedded $\mathcal{VB}$-subgroupoid $G(E') \hookrightarrow G(E)$ over $B' \hookrightarrow B$,

$$\begin{tikzcd} G(E') \arrow[r, "q_{G(E)}"] \arrow[d, shift left] \arrow[d, shift right] & G(A) \arrow[d, shift left] \arrow[d, shift right] \\ B' \arrow[r, "q_M"'] & M \end{tikzcd} .$$

This is done in [3] using the characterization of vector bundles via homogeneous structures (see [11]).

Proof of Theorem 4.8. The first part has been shown above. Recall from Appendix A the construction of $A(F_G) \to F_M$ and $j_G(A(F_G)) = F_A \to F_M$. Since $j_G \circ \tilde{a} = a^\dagger$, we find immediately that the core sections of F_A are $j_G \circ \tilde{a}$ for all $a \in \Gamma(K)$.

Let (A, F_A) be a foliated Lie algebroid with core K. The $\mathcal{VB}$-subgroupoid of $(TG, G; TM, M)$ integrating the subalgebroid $j_G(F_A) \to F_M$ of $A(TG) \to TM$ is a multiplicative subbundle $F_G \rightrightarrows F_M$ of $TG \rightrightarrows TM$ with core K. By Theorem 3.8, F_G is involutive. □

Theorem 4.9. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid and (A, F_A) the corresponding foliated algebroid. Then the infinitesimal ideal systems defined by $(G \rightrightarrows M, F_G)$ and (A, F_A) coincide.*

Proof. Since the Lie algebroid of $F_G \rightrightarrows F_M$ is a subalgebroid of $TA \rightrightarrows TM$ over F_M, the two involutive subbundles of TM coincide. We write (F_M, K, ∇^G) for the ideal system defined by $(G \rightrightarrows M, F_G)$ and (F_M, K, ∇^A) for the ideal system defined by (A, F_A). We have to show $\nabla^G = \nabla^A$.

Choose a ∇^G-parallel section $a \in \Gamma(A)$. Then $a^r \in \mathfrak{X}(G)$ is ∇^{F_G}-parallel, or, in other words, a^r preserves F_G. For $v \in F_M$, we have hence $T\operatorname{Exp}(ta)v \in F_G$ for all $t \in \mathbb{R}$ where this makes sense, and so $\beta_A(v) \in A(F_G)$. This shows that $\beta_a|_{F_M}$ is a section of $A(F_G)$, and so that $Ta|_{F_M}$ is a section of $F_A \to F_M$ (see Appendix A). As a consequence, we find $\tilde{\nabla}_X a \in \Gamma(K)$ for any connection $\tilde{\nabla}$ adapted to F_A and any $X \in \Gamma(F_M)$, and this finally leads to $\nabla^A_X a = 0$.

We have thus shown

$$\{a \in \Gamma(A) \mid a\ \nabla^G\text{-parallel}\} \subseteq \{a \in \Gamma(A) \mid a\ \nabla^A\text{-parallel}\}.$$

Since both connections are flat, it is easy to conclude from this that they have the same sets of parallel sections. Again by the flatness of the connections, one finds then that $\nabla^G = \nabla^A$. □

As a corollary of this and Theorem 4.5, we get the following result.

Corollary 4.10. *Let $G \rightrightarrows M$ be a source-simply connected Lie groupoid with Lie algebroid $A \to M$. Then multiplicative involutive distributions on G are in one-to-one correspondence with infinitesimal ideal systems in A.*

Example 4.11. Assume that H acts on a Lie groupoid G over M by groupoid automorphisms. Assume also that the action is free and proper. Starting from the data $(A, \mathcal{V}_M, 0, \nabla)$ where ∇ is the partial $\mathcal{V}_M$-connection on A determined by

$$[\xi_G, a^r] = (\nabla_{\xi_M} a)^r$$

for all $\xi \in \mathfrak{g}$ and $a \in \Gamma(A)$, the last theorem states that we recover exactly the foliated Lie algebroid $\mathcal{V}_A \longrightarrow \mathcal{V}_M$ obtained by applying the Lie functor to the foliated groupoid $\mathcal{V}_G \rightrightarrows \mathcal{V}_M$.

Example 4.12. Assume that $\mathfrak{g}$ is a Lie algebra, i.e. a Lie algebroid over a point. In this case, the tangent Lie algebroid $T\mathfrak{g}$ is also a Lie algebroid over a point, that is, $T\mathfrak{g}$ is a Lie algebra. It is easy to see that the Lie algebra structure on $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ is the semi-direct product Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on itself. Note also that the fact that a triple $(0, \mathfrak{f}, \nabla = 0)$ is an ideal system on $\mathfrak{g}$ is equivalent to saying that $\mathfrak{f} \subseteq \mathfrak{g}$ is an ideal.

The morphic involutive distribution $F_\mathfrak{g}$ associated to the ideal system $(0, \mathfrak{f}, 0)$ is given by $F_\mathfrak{g} = \mathfrak{g} \times \mathfrak{f}$. The property that $F_\mathfrak{g} = \mathfrak{g} \times \mathfrak{f}$ is a morphic involutive distribution is equivalent to saying that $\mathfrak{g} \times \mathfrak{f}$ is a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{g}$. In particular, if G is the connected and simply connected Lie group integrating $\mathfrak{g}$, we conclude that the foliated algebroid $F_\mathfrak{g} = \mathfrak{g} \times \mathfrak{f}$ integrates to a Lie subgroup $G \times \mathfrak{f}$ of the semi-direct Lie group $G \ltimes \mathfrak{g}$ determined by the adjoint action of G on its Lie algebra $\mathfrak{g}$. Using right (or left) translations, we get a subbundle

$F_G \subseteq TG$ which is involutive and multiplicative. Thus, in the case of Lie groups and Lie algebras, this recovers the results in [27, 15, 16].

5. Examples of ideal systems

In this section, we present several natural examples of ideal systems. These examples show that the ideal systems are the object that should be considered as the right notion of ideals in Lie algebroids.

5.1. **Regular Dirac structures and the kernel of the associated presymplectic groupoids.** Let (M, D) be a Dirac manifold. Recall that $(\mathsf{D} \to M, \mathrm{pr}_{TM}, [\cdot\,,\cdot])$ is then a Lie algebroid, where $[\cdot\,,\cdot]$ is the Courant Dorfman bracket on sections of $TM \oplus T^*M$.

Assume that the characteristic distribution $F_M \subseteq TM$, defined by

$$F_M(m) = \{v_m \in T_mM \mid (v_m, 0_m) \in \mathsf{D}(m)\}$$

for all $m \in M$, is a subbundle of TM. The involutivity of F_M follows from the properties of the Dirac structure. Set $K := F_M \oplus \{0\} \subseteq \mathsf{D}$. It is easy to check that K is a subalgebroid of D. Define

$$\begin{aligned}\nabla : \Gamma(F_M) \times \Gamma(\mathsf{D}/K) &\to \Gamma(\mathsf{D}/K)\\ \nabla_X \bar{d} &= \overline{[(X,0), d]}.\end{aligned}$$

This map is easily seen to be a well-defined, flat, partial F_M-connection on D/K, and the verification of the fact that $(\mathsf{D}, F_M, K, \nabla)$ is an ideal system on the Lie algebroid $\mathsf{D} \to M$ is straightforward.

We show that if $\mathsf{D} \to M$ integrates to a presymplectic groupoid $(G \rightrightarrows M, \omega_G)$ [5, 4], then $(\mathsf{D}, F_M, K, \nabla)$ integrates to the involutive subbundle $F_G = \ker \omega_G \subseteq TG$.

