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$\mathcal{VB}$-ALGEBROID MORPHISMS AND REPRESENTATIONS UP TO HOMOTOPY

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Abstract. We show in this paper that the correspondence between 2-term representations up to homotopy and $\mathcal{VB}$-algebroids, established in [6], holds also at the level of morphisms. This correspondence is hence an equivalence of categories. As an application, we study foliations and distributions on a Lie algebroid, that are compatible both with the linear structure and the Lie algebroid structure. In particular, we show how infinitesimal ideal systems in a Lie algebroid A are related with subrepresentations of the adjoint representation of A.

Contents

1. Introduction

There are several definitions of ideals in Lie algebroids. The most obvious one is the following: Let $(q\colon A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid. An ideal in A is a subbundle $I \subseteq A$ over M, such that the space of sections $\Gamma(I)$ is an ideal in $\Gamma(A)$ endowed with the Lie bracket $[\cdot,\cdot]$. The first immediate consequence of this definition is the inclusion $I \subseteq \ker(\rho)$, which shows that I is totally intransitive. This notion of ideal is hence not very useful. For instance, an ideal in this sense only corresponds to a surjective morphism of algebroids over the same base. Mackenzie defines *ideal systems* in his book [10] and shows that the kernel of a fibration of Lie algebroids is an ideal system in his sense. Two of the authors define in [9] an infinitesimal version of this. Infinitesimal ideal systems appear naturally in the study of multiplicative foliations on Lie groupoids, a subject which have drawn some attention in connection to geometric quantization of Poisson manifolds [7] and also in a modern approach to Cartan's work on Lie pseudogroups [5]. Multiplicative foliations on a Lie group are in one-to-one correspondence with ideals in its Lie algebra [13, 8]. An ideal in a Lie algebra is a subrepresentation of its adjoint representation. Hence, with the proper notion of adjoint representation of Lie algebroids, one expects infinitesimal ideal systems to be equivalent to some subrepresentations of the adjoint representation. This paper explains how infinitesimal ideal systems and the adjoint representation of Lie algebroid defined by Gracia-Saz and Mehta [6] and independently by Arias Abad and Crainic [1] are related.

The approach of Gracia-Saz and Mehta to study Lie algebroid representations is to view them as $\mathcal{VB}$-algebroids. For instance, flat A-connections on a vector bundle $E \to M$ are in one-to-one correspondence with $\mathcal{VB}$-algebroid structures on

$$\begin{array}{ccc} A \oplus E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & M. \end{array}$$

In general, flat A-superconnections correspond to splittings of a canonical short exact sequence of vector bundles, that is associated to the given $\mathcal{VB}$-algebroid. The tangent prolongation of a Lie algebroid is for instance a $\mathcal{VB}$-algebroid that corresponds to the adjoint representation by splitting (via the choice of a connection on A) the following exact sequence

$$0 \longrightarrow T^*M \otimes A \longrightarrow J^1(A) \longrightarrow A \longrightarrow 0,$$

where J^1A is the first order jet bundle of A. Note that flat superconnections are the two-term case of the representations up to homotopy defined by Arias Abad and Crainic in [1].

Our main result on $\mathcal{VB}$-algebroids is the existence of a one-to-one correspondence between morphisms of representation up to homotopy on 2-term complexes and morphisms of $\mathcal{VB}$-algebroids (Theorem 4.11). This reflects that the correspondence established by Gracia-Saz and Mehta in [6] is actually the correspondence of objects in an equivalence of categories between the category of representation up to homotopy of a given Lie algebroid and the category of $\mathcal{VB}$-algebroids with this Lie algebroid as side. Note that since the tensor product as in [1] of two 2-terms representations up to homotopy is a 3-term representation up to homotopy, which does not encode a $\mathcal{VB}$-algebroid structure, this equivalence of categories is not an equivalence of monoidal categories. There is no known notion of tensor products in the categories of Lie algebroids.