The map $\rho := \mathrm{pr}_{TM} : \mathsf{D} \to TM$ is the anchor of the Dirac structure D viewed as a Lie algebroid over M, and the map $\sigma := \mathrm{pr}_{T^*M} : \mathsf{D} \to T^*M$ defines an IM-2-form on the Lie algebroid D (see [5], [4]). Note that K is the kernel of σ and F_M is the kernel of $\sigma^t : TM \to \mathsf{D}^*$. The two-form $\Lambda := \sigma^*\omega_{\mathrm{can}} \in \Omega^2(\mathsf{D})$ is morphic in the sense that

$$\begin{tikzcd} T\mathsf{D} \arrow[r, "\Lambda^\sharp"] \arrow[d] & T^*\mathsf{D} \arrow[d] \\ TM \arrow[r, "-\sigma^t"'] & \mathsf{D}^* \end{tikzcd}$$

is a Lie algebroid morphism ([4]). See, for instance [23], for the Lie algebroid structure on $T^*\mathsf{D} \to \mathsf{D}^*$. If $\mathsf{D} \to M$ integrates to a presymplectic groupoid $(G \rightrightarrows M, \omega_G)$, the Lie algebroid $T^*\mathsf{D} \to \mathsf{D}^*$ is isomorphic to the Lie algebroid of the cotangent groupoid $T^*G \to \mathsf{D}^*$ and the map $\Lambda^\sharp$ integrates via the identifications $T\mathsf{D} \simeq A(TG)$ and $T^*\mathsf{D} \simeq A(T^*G)$ to the vector bundle map $\omega_G^\sharp$, that is a Lie groupoid morphism. See [4] for more details.

We show that the morphic involutive distribution $F_{\mathsf{D}} \subseteq T\mathsf{D}$ corresponding to $(\mathsf{D}, F_M, K, \nabla)$ is equal to the kernel of $\Lambda^\sharp$.

Let n be the dimension of M and k the rank of F_M. Then D is spanned locally by frames of n parallel sections, the first k of them spanning K. If d is a parallel section of D, we have $\pounds_X d \in \Gamma(K)$ for all $X \in \Gamma(F_M)$, that is, $\pounds_X(\sigma(d)) = 0$ for all $X \in \Gamma(F_M)$. Since $\mathbf{i}_X\sigma(d) = 0$ for all $X \in \Gamma(F_M)$, this yields $\mathbf{i}_X\mathbf{d}(\sigma(d)) = 0$ for all $X \in \Gamma(F_M)$. Hence, using this type of frames, we find with formulas (4.57) and (4.58) in [4], that the kernel of $\Lambda^\sharp$ is spanned by the restriction to F_M of the linear sections defined by parallel sections of D, and by the restrictions to F_M of the core sections defined by sections of K. Hence, by construction, the distribution F_{D} is the kernel of $\Lambda^\sharp$.

Since the kernel of $\omega_G^\sharp$ is multiplicative with Lie algebroid equal to the kernel of $\Lambda^\sharp$, this yields $F_G = \ker \omega_G^\sharp$.

Note that if $F_M \subseteq TM$ is simple, then the leaf space M/F_M has a natural Poisson structure such that the projection $(M, \mathsf{D}) \to (M/F_M, \pi)$ is a forward Dirac map. Under a completeness condition and if $F_G \subseteq TG$ is also simple, we get a Lie groupoid $G/F_G \rightrightarrows M/F_M$, with a natural symplectic structure ω such that the projection $\pi_G : G \to G/F_G$ satisfies $\pi_G^*\omega = \omega_G$. It would be interesting to study the relation between the integrability of the Poisson manifold $(M/F_M, \pi)$ and the completeness conditions on F_G (see [16]) so that the quotient $(G/F_G \rightrightarrows M/F_M, \omega)$ is a symplectic groupoid.

Note finally that, under the obvious smoothness and trivial holonomy conditions, the quotient of the Lie algebroid D by the ideal system (F_M, K, ∇) is the Lie algebroid $\mathsf{D}_\pi = \operatorname{graph}(\pi^\sharp : T^*(M/F_M) \to T(M/F_M))$.

5.2. **Foliated algebroids in the sense of Vaisman.** In [34], foliated Lie algebroids are defined as follows. A foliated Lie algebroid is a Lie algebroid $A \to M$ together with a subalgebroid B of A and an involutive subbundle $F_M \subseteq TM$ such that

(1) $\rho(B) \subseteq F_M$,
(2) A is locally spanned over $C^\infty(M)$ by *B-foliated cross sections*, i.e. sections a of A such that $[a, b] \in \Gamma(B)$ for all $b \in B$.

Recall our definition of ideal system on a Lie algebroid (Definition 1.1). Since the F_M-partial connection is flat, we get by Proposition 2.3 the existence of frames of parallel sections for A. By the properties of the connection, these are *K-foliated cross sections*. Since (1) is also satisfied by hypothesis, our ideal systems are foliated Lie algebroids in the sense of Vaisman if we set $B := K$.

The object that integrates the foliated algebroid in the sense of Vaisman is the right invariant image of B, which defines a subbundle of TG that is tangent to the s-fibers and invariant under left multiplication. This is exactly the intersection of our multiplicative subbundle $F_G \subseteq TG$, integrating (F_M, K, ∇), with $T^{\mathsf{s}}G$.

5.3. **The usual notion of ideals in Lie algebroids.** An ideal I in a Lie algebroid $A \to M$ is a subbundle over M such that $[a, i] \in \Gamma(I)$ for all $i \in \Gamma(I)$ and all $a \in \Gamma(A)$. The inclusion $I \subseteq \ker(\rho)$ follows immediately and shows that this definition of an ideal is very restrictive. In the other hand, usual ideals correspond obviously to the ideal systems $(F_M = 0, K = I, \nabla = 0)$ in A. Note that in this case, the quotient Lie algebroid A/I over $M/F_M = M$ is always defined. This is a trivial class of example for the results in the next section.

5.4. **The Bott connection and reduction by simple foliations.** The second standard example of a Lie algebroid is the tangent space TM of a smooth manifold M, endowed with the usual Lie bracket of vector fields and the identity Id_{TM} as anchor. Consider an involutive subbundle $F_M \subseteq TM$ and the Bott connection

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M)$$

associated to it. Then it is straightforward to check that the triple (F_M, F_M, ∇^{F_M}) is an ideal system in TM.

This ideal system corresponds to the subbundle of TM given by the tangent lift of F_M. The foliated groupoid associated to this ideal system is $(M \times M \rightrightarrows M, F_M \times F_M \rightrightarrows F_M)$ (we assume here for simplicity that M is simply connected).

5.5. **The ideal system associated to a fibration of Lie algebroids.** Let

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "q_A"'] & A' \arrow[d, "q_{A'}"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

be a fibration of Lie algebroids, i.e. the map f is a surjective submersion and $f^!\varphi : A \to \varphi^! A'$ is a surjective vector bundle morphism over the identity on A.

Then $K := \ker(\varphi) \subseteq A$, i.e.

$$K(m) = \left\{ a_m \in A_m \mid \varphi(a_m) = 0^{A'}_{f(m)} \right\}$$

is a subalgebroid of A and $T^f M \subseteq TM$ is an involutive subbundle. The equality $Tf \circ \rho = \rho' \circ \varphi$ yields immediately $\rho(K) \subseteq F_M$.

Define a connection $\nabla^\varphi : \Gamma(T^f M) \times \Gamma(A/K) \to \Gamma(A/K)$ by setting $\nabla^\varphi_X \bar{a} = 0$ for all sections $a \in \Gamma(A)$ that are (φ, f)-related to some section $a' \in \Gamma(A')$, i.e. such that $\varphi \circ a = a' \circ f$. Then the properties of the Lie algebroid morphism (φ, f) imply that $(T^f M, K, \nabla^\varphi)$ is an ideal system in A.

By the results in the next section, we can roughly say that any ideal system can be constructed this way.

6. The leaf space of a foliated algebroid

Assume that (F_M, K, ∇) is an ideal system in A. Then there is an induced involutive subbundle $F_A \subseteq TA$ as in Corollary 4.4. We will show that if the leaf space M/F_M is a smooth manifold such that the quotient map $\pi_M : M \to M/F_M$ is a surjective submersion, and if ∇ has trivial holonomy, then there is an induced Lie algebroid structure $([q_A] : A/F_A \to M/F_M, [\rho], [\cdot\,,\cdot]_{A/F_A})$ such that the projection

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

is a Lie algebroid morphism. Furthermore, if $A \to M$ integrates to a Lie groupoid $G \rightrightarrows M$ and the completeness and regularity conditions for the leaf space $G/F_G \rightrightarrows M/F_M$ to be a Lie groupoid are satisfied (see [16]), then $A/F_A \to M/F_M$ is isomorphic to the Lie algebroid of $G/F_G \rightrightarrows M/F_M$. We will see that this reduction process is in reality a reduction by the ideal system (F_M, K, ∇) in A.

The class of $a_m \in A_m$ will be written $[a_m] \in A/F_A$, and, in the same manner, the class of $m \in M$ will be denoted by $[m] \in M/F_M$. The class of $a_m \in A_m$ in A/K will be written $\bar{a}_m$.

Proposition 6.1. *Let (F_M, K, ∇) be an ideal system in $A \to M$ and $F_A \subseteq TA$ the corresponding morphic involutive distribution as in Corollary 4.4.*

(1) *The map $\pi : A \to A/F_A$ factors as a composition*

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "\pi"] & \\ A/K \arrow[r, "\bar{\pi}"'] & A/F_A \end{tikzcd}$$

That is, we have $\pi(a_m + k_m) = \pi(a_m)$ for all $a_m \in A$ and $k_m \in K(m)$.

(2) *The equivalence relation $\sim:=\sim_{F_A}$ on A can be described as follows.*

$$a_m \sim a_n \Leftrightarrow \begin{array}{c}\textit{There exist linear sections } (X_1,\bar{X}_1),\ldots,(X_r,\bar{X}_r) \textit{ of } F_A \to A\\ \textit{with flows } \phi^1,\ldots,\phi^r \textit{ such that}\\ a_m \in \phi^1_{t_1}\circ\ldots\circ\phi^r_{t_r}(a_n)+K(m)\\ \textit{for some } t_1,\ldots,t_r\in\mathbb{R}.\end{array} \tag{6.7}$$

(3) *The map q_A induces a map $[q_A]: A/F_A \to M/F_M$ such that*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

commutes.

(4) *Let $a \in \Gamma(A)$ be such that $\bar{a} \in \Gamma(A/K)$ is ∇-parallel. Let $U := \mathrm{Dom}(a) \subseteq M$. Then there is an induced map $[a]: \pi_M(U) \to A/F_A$ such that*

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "\pi_M"'] & A \arrow[d, "\pi"] \\ \pi_M(U) \arrow[r, "{[a]}"'] & A/F_A \end{tikzcd}$$

commutes. The map $[a]$ is a section of $[q_A]$:

$$[q_A]\circ[a] = \mathrm{Id}_{\pi_M(U)}.$$

Proof. (1) Recall that all the core sections $k^\uparrow \in \mathfrak{X}(A)$ with $k \in \Gamma(K)$ are sections of F_A. Choose $a_m \in A$ and $k_m \in K(m)$. Then there exists a section $k \in \Gamma(K)$ with $k(m) = k_m$. The flow $\phi^{k^\uparrow}$ of $k^\uparrow$ is given by $\phi_t^{k^\uparrow}(a) = a + tk(q_A(a))$ for all $a \in A$ and $t \in \mathbb{R}$. Hence, we have $a_m \sim a_m + tk(m) = a_m + tk_m$ for all $t \in \mathbb{R}$, and in particular $a_m \sim a_m + k_m$. The map $\bar{\pi}: A/K \to A/F_A$, $\bar{\pi}(\bar{a}_m) = [a_m]$ is hence well-defined and the diagram commutes.

(2) Since the family of linear sections of F_A and the family of core sections of F_A span together F_A, its leaves are the accessible sets of these two families of vector fields ([29, 30, 31], see [26] for a review of these results). Hence, two points a_m and a_n in A are in the same leaf of F_A if they can be joined by finitely many curves along flow lines of core sections $k^\uparrow$ for $k \in \Gamma(K)$ and linear vector fields $X \in \Gamma(F_A)$. By the involutivity of F_A, we have $D_X k \in \Gamma(K)$ for all $k \in \Gamma(K)$ and linear vector fields $X \in \Gamma(F_A)$. Hence, by Lemma B.3, we get that K is invariant under the flow lines of linear vector fields with values in F_A. That is, using the fact that ϕ_t^X is a vector bundle morphism, we have

$$\left(\phi_t^X \circ \phi_s^{k^\uparrow}\right)(a_m) \in \phi_t^X(a_m + K(m)) = \phi_t^X(a_m) + K\left(\phi_t^{\bar{X}}(m)\right)$$

for all $a_m \in A$, $t \in \mathbb{R}$ where this makes sense and $s \in \mathbb{R}$. Since

$$\phi_s^{k^\uparrow} \circ \phi_t^X(a_m) \in \phi_t^X(a_m) + K\left(\phi_t^{\bar{X}}(m)\right),$$

the proof is finished.

(3) Assume that $a_m \sim a_n$ for some elements $a_m, a_n \in A$. Then there exists, without loss of generality, one linear vector field $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$, an element $k_m \in K(m)$ and $t \in \mathbb{R}$ such that $a_m = \phi_t^X(a_n) + k_m$. We have then immediately

$$m = q_A(a_m) = \left(q_A \circ \phi_t^X\right)(a_n) = \phi_t^{\bar{X}}(n),$$

which shows $m \sim_{F_M} n$.

(4) Assume first that $\bar{a}$ does not vanish on its domain of definition. Since $\bar{a}$ is ∇-parallel, we have $D_X a \in \Gamma(K)$ for all linear vector fields $X \in \Gamma(F_A)$, $X \sim_{q_A} \bar{X} \in \Gamma(F_M)$. By Lemma B.3, this yields

$$\phi_t^X(a(m)) \in a\left(\phi_t^{\bar{X}}(m)\right) + K\left(\phi_t^{\bar{X}}(m)\right) \tag{6.8}$$

for all $t \in \mathbb{R}$ where this makes sense and consequently

$$\pi(a(m)) = \pi\left(a\left(\phi_t^{\bar{X}}(m)\right)\right).$$

Since F_M is spanned by projections $\bar{X}$ of linear vector fields $X \in \Gamma(F_A)$, this shows that a projects to the map $[a]$ that is defined by the diagram.

In general, we have $a = \sum_{i=1}^n f_i a_i$ on an open set U with non-vanishing ∇-parallel sections $a_1, \ldots, a_n$ of A such that $a_1, \ldots, a_r \in \Gamma(K)$ for some $r \leq n$ and functions $f_1, \ldots, f_n \in C^\infty(U)$ such that $f_{r+1}, \ldots, f_n$ are F_M-invariant. This yields using (6.8):

$$\begin{aligned}\phi_t^X(a(m)) &= \phi_t^X\left(\sum_{i=1}^n f_i(m) a_i(m)\right) \\ &\in \sum_{i=r+1}^n f_i\left(\phi_t^{\bar{X}}(m)\right)\phi_t^X(a_i(m)) + K\left(\phi_t^{\bar{X}}(m)\right) \\ &= a\left(\phi_t^{\bar{X}}(m)\right) + K\left(\phi_t^{\bar{X}}(m)\right)\end{aligned} \tag{6.9}$$

and we get the statement in the same manner as above.

We have

$$([q_A] \circ [a]) \circ \pi_M = [q_A] \circ \pi \circ a = \pi_M \circ q_A \circ a = \pi_M \circ \mathrm{Id}_M = \pi_M,$$

which shows the last claim since π_M is surjective.

□

Corollary 6.2. *Let (F_M, K, ∇) be an ideal system in a Lie algebroid $A \to M$. Choose $\bar{a}_m$ and $\bar{a}_n$ in A/K.*

1. *$\bar{\pi}(\bar{a}_m) = \bar{\pi}(\bar{a}_n)$ if and only if $\overline{a_m} \in A/K$ is the ∇-parallel transport of $\overline{a_n}$ over a piecewise smooth path along the foliation defined by F_M on M.*
2. *If ∇ has trivial holonomy, then $\bar{\pi}(\overline{a_m}) = \bar{\pi}(\overline{a'_m})$ if and only if $\overline{a_m} = \overline{a'_m}$.*

Remark 6.3. We find as a consequence of the last corollary that, if ∇ has trivial holonomy, then the map $\bar{\pi}$ is bijective in every fiber.

Proof. (1) Assume first that $\bar{\pi}(\bar{a}_m) = \bar{\pi}(\bar{a}_n)$. Then there exists without loss of generality one linear vector field $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$ and $t \in \mathbb{R}$ such that $\bar{a}_m = \overline{\phi_t^X(a_n)}$.