We apply our results on morphisms to the inclusion of distributions inside the tangent bundle (see Theorem 5.19). We obtain the equivalence of infinitesimal ideal systems in a Lie algebroid with subrepresentations of its adjoint and double representations up to homotopy. We also discuss the case of general (non-integrable) subbundles of the tangent of a Lie algebroid. In that case, the representations up to homotopy lead to a new interpretation of the infinitesimal description of multiplicative distributions obtained by [5] (see Theorem 5.17).

This paper is organized as follows. Sections 2 and 3 recall background knowledge on representations up to homotopy and double vector bundles. Section 4 establishes the one-to-one correspondence between $\mathcal{VB}$-algebroid morphisms and morphisms of representations up to homotopy on 2-term complexes. We revisit Lie bialgebroids and IM-2 forms from the perspective of morphisms of representations up to homotopy. Section 5 studies (integrable and non integrable) subbundles of the tangent of a Lie algebroid, that are compatible with the linear and with the Lie algebroid structure.

We show in the appendix that the dictionary between $\mathcal{VB}$-algebroids and 2-term representations up to homotopy is compatible with dualizations in each category.

It would be interesting to describe the VB-algebroid counterpart of a quasi-isomorphism of 2-terms representations up to homotopy. In a work in preparation, del Hoyo and the third author relate quasi-isomorphisms of *Lie groupoid* 2-term representations up to homotopy to a proper notion of Morita equivalence of VB-groupoids.

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2. Representations up to homotopy of Lie algebroids

We recall here some background material on representations up to homotopy. We mostly follow [1].

2.1. **Definition and examples.**

Let $E \to M$ be a vector bundle and $V = \bigoplus_{k\in\mathbb{Z}} V_k$ a graded vector bundle. The space of V-valued E-differential forms, $\Omega(E;V) := \Gamma(\wedge^\bullet E^* \otimes V)$, has a grading given by

$$\Omega(E;V)_k = \bigoplus_{i+j=k} \Gamma(\wedge^i E^* \otimes V_j)$$

and a natural (graded-)module structure over the algebra $\Omega(E) := \Gamma(\wedge^\bullet E^*)$.

If $W = \bigoplus_{k\in\mathbb{Z}} W_k$ is a second graded vector bundle, $\underline{\mathrm{Hom}}(V,W)$ is the graded vector bundle whose degree k part is

$$\underline{\mathrm{Hom}}(V,W)_k = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(V_i, W_{i+k}).$$

Let now $(A, [\cdot,\cdot], \rho_A)$ be a Lie algebroid over M.

Definition 2.1. *A homogeneous A-connection ∇ on the graded vector bundle $V = \bigoplus_{k\in\mathbb{Z}} V_k$ is an A-connection $\nabla : \Gamma(A) \times \Gamma(V) \to \Gamma(V)$ such that ∇_a preserves $\Gamma(V_k)$, for all $k \in \mathbb{Z}$ and $a \in \Gamma(A)$. Equivalently, an A-connection on V is given by a family $\{\nabla^k\}_{k\in\mathbb{Z}}$, where each ∇^k is an A-connection on V_k.*

From now on, we assume that all connections on graded vector bundles are homogeneous.

Definition 2.2. *Let V be a graded vector bundle. A representation up to homotopy of A on V is a degree one map $\mathcal{D} : \Omega(A;V)_\bullet \to \Omega(A;V)_{\bullet+1}$ such that $\mathcal{D}^2 = 0$ and*

$$\mathcal{D}(\alpha \wedge \omega) = d_A\alpha \wedge \omega + (-1)^k \alpha \wedge \mathcal{D}(\omega), \text{ for } \alpha \in \Omega^k(A),\ \omega \in \Omega(A;V), \tag{2.1}$$

where $d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ is the Lie algebroid differential

$$\begin{aligned} d_A\alpha(a_1,\ldots,a_{k+1}) = &\sum_{i=1}^{k}(-1)^{i+1}\mathcal{L}_{\rho_A(a_i)}\alpha(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_k) \\ &+ \sum_{1\le i<j\le k}(-1)^{i+j}\alpha([a_i,a_j],a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{j-1},a_{j+1},\ldots,a_k). \end{aligned}$$

Definition 2.3. A morphism between two representations up to homotopy of A is a degree zero $\Omega(A)$-linear map

$$\Omega(A;V) \to \Omega(A;W)$$

which intertwines the differentials $\mathcal{D}_W$ and $\mathcal{D}_V$. We denote it by $(A,V) \Rightarrow (A,W)$.