Consider the curve $a : [0, t] \to A$ over $c := \phi_\cdot^{\bar{X}}(n)$ defined by

$$a(\tau) = \phi_\tau^X(a_n)$$

for $\tau \in [0, t]$. For each $\tau \in [0, t]$, we find a parallel section a^τ of A and $\varepsilon_\tau > 0$ such that $\phi_\cdot^X(a_n)$ is defined on $(-\varepsilon_\tau, \tau + \varepsilon_\tau)$, $\phi_\cdot^X(\tau - \varepsilon_\tau, \tau + \varepsilon_\tau) \subseteq \mathrm{Dom}(a^\tau)$ and $\overline{a^\tau(c(\tau))} = \overline{a(\tau)}$. Since ϕ_s^X preserves K for all s where defined, we get then

$$\begin{aligned}\overline{a^\tau(c(s))} = \overline{a^\tau(\phi_s^{\bar{X}}(n))} &= \overline{a^\tau(\phi_{s-\tau}^{\bar{X}}\phi_\tau^{\bar{X}}(n))} \overset{(6.9)}{=} \overline{\phi_{s-\tau}^X a^\tau(c(\tau))} \\ &= \overline{\phi_{s-\tau}^X a(\tau)} = \overline{\phi_{s-\tau}^X \phi_\tau^X(a_n)} = \overline{\phi_s^X(a_n)} = \overline{a(s)}\end{aligned}$$

for $s \in (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$. This yields $\nabla_{\bar{X}(c(\tau))}\bar{a} = 0$ for all τ.

Conversely, assume that $\overline{a_m} \in A/K$ is the ∇-parallel transport of $\overline{a_n}$ over a piecewise smooth path along a path lying in the leaf of F_M through n. Without loss of generality, this path is a segment of a flow curve of a vector field $\bar{X} \in \Gamma(F_M)$,

$m=\phi_t^{\bar{X}}(n)$ for some $t\in\mathbb{R}$, and there exists a ∇-parallel section a of A such that $\overline{a(m)}=\overline{a_m}$ and $\overline{a(n)}=\overline{a_n}$. Choose any linear vector field $X\in\Gamma(F_A)$ over $\bar{X}$. Then we get as in the proof of Proposition 6.1, 4) that

$$\overline{a(m)}=\overline{a\left(\phi_t^{\bar{X}}(n)\right)}=\overline{\phi_t^X(a(n))}$$

and hence $a_m\sim a_n$ by Proposition 6.1, 2).

(2) This is immediate since here, parallel transport does not depend on the path along the leaf of F_M through m.

□

Using Proposition 6.1, we show that if M/F_M is a smooth manifold and the projection is a submersion, and if ∇ has no holonomy, then the quotient A/F_A is a vector bundle over M/F_M.

Proposition 6.4. *Let (F_M,K,∇) be an ideal system in a Lie algebroid $A\to M$. Assume that M/F_M is a smooth manifold such that the projection is a submersion, and that the connection ∇ has no holonomy. The quotient space A/F_A inherits a vector bundle structure over M/F_M such that the projection (π,π_M) is a vector bundle morphism.*

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/F_A \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle [q_A]} \\ M & \xrightarrow[\pi_M]{} & M/F_M \end{array}$$

Proof. Since the flows of linear vector fields are vector bundle morphisms over the flows of their projection in TM, it is easy to see that for each $m\in M$, $(A/F_A)_{[m]}=[q_A]^{-1}[m]$ inherits the structure of a vector space.

Choose a local frame for A/K of ∇-parallel sections $\bar{a}_1,\dots,\bar{a}_k$ defined on a foliated chart domain $U\subseteq M$ for F_M, $k=\operatorname{rank}(A/K)$. Write $q_{A/K}:A/K\to M$ for the vector bundle projection and consider the local trivialization of A/K:

$$\begin{array}{cccc} \Phi: & q_{A/K}^{-1}(U) & \to & U\times\mathbb{R}^k \\ & \overline{b_m} & \mapsto & (m,\xi_1(b_m),\dots,\xi_k(b_m)), \end{array}$$

where $b_m=\sum_{i=1}^k\xi_i(b_m)a_i(m)+c_m$ with some $c_m\in K(m)$.

By Corollary 6.2, we find that for $\bar{b}_m,\bar{b}_n\in q_{A/K}^{-1}(U)$, the equality

$$\bar{\pi}\left(\overline{b_m}\right)=\bar{\pi}\left(\overline{b_n}\right)$$

implies

$$\xi_i(b_m)=\xi_i(b_n)\text{ for }i=1,\dots,k,$$

since $\bar{b}_m$ is the parallel transport of $\bar{b}_n$ along any path in the leaf of F_M through m and n, and so in particular along a path in U. That is, the map Φ factors to a well-defined map

$$[\Phi]:[q_A]^{-1}(\bar{U})\to\bar{U}\times\mathbb{R}^k$$

such that

$$[\Phi]\circ\bar{\pi}=(\pi_M\times\mathrm{Id}_{\mathbb{R}^k})\circ\Phi.$$

It is easy to see that $(A/F_A)_{[m]}\simeq(A/K)_m$ and so that $[\Phi]$ is an isomorphism in every fiber.

The map $[\Phi]$ is the projection to A/F_A of the "well chosen" local trivialization $q_{A/K}\times\xi_1\times\dots\times\xi_k$ of A/K. Since by Proposition 2.3, we can cover A/K by this type of ∇-invariant trivializations, we find that we can construct trivializations for A/F_A, which is hence shown to be a vector bundle over $\bar{M}$. □

Remark 6.5. Corollary 6.2 implies that the quotient space $\bar{\pi} : A/K \to A/F_A$ is the quotient by the equivalence relation given by parallel transport. Constructions like this were made in [35], see also [18]. This idea will be used (implicitly) in the proofs of the following statements.

Note that this shows also that the data (A, F_M, K, ∇) is an infinitesimal version of the ideal systems as in [23], and the methods of construction of both quotient algebroids are similar.

Example 6.6. In the situation of Example 5.4, if the foliation defined by F_M on M is simple, i.e. if the leaf space M/F_M is a smooth manifold, then the reduced algebroid $TM/(TM, F_M, F_M, \nabla^{F_M}) \to M/F_M$ is isomorphic to the tangent space $T(M/F_M) \to M/F_M$.

To see this, note that a vector field $X \in \mathfrak{X}(M)$ is ∇^{F_M}-parallel if and only if $[X, \Gamma(F_M)] \subseteq \Gamma(F_M)$. But this implies that X is pr-related to a vector field $\bar{X} \in \Gamma(M/F_M)$, where $\mathrm{pr} : M \to M/F_M$ is the projection (Lemma 2.4).

If $v_m \in TM$ is $(TM, F_M, F_M, \nabla^{F_M})$-equivalent to $w_n \in TM$, then there exists without loss of generality one ∇-parallel vector field $X \in \mathfrak{X}(M)$ such that $X(m) = v_m$ and $X(n) = w_n$. This shows that the map

$$\Phi : TM/(TM, F_M, F_M, \nabla^{F_M}) \to T(M/F_M), \qquad [v] \to T\mathrm{pr}(v)$$

is a well-defined surjective vector bundle morphism over the identity. Since

$$\mathrm{rank}(TM/(TM, F_M, F_M, \nabla^{F_M})) = \mathrm{rank}(T(M/F_M)),$$

it is an isomorphism.

It will be easy to see from the following constructions that the Lie algebroid structures coincide.

Note that, by construction, we have the following vector bundle morphism

$$\begin{array}{ccc} A/K & \xrightarrow{\bar{\pi}} & A/F_A \\ {\scriptstyle q_{A/K}}\downarrow & & \downarrow{\scriptstyle [q_A]} \\ M & \xrightarrow[\pi_M]{} & M/F_M \end{array} , \qquad \bar{a}_m \mapsto [a_m]$$

which is an isomorphism in every fiber, and we find that for each local section α of A/F_A defined on $\bar{U} \subseteq \bar{M}$, there exists a ∇-parallel section a of A defined on $\pi_M^{-1}(\bar{U})$ such that $\pi \circ a = \alpha \circ \pi_M$, i.e. $\alpha = [a]$.

Proposition 6.7. *Let (F_M, K, ∇) be an ideal system in A and assume that the quotient space $\bar{M} = M/F_M$ is a smooth manifold and ∇ has trivial holonomy. Then there is an induced map $[\rho] : A/F_A \to T\bar{M}$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TM \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle T\pi_M} \\ A/F_A & \xrightarrow[{[\rho]}]{} & T\bar{M} \end{array}$$

commutes.

Remark 6.8. If $a \in \Gamma(A)$ is ∇-parallel, then $\rho(a) \in \mathfrak{X}(M)$ is ∇^{F_M}-parallel and π_M-related to $[\rho][a] \in \mathfrak{X}(\bar{M})$.

Proof. Define $[\rho] : A/F_A \to T(M/F_M)$ by

$$[\rho]([a_m]) = T_m\pi_M(\rho(a_m)) \in T_{[m]}(M/F_M) \simeq T_mM/F_M(m).$$

To see that $[\rho]$ is well-defined, recall first that $\rho(K) \subseteq F_M$. If $[a_m] = [a_n]$, then $a_m = k_m + \phi_t^X(a_n)$ for some linear section $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$, $t \in \mathbb{R}$ and $k_m \in K(m)$.

As in the proof of Corollary 6.2 , consider the curve $a : [0,t] \to A$ over $c := \phi_{\cdot}^{\bar{X}}(n)$ defined by

$$a(\tau) = \phi_\tau^X(a_n)$$

for $\tau \in [0,t]$. Then $\nabla_{\bar{X}(c(\tau))}\bar{a} = 0$ for all τ. For each $\tau \in [0,t]$, we find $\varepsilon_\tau > 0$ and a parallel section a^τ of A such that $\overline{a^\tau(c(s))} = \overline{a(s)}$ for $s \in [\tau-\varepsilon_\tau, \tau+\varepsilon_\tau]$. Then, $\rho \circ a^\tau$ is ∇^{F_M}-parallel, and we get by Lemma 2.4 that $(\rho \circ a^\tau) \sim_{\pi_M} Y^\tau$ for some $Y^\tau \in \mathfrak{X}(\bar{M})$.

Since $[c(\tau-\varepsilon_\tau)] = [c(s)]$ for all $s \in [\tau-\varepsilon_\tau, \tau+\varepsilon_\tau]$, we have then

$$\begin{aligned} T_{c(\tau-\varepsilon_\tau)}\pi_M(\rho(a(\tau-\varepsilon_\tau))) &= T_{c(\tau-\varepsilon_\tau)}\pi_M(\rho(a^\tau(c(\tau-\varepsilon_\tau)))) \\ &= Y^\tau([c(\tau-\varepsilon_\tau)]) = Y^\tau([c(s)]) = T_{c(s)}\pi_M(\rho(a(s))) \end{aligned}$$

for all $s \in (\tau-\varepsilon_\tau, \tau+\varepsilon_\tau)$. Since $[0,t]$ is covered by (finitely many) intervals like this, we get $T_m\pi_M(\rho(a_m)) = T_m\pi_M(\rho(a(0))) = T_n\pi_M(\rho(a_n))$, which shows that $[\rho]$ is well-defined. □

Now we will define a Lie bracket on the space of sections of A/F_A. For $\alpha, \beta \in \Gamma(A/F_A)$, choose ∇-parallel sections $a, b \in \Gamma(A)$ such that $\alpha \circ \pi_M = \pi \circ a$ and $\beta \circ \pi_M = \pi \circ b$. Then $[a,b]$ is ∇-parallel by the properties of ∇ and we can define

$$\Big[\alpha, \beta\Big]_{A/F_A} \in \Gamma(A/F_A)$$

by

$$\Big[\alpha, \beta\Big]_{A/F_A} = [\,[a,b]\,]$$

or in other words

$$\Big[[a],[b]\Big]_{A/F_A} \circ \pi_M = \pi \circ [a,b]$$

for all ∇-parallel sections $a, b \in \Gamma(A)$. By the properties of ∇, this definition does not depend on the choice of the ∇-parallel sections (which can be made up to sections of K) and by definition and with Remark 6.8, and Example 6.6 we get the following result.

Proposition 6.9. *Let (F_M, K, ∇) be an ideal system in A and assume that the quotient space $\bar{M} = M/F_M$ is a smooth manifold and ∇ has trivial holonomy. Then for all ∇-parallel sections $a, b \in \Gamma(A)$, we have*

$$[\rho]\left(\Big[[a],[b]\Big]_{A/F_A}\right) = \Big[[\rho][a], [\rho][b]\Big]_{T\bar{M}},$$

where the bracket on the right-hand side is the Lie bracket on vector fields on $\bar{M}$, and the bracket on the left-hand side is defined as above.

We can now complete the proof of the following theorem.

Theorem 6.10. *Let (F_M, K, ∇) be an ideal system in a Lie algebroid A. Assume that $\bar{M} = M/F_M$ is a smooth manifold and that ∇ has trivial holonomy. Then the triple $(A/F_A = (A/K)/\nabla, [\rho], [\cdot\,,\cdot]_{A/F_A})$ is a Lie algebroid over $\bar{M}$ such that the projection (π, π_M) is a Lie algebroid morphism.*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

Proof. The Jacobi identity follows immediately from the properties of the Lie bracket $[\cdot\,,\cdot]$ on $\Gamma_M(A)$ and the definition of $[\cdot\,,\cdot]_{A/F_A}$. For the Leibniz identity choose $[a], [b] \in \Gamma(A/F_A)$ corresponding to ∇-parallel sections $a, b \in \Gamma_M(A)$, and $f \in C^\infty(\bar{M})$. We have then $\pi_M^* f \in$

$C^\infty(M)^{F_M}$ and $f\cdot[b]$ corresponds to the ∇-parallel section $(\pi_M^*f)\cdot b$ of A (see Lemma 2.4). We have hence

$$\begin{aligned}[][a], f\cdot[b]]_{A/F_A}\circ\pi_M &= \pi\circ[a,(\pi_M^*f)\cdot b]\\ &= \pi\circ(\pi_M^*f\cdot[a,b]+\rho(a)(\pi_M^*f)\cdot b)\\ &= \pi\circ(\pi_M^*f\cdot[a,b]+\pi_M^*([\rho][a](f))\cdot b)\\ &= \big(f\cdot[[a],[b]]_{A/F_A}+[\rho][a](f)\cdot[b]\big)\circ\pi_M,\end{aligned}$$

where we have used Proposition 6.7 in the third equality.

The fact that (π,π_M) is compatible with the Lie algebroid brackets is immediate by construction and the compatibility of the anchor maps is given by the definition of $[\rho]$. □

Example 6.11.
(1) In the situation of Example 5.1, assume that the leaf space of the foliation defined by F_M is a smooth manifold. Then the reduced Lie algebroid constructed as above is the graph of the Poisson structure that is induced by the Dirac structure D on M/F_M.
(2) In the case of an ideal in the usual sense as in Example 5.3, the reduced algebroid is just the induced structure on $A/I\to M$.
(3) As already mentionned, the reduced algebroid in Example 5.4 is the tangent space of the leaf space of the foliation defined by F_M.
(4) In the case of the kernel of a fibration as in Example 5.5, the reduced algebroid is the Lie algebroid structure on $A'\to M'$.
(5) Assume that $F_M=TM$ (this is the special case of ideal systems studied in [7]). Then the quotient Lie algebroid is a Lie algebra.

Assume now that $F_G\subseteq TG$ is multiplicative and involutive on $G\rightrightarrows M$ such that the leaf space G/F_G is a Lie groupoid over the leaf space M/F_M (there are topological conditions for this to be true, see [16]). The multiplicative involutive distribution F_G determines an ideal system (F_M,K,∇) in the Lie algebroid A of $G\rightrightarrows M$ and, under the trivial holonomy condition on ∇, a Lie algebroid $(A/F_A,[\rho],[\cdot,\cdot]_{A/F_A})$ as in the preceding theorem. We conclude this subsection with the comparison of this Lie algebroid with the Lie algebroid of the quotient groupoid $G/F_G\rightrightarrows M/F_M$.

Theorem 6.12. *Let (F_M,K,∇) be an ideal system in A. Assume that A integrates to a Lie groupoid $G\rightrightarrows M$, and F_A to a multiplicative involutive distribution F_G on G. If G/F_G and M/F_M are smooth manifolds, ∇ has trivial holonomy and F_G is such that $G/F_G\rightrightarrows M/F_M$ is a Lie groupoid, then we have*

$$A(G/F_G)=A/F_A,$$

where A/F_A is equipped with the Lie algebroid structure in the previous theorem.

Remark 6.13. It would be interesting to study the relation between the trivial holonomy property of ∇ and the condition of F_G for $G/F_G\rightrightarrows M/F_M$ to be a Lie groupoid.