In this paper we are mostly concerned with representations up to homotopy on graded vector bundles V concentrated in degree 0 and 1. These are called *2-term graded vector bundles* and the representations up to homotopy of A on 2-term graded vector bundles form a category which we denote by $\mathbb{R}\mathrm{ep}^2(A)$. We denote by $\mathrm{Rep}^2(A)$ the set of isomorphism classes of objects of $\mathbb{R}\mathrm{ep}^2(A)$.

For a 2-term representation up to homotopy $V \in \mathbb{R}\mathrm{ep}^2(A)$, the derivation property (2.1) implies that the differential $\mathcal{D} : \Omega(A;V) \to \Omega(A;V)$ is determined by

(1) a bundle map $\partial : V_0 \to V_1$;
(2) an A-connection ∇ on V compatible with ∂ (i.e. $\partial \circ \nabla^0 = \nabla^1 \circ \partial$);
(3) an element $K \in \Omega^2(A, \underline{\mathrm{End}}(V)_{-1}) = \Omega^2(A, \mathrm{Hom}(V_1, V_0))$ such that $d_{\nabla^{\mathrm{End}}}K = 0$ and the diagram below commutes

$$\begin{tikzcd} V_0 \arrow[r, "\partial"] \arrow[d, "R_{\nabla^0}"'] & V_1 \arrow[d, "R_{\nabla^1}"] \arrow[dl, "-K"'] \\ V_0 \arrow[r, "\partial"'] & V_1 \end{tikzcd}$$

where R_{∇^i} is the curvature of ∇^i, for $i = 0,1$. We say that (∂, ∇, K) are *the structure operators* for $V \in \mathbb{R}\mathrm{ep}^2(A)$.

We refer to [1] for a detailed exposition of the correspondence $\mathcal{D} \mapsto (\partial, \nabla, K)$ (pointing out that our sign convention for K is different from the one in [1]).

For $V, W \in \mathbb{R}\mathrm{ep}^2(A)$, a morphism $(A,V) \Rightarrow (A,W)$ is determined by a triple (ϕ_0, ϕ_1, Φ), where $\phi_0 : V_0 \to W_0$, $\phi_1 : V_1 \to W_1$ are bundle maps and $\Phi \in$

$\Omega^1(A; \mathrm{Hom}(V_1, W_0))$, satisfying

$$\phi_1 \circ \partial_V = \partial_W \circ \phi_0, \tag{2.2}$$

$$\nabla_a^{\mathrm{Hom}}(\phi_0, \phi_1) = (\Phi_a \circ \partial_V,\, \partial_W \circ \Phi_a) \quad \text{for all } a \in \Gamma(A) \tag{2.3}$$

and

$$d_{\nabla^{\mathrm{Hom}}}\Phi = \phi_0 \circ K_V - K_W \circ \phi_1, \tag{2.4}$$

where ∇^{Hom} is the A-connection on $\underline{\mathrm{Hom}}(V, W)$ (see [1] for more details).

In the following, given vector bundles E, E' over M, we denote by $E_{[0]} \oplus E'_{[1]}$ the graded vector bundle consisting of E in degree 0 and of E' in degree 1.

Example 2.4 (Double representation)**.** Let $B \to M$ be a vector bundle and consider the graded vector bundle $V = B_{[0]} \oplus B_{[1]}$. Any connection $\nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)$ induces a representation up to homotopy of TM on V by taking $\partial = \mathrm{id}_B$, $\nabla^0 = \nabla^1 = \nabla$ and $K = -R_\nabla$, the curvature of ∇. The isomorphism class of this representation does not depend on the choice of ∇ and is called the *double representation of TM on B*. We denote it by $\mathcal{D}(B) \in \mathrm{Rep}^2(TM)$ and the representation itself by $\mathcal{D}_\nabla(B) \in \mathbb{R}\mathrm{ep}^2(A)$.