Proof of Theorem 6.12. Let $\pi_G: G\to G/F_G$ be the projection, and $[\mathsf{s}],[\mathsf{t}]$ the source and target maps of $G/F_G\rightrightarrows M/F_M$. Recall from Theorem 3.6 that a section $a\in\Gamma(A)$ is ∇-parallel if and only if $[a^r,\Gamma(F_G)]\subseteq\Gamma(F_G)$ and the vector field a^r is then π_G-related to a vector field $\overline{a^r}\in\mathfrak{X}(G/F_G)$. We have

$$T[\mathsf{s}]\circ\overline{a^r}\circ\pi_G=T[\mathsf{s}]\circ T\pi_G\circ a^r=T\pi_G(T\mathsf{s}\circ a^r)=0,$$

which shows that $\overline{a^r}$ is tangent to the $[\mathsf{s}]$-fibers. By Lemma 3.18 in [16], we get

$$\overline{a^r}([g])=T_g\pi_G(a^r(g))=T_g\pi_G(a(\mathsf{t}(g))\star 0_g)=T_{\mathsf{t}(g)}\pi_G(a(\mathsf{t}(g)))\star 0_{[g]}=\overline{a^r}([\mathsf{t}][g])\star 0_{[g]},$$

which shows that $\overline{a^r}=\tilde a^r$ for $\tilde a:=\overline{a^r}|_{M/F_M}\in\Gamma(A(G/F_G))$.

Since (π_G, π_M) is a Lie groupoid morphism

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] & G/F_G \arrow[d, shift right, "{[\mathsf{t}]}"'] \arrow[d, shift left, "{[\mathsf{s}]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

the map $A(\pi_G) = T\pi_G|_A$

$$\begin{tikzcd} A \arrow[r, "A(\pi_G)"] \arrow[d] & A(G/F_G) \arrow[d] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

is a Lie algebroid morphism and $K = \ker(A(\pi_G))$. For any ∇-parallel section $a \in \Gamma(A)$ we have $a^r \sim_{\pi_G} \tilde{a}^r$ and hence

$$A(\pi_G) \circ a = T\pi_G a = \tilde{a} \circ \pi_M. \tag{6.10}$$

Define the map

$$\begin{tikzcd} A/F_A \arrow[r, "\Psi"] \arrow[d] & A(G/F_G) \arrow[d] \\ \bar{M} \arrow[r, "\mathrm{Id}_M"'] & M/F_M \end{tikzcd}$$

by

$$\Psi([a_m]) = A(\pi_G)(a_m)$$

for all $a_m \in A$. To see that this does not depend on the representative, use $K = \ker(A(\pi_G))$ and recall that $a_m \sim a_n$ if and only if $\bar{a}_m$ is the ∇-parallel transport of $\bar{a}_n$ along a path lying in the leaf through m of F_M (Corollary 6.2). Without loss of generality, there exists a ∇-parallel section $a \in \Gamma(A)$ such that $a(m) = a_m$ and $a(n) = a_n + k_n$ for some $k_n \in K(n)$. Then, using (6.10), we get

$$\begin{aligned} \Psi([a_m]) &= A(\pi_G)(a_m) = \left(A(\pi_G) \circ a\right)(m) = \left(\tilde{a} \circ \pi_M\right)(m) \\ &= \left(\tilde{a} \circ \pi_M\right)(n) = A(\pi_G)(a_n) = \Psi([a_n]). \end{aligned}$$

Hence, Ψ is a well-defined vector bundle morphism over the identity on M/F_M. Furthermore, the considerations above show that for any ∇-parallel section a of A and corresponding section $[a]$ of A/F_A, we get

$$\Psi \circ [a] = \tilde{a}.$$

The compatibility of the Lie algebroid brackets and anchors is then immediate by the construction of A/F_A, and the fact that $A(\pi_G)$ is a Lie algebroid morphism. □

Example 6.14. In the situation of Example 5.4 with M simply connected, the foliated Lie groupoid integrating the ideal system was $(M \times M \rightrightarrows M, F_M \times F_M)$. It is easy to check that the leaf space of the foliation defined by $F_M \times F_M$ is the groupoid $M/F_M \times M/F_M \rightrightarrows M/F_M$ (see also [16]), hence a Lie groupoid if M/F_M is a smooth manifold. As we have seen above, the reduced Lie algebroid $TM \to M$ by the ideal system is equal to $T(M/F_M)$. This is the Lie alegbroid of $M/F_M \times M/F_M \rightrightarrows M/F_M$.

Appendix A. The Lie algebroid of the tangent groupoid

If $G \rightrightarrows M$ is a Lie groupoid with Lie algebroid A, then we can consider the Lie algebroid $q_{A(TG)} : A(TG) \to TM$ of the tangent Lie groupoid $TG \rightrightarrows TM$. Since the projection $p_G : TG \to G$ is a Lie groupoid morphism, we have a Lie algebroid morphism $A(p_G) : A(TG) \to A$ over $p_M : TM \to M$:

$$
\begin{tikzcd}
A(TG) \arrow[r, "A(p_G)"] \arrow[d, "q_{A(TG)}"'] & A \arrow[d, "q_A"] \\
TM \arrow[r, "p_M"'] & M
\end{tikzcd}
\tag{1.11}
$$

Let a be a section of A, choose $v \in TM$ and consider the curve $\gamma : (-\varepsilon, \varepsilon) \to TG$ defined by

$$\gamma(t) = T\,\mathrm{Exp}(ta)v$$

for ε small enough. Then we have $\gamma(0) = v$ and $T\mathsf{s}(\gamma(t)) = v$ for all $t \in (-\varepsilon, \varepsilon)$. Hence, $\dot\gamma(0) \in A_v(TG)$ and we can define a **linear** section $\beta_a : TM \to A(TG)$ by

$$\beta_a(v) = \left.\frac{d}{dt}\right|_{t=0} T\,\mathrm{Exp}(ta)v \tag{1.12}$$

for all $v \in TM$. It is easy to check that $\beta_a^r \in \mathfrak{X}(TG)^r$ is the complete lift of a^r (see [23]). In particular the flow of β_a^r is $TL_{\mathrm{Exp}(\cdot a)}$, and (β_a, a) is a morphism of vector bundles.

In the same manner, we can consider $v_m \in TM$, $a \in A_m$ and the curve $\gamma : \mathbb{R} \to TG$ defined by

$$\gamma(t) = v + ta,$$

where TM and A are seen as subsets of TG, $T_MG = TM \oplus A$. We have again $\gamma(0) = v$ and $T\mathsf{s}(\gamma(t)) = v$ for all t, which yields $\dot\gamma(0) \in A_v(TG)$. Given $a \in \Gamma_M(A)$, we define a **core** section $\tilde a$ of $A(TG)$ by

$$\tilde a(v) = \left.\frac{d}{dt}\right|_{t=0} v + ta(p_M(v)) \tag{1.13}$$

for all $v \in TM$. We have for $v_g \in T_gG$ with $T_g\mathsf{t}(v_g) = v_m$:

$$\tilde a^r(v_g) = \tilde a(v_m) \star 0_{v_g} = \left.\frac{d}{dt}\right|_{t=0} v_g + ta^r(g).$$

The vector bundle $A(TG)$ is spanned by the two types of sections β_a and $\tilde a$, for $a \in \Gamma_M(A)$, and, using the flows of β_a^r and $\tilde b^r \in \mathfrak{X}^r(TG)$, it is easy to check that the equalities

$$[\beta_a, \beta_b]_{A(TG)} = \beta_{[a,b]}, \quad \left[\beta_a, \tilde b\right]_{A(TG)} = \widetilde{[a,b]}, \quad \left[\tilde a, \tilde b\right]_{A(TG)} = 0$$

hold for all $a, b \in \Gamma_M(A)$.

There exists a natural injective bundle map

$$\iota_A : A \longrightarrow TG \tag{1.14}$$

over $\epsilon : M \to G$. The canonical involution $J_G : TTG \longrightarrow TTG$ restricts to an isomorphism of Lie algebroids $j_G : TA \longrightarrow A(TG)$. More precisely, there exists a commutative diagram

$$
\begin{tikzcd}
TA \arrow[r, "j_G"] \arrow[d, "T\iota_A"'] & A(TG) \arrow[d, "\iota_{A(TG)}"] \\
TTG \arrow[r, "J_G"'] & TTG
\end{tikzcd}
\tag{1.15}
$$

We check the following identities:

(1) $j_G \circ Ta = \beta_a$ and
(2) $j_G \circ a^\dagger = \tilde{a}$,

where $a^\dagger$ is defined as in (2.3). First, we have for $v_m = \dot{c}(0) \in TM$:

$$\begin{aligned} j_G(T_m a v_m) =& j_G\left(\left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \operatorname{Exp}(sa)c(t)\right) \\ =& \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(sa)c(t) = \beta_a(v_m). \end{aligned}$$

In the same manner, we compute

$$\begin{aligned} j_G\left(T_m 0^A v_m + \left.\frac{d}{ds}\right|_{s=0} s a_m\right) =& j_G\left(\left.\frac{d}{ds}\right|_{s=0} sa(c(s))\right) = j_G\left(\left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(tsa)c(s)\right) \\ =& \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \operatorname{Exp}(tsa)c(s) = \left.\frac{d}{dt}\right|_{t=0} v_m + ta(m), \end{aligned}$$

which proves the second equality.

The identity

$$\rho_{A(TG)} \circ j_G = J_M \circ T\rho_A = \rho_{TA}$$

is verified easily on these linear and core sections. This shows that the Lie algebroid $A(TG) \to TM$ of the tangent groupoid is canonically isomorphic to the tangent Lie algebroid $TA \to TM$ of A.

Appendix B. Invariance of bundles under flows

We prove here a result that is standard, but the proof of which is difficult to find in the literature.

Theorem B.1. *Let M be a smooth manifold and E be a subbundle of the direct sum vector bundle $\mathbb{T}M := TM \oplus T^*M$. Let $Z \in \mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t. If*

$$\pounds_Z e \in \Gamma(E) \quad \textit{for all} \quad e \in \Gamma(E),$$

then

$$\phi_t^* e \in \Gamma(E) \quad \textit{for all} \quad e \in \Gamma(E) \quad \textit{and} \quad t \in \mathbb{R} \textit{ where this makes sense.}$$

Corollary B.2. *Let F be a subbundle of the tangent bundle TM of a smooth manifold M. Let $Z \in \mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t. If*

$$[Z, \Gamma(F)] \subseteq \Gamma(F),$$

then

$$T_m\phi_t F(m) = F(\phi_t(m))$$

for all $m \in M$ and t where this makes sense.

Proof. Choose $X \in \Gamma(F)$ and $m \in M$. Then, by Theorem B.1, we have $T\phi_t \circ X \circ \phi_{-t} = \phi_{-t}^* X \in \Gamma(F)$ for all t where this makes sense, and hence:

$$T_m\phi_t X(m) = \left(\phi_{-t}^* X\right)(\phi_t(m)) \in F(\phi_t(m)).$$

□

Proof of Theorem B.1. The subbundle E of $\mathbb{T}M$ is an embedded submanifold of $\mathbb{T}M$. For each section σ of $\mathbb{T}M$, the smooth function $l_\sigma : \mathbb{T}M \to \mathbb{R}$ is defined by

$$l_\sigma(v,\alpha) = \langle \sigma(p(v,\alpha)), (v,\alpha)\rangle$$

for all $(v,\alpha) \in \mathbb{T}M$, where $p : \mathbb{T}M \to M$ is the projection. For all $e \in E$, the tangent space T_eE of the submanifold E of $\mathbb{T}M$ is equal to

$$\ker\left\{\mathbf{d}_e l_\sigma \mid \sigma \in \Gamma\left(E^\perp\right)\right\},$$

where $E^\perp$ is the orthogonal space to E relative to the canonical symmetric fiberwise pairing on $TM \oplus T^*M$:

$$\langle (v_m,\alpha_m),(w_m,\beta_m)\rangle = \alpha_m(w_m) + \beta_m(v_m)$$

for all $v_m, w_m \in T_mM$, $\alpha_m, \beta_m \in T_m^*M$, $m \in M$.

Consider the complete lift $\tilde{Z}$ to $\mathbb{T}M$ of Z, i.e. the vector field $\tilde{Z} \in \mathfrak{X}(\mathbb{T}M)$ defined by

$$\tilde{Z}(l_\sigma) = l_{\pounds_Z\sigma} \qquad \text{and} \qquad \tilde{Z}(p^*f) = p^*(Z(f))$$

for all $\sigma \in \Gamma(\mathbb{T}M)$ and $f \in C^\infty(M)$ (see [23]).

Choose $e \in E$ and $\sigma \in \Gamma\left(E^\perp\right)$. Then we have $\pounds_Z\sigma \in \Gamma\left(E^\perp\right)$ since for all $\tau \in \Gamma(E)$:

$$\langle \pounds_Z\sigma, \tau\rangle = Z\left(\langle\sigma,\tau\rangle\right) - \langle \sigma, \pounds_Z\tau\rangle = 0.$$

This leads to

$$(\mathbf{d}_e l_\sigma)(\tilde{Z}(e)) = \left(\tilde{Z}(l_\sigma)\right)(e) = l_{\pounds_Z\sigma}(e) = 0.$$

Hence, the vector field $\tilde{Z}$ is tangent to E on E. As a consequence, its flow curves starting at points of e remain in the submanifold E.

It is easy to check that the flow Φ_t of the vector field $\tilde{Z}$ is equal to $(T\phi_t, (\phi_{-t})^*)$, i.e.,

$$\Phi_t(v_m,\alpha_m) = (T_m\phi_t(v_m), \alpha_m \circ T_{\phi_t(m)}\phi_{-t})$$

for all $(v_m,\alpha_m) \in \mathbb{T}_mM$. Choose a section $(X,\alpha) \in \Gamma(E)$ and a point $m \in M$. We find

$$(\phi_t^*(X,\alpha))(m) = \left(T_{\phi_t(m)}\phi_{-t}X(\phi_t(m)), \alpha_{\phi_t(m)} \circ T_m\phi_t\right) = \Phi_{-t}\left((X,\alpha)(\phi_t(m))\right) \in E(m)$$

since $(X,\alpha)(\phi_t(m)) \in E(\phi_t(m))$. Thus, we have shown that $\phi_t^*(X,\alpha)$ is a section of E. □

Assume now that $q_A : A \to M$ is a vector bundle, and consider a linear vector field X on A, i.e. the map $X : A \to TA$ is a vector bundle morphism over $\bar{X} : M \to TM$ such that $X \sim_{q_A} \bar{X}$. Let $\phi_\cdot^X$ be the flow of X and $\phi_\cdot^{\bar{X}}$ the flow of $\bar{X}$. Then $\phi_t^X : A \to A$ is a vector bundle morphism over $\phi_t^{\bar{X}}$ for all $t \in \mathbb{R}$ where this is defined.

Note that for any $a \in \Gamma(A)$, the section $D_Xa \in \Gamma(A)$ is defined by

$$(D_Xa)(m) = \left.\frac{d}{dt}\right|_{t=0} \phi_{-t}^X(a(\phi_t^{\bar{X}}(m)))$$

for all $m \in M$. In the same manner, if $\varphi \in \Gamma(A^*)$, we can define

$$(D_X\varphi)(m) = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^X)^*(\varphi(\phi_t^{\bar{X}}(m)))$$

for all $m \in M$. We have then $\varphi(a) \in C^\infty(M)$, and

$$\bar{X}(m)(\varphi(a)) = \varphi(D_Xa)(m) + (D_X\varphi)(a)(m). \tag{2.16}$$

We can now show the following lemma.

Lemma B.3. *Let A be a vector bundle and $B \subseteq A$ a subbundle.*

(1) *If $(X,\bar{X})$ is a linear vector field on A such that*

$$D_Xb \in \Gamma(B)$$

for all $b \in \Gamma(B)$, then $\phi_t^X(b_m) \in B\left(\phi_t^{\bar{X}}(m)\right)$ for all $b_m \in B_m$.

(2) *Assume furthermore that $a \in \Gamma(A)$ is such that $a(m)$ is linearly independent to $B(m)$ for all m in* $\mathrm{Dom}(a)$ *and*
$$D_X a \in \Gamma(B).$$
Then
$$\phi_t^X(a(m)) \in a\left(\phi_t^{\bar{X}}(m)\right) + B\left(\phi_t^{\bar{X}}(m)\right)$$
for all $m \in U$ and $t \in \mathbb{R}$ where this makes sense.