Example 2.5 (Adjoint representation)**.** *Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over M. Any connection $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ on A induces a representation up to homotopy of A on $V = A_{[0]} \oplus TM_{[1]}$ in the following manner. The map ∂ is just the anchor $\rho_A : A \to TM$. The A-connection ∇^{bas} on V (called the basic connection) has degree zero and degree one parts given by*

$$\begin{array}{rccl} \nabla^{\mathrm{bas}}: & \Gamma(A) \times \Gamma(A) & \longrightarrow & \Gamma(A) \\ & (a, b) & \longmapsto & [a, b]_A + \nabla_{\rho_A(b)} a. \end{array}$$

and

$$\begin{array}{rccl} \nabla^{\mathrm{bas}}: & \Gamma(A) \times \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ & (a, X) & \longmapsto & [\rho_A(a), X]_A + \rho_A(\nabla_X a), \end{array}$$

respectively. The element K is the basic curvature $R_\nabla^{\mathrm{bas}} \in \Omega^2(A, \mathrm{Hom}(TM, A))$ defined by

$$R_\nabla^{\mathrm{bas}}(a, b)(X) = \nabla_X[a, b] - [\nabla_X a, b] - [a, \nabla_X b] + \nabla_{\nabla_a^{\mathrm{bas}} X}\, b - \nabla_{\nabla_b^{\mathrm{bas}} X}\, a.$$

As before, the isomorphism class of this representation does not depend on the choice of ∇ and it is called the adjoint representation of A. We denote it by $\mathrm{ad} \in \mathrm{Rep}^2(A)$ and the representation itself by $\mathrm{ad}_\nabla \in \mathbb{R}\mathrm{ep}^2(A)$.

Given a 2-term representation $V \in \mathbb{R}\mathrm{ep}^2(A)$ with structure operators (∂, ∇, K) of A on $V = V_0 \oplus V_1$, its dual is the representation $V^\top \in \mathbb{R}\mathrm{ep}^2(A)$, where $V_0^\top = V_1^*$, $V_1^\top = V_0^*$, with structure operators given by

$$\partial_{V^\top} = \partial^*,\ \ \nabla^{V^\top} = \nabla^* \text{ and } K_{V^\top} = -K^* \tag{2.5}$$

where ∇^* is the A-connection dual to ∇, given by

$$\langle \nabla_a^* \xi, v\rangle + \langle \xi, \nabla_a v\rangle = \mathcal{L}_{\rho_A(a)}\langle \xi, v\rangle,\ \forall v \in \Gamma(V),\, \xi \in \Gamma(V^\top). \tag{2.6}$$

Example 2.6 (Coadjoint representation)**.** *The coadjoint representation is the representation of A on $T^*M_{[0]} \oplus A^*_{[1]}$ dual to the adjoint representation. It is denoted by $\mathrm{ad}^\top(A) \in \mathrm{Rep}^2(A)$.*

2.2. **Pullbacks.** We define here morphisms between 2-term representations up to homotopy of different Lie algebroids. Let $(A', [\cdot,\cdot]_{A'}, \rho_{A'})$ be another Lie algebroid over M and $T : A \to A'$ a Lie algebroid morphism over id_M. Choose $W \in \mathbb{R}\mathrm{ep}^2(A')$ with structure operators (∂, ∇, K).

Define $\nabla^T : \Gamma(A) \times \Gamma(W) \to \Gamma(W)$ to be the A-connection given by

$$\nabla^T_a w := \nabla_{T(a)} w \tag{2.7}$$

for $a \in \Gamma(A)$ and $w \in \Gamma(W)$ and $T^*K \in \Omega^2(A; \mathrm{Hom}(W_1, W_0))$ by

$$T^*K(a_1, a_2) = K(T(a_1), T(a_2)),\ (a_1, a_2) \in A \times_M A. \tag{2.8}$$

Lemma 2.7. *The triple $(\partial, \nabla^T, T^*K)$ defines structure operators for a representation up to homotopy of A on W which is called the pullback of W by T and it is denoted by $T^!W \in \mathbb{R}\mathrm{ep}^2(A)$.*

Proof. We leave the details to the reader. □

Example 2.8. *If T is the inclusion of a Lie subalgebroid $A \hookrightarrow A'$, the pullback $T^!W$ is just the restriction of the representation to A.*

The usefulness of taking pullbacks is that it allows one to define morphisms between representations up to homotopy of different algebroids.

Definition 2.9. *Let $W \in \mathbb{R}\mathrm{ep}^2(A')$ and $V \in \mathbb{R}\mathrm{ep}^2(A)$ be representations up to homotopy. We define a morphism $(A, V) \Rightarrow (A', W)$ over a Lie algebroid morphism $T : A \to A'$ to be an usual morphism $(A, V) \Rightarrow (A, T^!W)$ (as given in Definition 2.3).*

Remark 2.10. The pullback operation can be defined for arbitrary representations up to homotopy. It was already defined in this generality for representations up to homotopy of Lie groupoids in [1]. Also, the pullback can be extended to morphisms and we get a functor $T^! : \mathbb{R}\mathrm{ep}^2(A') \to \mathbb{R}\mathrm{ep}^2(A)$.

3. Double vector bundles.

We briefly recall the definitions of double vector bundles, of some of their special sections and of their morphisms. We refer to [10] for a more detailed treatment (see also [6] for a treatment closer to ours). We also classify subbundles of double vector bundles.

3.1. Preliminaries.

Definition 3.1. *A double vector bundle is a commutative square*

$$\begin{tikzcd} D \arrow[r, "q_B^D"] \arrow[d, "q_A^D"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

satisfying the following three conditions:

DV1. *all four sides are vector bundles;*
DV2. *q_B^D is a vector bundle morphism over q_A;*
DV3. *$\underset{B}{+} : D \times_B D \to D$ is a vector bundle morphism over $+ : A \times_M A \to A$, where $\underset{B}{+}$ is the addition map for the vector bundle $D \to B$.*

Given a double vector bundle $(D; A, B; M)$, the vector bundles A and B are called the *side bundles*. The zero sections are denoted by $0^A : M \to A$, $0^B : M \to B$, ${}^A0 : A \to D$ and ${}^B0 : B \to D$. Elements of D are written $(d; a, b; m)$, where $d \in D$, $m \in M$ and $a = q_A^D(d) \in A_m$, $b = q_B^D(d) \in B_m$.

The *core* C of a double vector bundle is the intersection of the kernels of q_A^D and q_B^D. It has a natural vector bundle structure over M, the projection of which we call $q_C : C \to M$. The inclusion $C \hookrightarrow D$ is usually denoted by

$$C_m \ni c \longmapsto \overline{c} \in (q_A^D)^{-1}(0_m^A) \cap (q_B^D)^{-1}(0_m^B).$$

Definition 3.2. *Let $(D; A, B; M)$ and $(D'; A', B'; M)$ be two double vector bundles. A double vector bundle morphism $(F; F_{\mathrm{ver}}, F_{\mathrm{hor}}; f)$ from D to D' is a commutative cube*

$$\begin{tikzcd}[row sep=small, column sep=small]
& B \arrow[rr, "F_{\mathrm{hor}}"] \arrow[dd] & & B' \arrow[dd] \\
D \arrow[ur] \arrow[rr, "F", near start] \arrow[dd] & & D' \arrow[ur] \arrow[dd] & \\
& M \arrow[rr, "f", near start] & & M \\
A \arrow[ur] \arrow[rr, "F_{\mathrm{ver}}"'] & & A' \arrow[ur] &
\end{tikzcd}$$

where all the faces are vector bundle morphisms.

Given a double vector bundle morphism $(F; F_{\mathrm{ver}}, F_{\mathrm{hor}}; f)$, its restriction to the core bundles induces a vector bundle morphism $F_c : C \to C'$. In the following, we are mainly interested in double vector bundle morphisms where $f = \mathrm{id}_M : M \to M$. In this case, we omit the reference to f and denote a double vector bundle morphism by $(F; F_{\mathrm{ver}}, F_{\mathrm{hor}})$.