Proof. (1) We check that the vector field X is tangent to B on points in B. Let $\varphi \in \Gamma(A^*)$ be a section of B°, i.e. $\varphi_m(b_m) = 0$ for all $b_m \in B$. Let $l_\varphi \in C^\infty(A)$ be the linear function defined by φ. By (2.16), we have then $D_X\varphi \in \Gamma(B^\circ)$. Choose $b_m \in B$. We have then
$$\begin{aligned}\mathbf{d}_{b_m} l_\varphi(X(b_m)) &= \left.\frac{d}{dt}\right|_{t=0} l_\varphi(\phi_t^X(b_m)) = \left.\frac{d}{dt}\right|_{t=0} \varphi_{\phi_t^{\bar{X}}(m)}(\phi_t^X(b_m)) \\ &= (D_X\varphi)(b_m) = 0.\end{aligned}$$
Thus, X is tangent to B on B and the flow of X preserves B.

(2) Assume now that $(b_1, \ldots, b_k)$ is a local frame for B on an open set $U \subseteq M$. Complete this frame to a local frame $(b_1, \ldots, b_n)$ for A defined on an open U such that $b_{k+1} := a \in \Gamma(A)$. Let $\varphi_1, \ldots, \varphi_n$ be a frame for A^* that is dual to $(b_1, \ldots, b_n)$, i.e. such that $(\varphi_{k+1}, \ldots, \varphi_n)$ is a frame for B° and $\varphi_{k+1}(a) = 1$. Then, the closed submanifold C of $A|_U$ defined by $C \cap A_m = a(m) + B_m$ is the level set with value $(1, 0, \ldots, 0)$ of the function
$$(l_{\varphi_{k+1}}, \ldots, l_{\varphi_n}) : A|_U \to \mathbb{R}^{n-k}.$$
Since $D_X a \in \Gamma(B)$ for the linear vector field $(X, \bar{X})$ on A, we get
$$0 = \bar{X}(\varphi_i(a)) = \varphi_i(D_X a) + D_X\varphi_i(a) = 0 + D_X\varphi_i(a)$$
for $i = k+1, \ldots, n$ and this yields as before for all $b_m \in B$:
$$\begin{aligned}\mathbf{d}_{a(m)+b_m} l_{\varphi_i}(X(a(m)+b_m)) &= \left.\frac{d}{dt}\right|_{t=0} l_{\varphi_i}(\phi_t^X(a(m)+b_m)) \\ &= (D_X\varphi_i)(a(m)+b_m) = 0.\end{aligned}$$
Hence, X is tangent to C on points of C. That is, the flow of X preserves C.

□

References

1. K. A. Behrend, *On the de Rham cohomology of differential and algebraic stacks*, Adv. Math. **198** (2005), no. 2, 583–622.
2. R. Bott, *Lectures on characteristic classes and foliations. Notes by Lawrence Conlon. Appendices by J. Stasheff.*, Lectures algebraic diff. Topology, Lect. Notes Math. 279, 1-94 (1972)., 1972.
3. H. Bursztyn, A. Cabrera, and M. del Hoyo, *Vector bundles over Lie groupoids and algebroids*, arXiv:1410.5135.
4. H. Bursztyn, A. Cabrera, and C. Ortiz, *Linear and multiplicative 2-forms*, Lett. Math. Phys. **90** (2009), no. 1-3, 59–83.
5. H. Bursztyn, M. Crainic, A. Weinstein, and C. Zhu, *Integration of twisted Dirac brackets*, Duke Math. J. **123** (2004), no. 3, 549–607.
6. A. Coste, P. Dazord, and A. Weinstein, *Groupoïdes symplectiques*, Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2, Publ. Dép. Math. Nouvelle Sér. A, vol. 87, Univ. Claude-Bernard, Lyon, 1987, pp. i–ii, 1–62.
7. M. Crainic, M.A. Salazar, and I. Struchiner, *Linearization of multiplicative forms*, arXiv:1210.2277 (2012).
8. T. Drummond, M. Jotz, and C. Ortiz, *VB-algebroid morphisms and representations up to homotopy*, Differential Geometry and its Applications **40** (2015), 332–357.
9. S. Evens, J.-H. Lu, and A. Weinstein, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*, Quart. J. Math. Oxford Ser. (2) **50** (1999), no. 200, 417–436. MR 1726784 (2000i:53114)

10. R. L. Fernandes, *Lie algebroids, holonomy and characteristic classes*, Adv. Math. **170** (2002), no. 1, 119–179. MR 1929305 (2004b:58023)
11. J. Grabowski and M. Rotkiewicz, *Higher vector bundles and multi-graded symplectic manifolds*, J. Geom. Phys. **59** (2009), no. 9, 1285–1305.
12. A. Gracia-Saz and R. A. Mehta, *Lie algebroid structures on double vector bundles and representation theory of Lie algebroids*, Adv. Math. **223** (2010), no. 4, 1236–1275.
13. E. Hawkins, *A groupoid approach to quantization*, J. Symplectic Geom. **6** (2008), no. 1, 61–125. MR 2417440 (2009j:46167)
14. M. Jotz, *Dirac Group(oid)s and Their Homogeneous Spaces*, Ph.D. thesis, EPFL, available at http://library.epfl.ch/theses/?nr=5064, Lausanne, 2011.
15. ———, *Dirac Lie groups, Dirac homogeneous spaces and the theorem of Drinfeld*, Indiana Univ. Math. J. **60** (2011), 319–366.
16. ———, *The leaf space of a multiplicative foliation*, J. Geom. Mech. **4** (2012), no. 3, 313–332.
17. ———, *Homogeneous spaces of Dirac groupoids*, New version of the preprint arXiv:1009.0713, available at http://www.jotz-lean.staff.shef.ac.uk/ (2013).
18. M. Jotz, T. S. Ratiu, and M. Zambon, *Invariant frames for vector bundles and applications*, Geom. Dedicata **158** (2012), 23–34. MR 2922701
19. M. Jotz Lean, *Dorfman connections and Courant algebroids*, arXiv:1209.6077 (2013).
20. C. Laurent-Gengoux, M. Stiénon, and P. Xu, *Holomorphic Poisson manifolds and holomorphic Lie algebroids*, Int. Math. Res. Not. IMRN (2008), Art. ID rnn 088, 46.
21. ———, *Integration of holomorphic Lie algebroids*, Math. Ann. **345** (2009), no. 4, 895–923.
22. K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry. II*, Adv. Math. **154** (2000), no. 1, 46–75.
23. ———, *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005.
24. K. C. H. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids.*, Duke Math. J. **73** (1994), no. 2, 415–452.
25. ———, *Classical lifting processes and multiplicative vector fields*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 193, 59–85.
26. J.-P. Ortega and T.S. Ratiu, *Momentum Maps and Hamiltonian Reduction.*, Progress in Mathematics (Boston, Mass.) 222. Boston, MA: Birkhäuser. xxxiv, 497 p. , 2004.
27. C. Ortiz, *Multiplicative Dirac structures on Lie groups.*, C. R., Math., Acad. Sci. Paris **346** (2008), no. 23-24, 1279–1282.
28. ———, *Multiplicative Dirac Structures*, Ph.D. thesis, Instituto de Matemática Pura e Aplicada, 2009.
29. P. Stefan, *Accessible sets, orbits, and foliations with singularities*, Proc. London Math. Soc. (3) **29** (1974), 699–713.
30. ———, *Integrability of systems of vector fields*, J. London Math. Soc. (2) **21** (1980), no. 3, 544–556.
31. H.J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.
32. X. Tang, *Deformation quantization of pseudo-symplectic (Poisson) groupoids*, Geom. Funct. Anal. **16** (2006), no. 3, 731–766.
33. W. M. Tulczyjew, *The Legendre transformation*, Ann. Inst. H. Poincaré Sect. A (N.S.) **27** (1977), no. 1, 101–114.
34. I. Vaisman, *Foliated Lie and Courant algebroids*, Mediterr. J. Math. **7** (2010), no. 4, 415–444.
35. M. Zambon, *Reduction of branes in generalized complex geometry*, J. Symplectic Geom. **6** (2008), no. 4, 353–378.

Department of Mathematics, University of California, Berkeley, CA 94720.
E-mail address: `madeleine.jotz@gmail.com`

Departamento de Matemática, Universidade Federal do Paraná, Setor de Ciencias Exatas - Centro Politecnico, 81531-990 Curitiba, Brazil
E-mail address: `cristian.ortiz@ufpr.br`