Given a double vector bundle $(D; A, B; M)$, the space of sections $\Gamma(B, D)$ is generated as a $C^\infty(B)$-module by two distinguished classes of sections (see [11]), the *linear* and the *core sections* which we now describe.

Definition 3.3. *For a section $c : M \to C$, the corresponding core section $\hat{c} : B \to D$ is defined as*

$$\hat{c}(b_m) = {}^B0_{b_m} \underset{A}{+} \overline{c(m)}, \; m \in M, \, b_m \in B_m. \tag{3.1}$$

We denote the space of core sections by $\Gamma_c(B, D)$.

Definition 3.4. *A section $\mathcal{X} \in \Gamma(B, D)$ is called linear if $\mathcal{X} : B \to D$ is a bundle morphism from $B \to M$ to $D \to A$. The space of linear sections is denoted by $\Gamma_\ell(B, D)$.*

The space of linear sections is a locally free $C^\infty(M)$-module (see e.g. [6]). Hence, there is a vector bundle $\widehat{A}$ over M such that $\Gamma_\ell(B, D)$ is isomorphic to $\Gamma(\widehat{A})$ as $C^\infty(M)$-modules. Note that for a linear section $\mathcal{X}$, there exists a section $\mathcal{X}_0 : M \to A$ such that $q_A^D \circ \mathcal{X} = \mathcal{X}_0 \circ q_B$. The map $\mathcal{X} \mapsto \mathcal{X}_0$ induces a short exact sequence of vector bundles

$$0 \longrightarrow B^* \otimes C \hookrightarrow \widehat{A} \longrightarrow A \longrightarrow 0, \tag{3.2}$$

where for $T \in \Gamma(B^* \otimes C)$, the corresponding section $\widehat{T} \in \Gamma_\ell(B, D)$ is given by

$$\widehat{T}(b_m) = {}^B0_{b_m} \underset{A}{+} \overline{T(b_m)}. \tag{3.3}$$

We call splittings $h : A \to \widehat{A}$ of the short exact sequence (3.2) *horizontal lifts.*

Example 3.5. *Let A, B, C be vector bundles over M and consider $D = A \oplus B \oplus C$. With the vector bundle structures $D = q_A^!(B \oplus C) \to A$ and $D = q_B^!(A \oplus C) \to B$, one has that $(D; A, B; M)$ is a double vector bundle called the trivial double vector bundle with core C. The core sections are given by*

$$b_m \mapsto (0^A_m, b_m, c(m)), \text{ where } m \in M,\, b_m \in B_m,\, c \in \Gamma(C).$$

The space of linear sections $\Gamma_\ell(B, D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, T) : b_m \mapsto (a(m), b_m, T(b_m)), \text{ where } T \in \Gamma(B^* \otimes C),\, a \in \Gamma(A).$$

The canonical inclusion $\Gamma(A) \hookrightarrow \Gamma_\ell(B, D)$ is a horizontal lift.

Let A', B', C' be another triple of vector bundles over M and consider the corresponding trivial double vector bundle with core C', $D' = A' \oplus B' \oplus C'$. Any double vector bundle morphism $(F; F_{\rm ver}, F_{\rm hor})$ from D to D' is given by

$$(a, b, c) \mapsto (F_{\rm ver}(a), F_{\rm hor}(b), F_c(c) + \Phi_a(b)) \tag{3.4}$$

where $F_c : C \to C'$ is a vector bundle morphism and $\Phi \in \Gamma(A^* \otimes B^* \otimes C')$.

A *decomposition* for a double vector bundle $(D; A, B; M)$ is an isomorphism σ of double vector bundles from the trivial double vector bundle with core C to D covering the identities on the side bundles A, B and inducing the identity on the core C. The space of decompositions for D will be denoted by $\mathrm{Dec}(D)$. We recall now how this is an affine space over $\Gamma(A^* \otimes B^* \otimes C)$. Given an element $\Phi \in \Gamma(A^* \otimes B^* \otimes C)$, consider the double vector bundle morphism

$$\begin{array}{rccc} I_\Phi : & A \oplus B \oplus C & \longrightarrow & A \oplus B \oplus C \\ & (a, b, c) & \longmapsto & (a, b, c + \Phi_a(b)) \end{array} \tag{3.5}$$

obtained from (3.4) by taking $F_{\rm ver}$, $F_{\rm hor}$, F_c to be the identity morphisms. For a decomposition σ,

$$\Phi \cdot \sigma := \sigma \circ I_\Phi \tag{3.6}$$

defines the affine structure on $\mathrm{Dec}(D)$.

Remark 3.6. The space of horizontal lifts is also affine over $\Gamma(A^* \otimes B^* \otimes C)$ (this follows directly from the definition of horizontal lifts). There is a natural one-to-one correspondence between decompositions and horizontal lifts for D [6]. Concretely, given a horizontal lift h, the decomposition $\sigma_h : A \oplus B \oplus C \to D$ is given by

$$\sigma_h(a_m, b_m, c_m) = h(a)(b_m) \underset{B}{+} ({}^B0_{b_m} \underset{A}{+} \overline{c_m}), \tag{3.7}$$

where $m \in M$ and $a \in \Gamma(A)$ is any section with $a(m) = a_m$. Conversely, given a decomposition $\sigma : A \oplus B \oplus C \to D$, the map $h_\sigma : \Gamma(A) \to \Gamma_\ell(B, D)$,

$$h_\sigma(a)(b_m) = \sigma(a(m), b_m, 0^C_m),\ m \in M, \tag{3.8}$$

is a horizontal lift. The map $h \mapsto \sigma_h$ and its inverse $\sigma \mapsto h_\sigma$ are affine.

Example 3.7. *For a vector bundle $B \to M$,*

$$\begin{array}{ccc} TB & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ TM & \longrightarrow & M \end{array}$$

is a double vector bundle with core bundle $B \to M$. The core section corresponding to $b \in \Gamma(B)$ is the vertical lift $b^\uparrow : B \to TB$. One has that

$$b^\uparrow(\ell_\psi) = \langle \psi, b\rangle \circ q_B \quad \text{and} \quad b^\uparrow(f \circ q_B) = 0,$$

where $\ell_\psi, f \circ q_B \in C^\infty(B)$ are the linear function and the pullback function corresponding to $\psi \in \Gamma(B^*)$ and $f \in C^\infty(M)$, respectively. An element of $\Gamma_\ell(B, TB)$ is called a linear vector field. It is well-known (see e.g. [10]) that a linear vector field $X : B \to TB$ covering $x : M \to TM$ corresponds to a derivation $L : \Gamma(B^*) \to \Gamma(B^*)$ having x as its symbol. The precise correspondence is given by

$$X(\ell_\psi) = \ell_{L(\psi)} \quad \text{and} \quad X(f \circ q_B) = \mathcal{L}_x(f) \circ q_B.$$

Hence, the choice of a horizontal lift for $(TB; TM, B; M)$ is equivalent to the choice of a connection on B^*. For convenience, we shall prefer working with the dual connection on B (see (2.6)). In this case, one can identify $\mathrm{Dec}(TB)$ with the space of connections on B.

Example 3.8. *Let $A \to M$ be a vector bundle and consider $TA \to TM$ as the horizontal side bundle of the tangent double,*

$$\begin{array}{ccc} TA & \longrightarrow & TM \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & M. \end{array}$$

For any $a \in \Gamma(A)$, $Ta : TM \to TA$ is a linear section covering a itself. Yet, the map $a \mapsto Ta$ splits (3.2) only at the level of sections, as it fails to be $C^\infty(M)$-linear. The choice of a connection ∇ on A restores the $C^\infty(M)$-linearity and induces a horizontal lift by

$$h(a)(x) = Ta(x) \underset{TM}{+} (T0(x) - \overline{\nabla_x a}), \; x \in TM, \, a \in \Gamma(A). \tag{3.9}$$

The associated decomposition $\sigma_h \in \mathrm{Dec}(TA)$ coincides with the one induced by ∇ as in Example 3.7.

3.2. **Dualization of double vector bundles.** Given a double vector bundle $(D; A, B; M)$ with core C, its *horizontal dual* is the double vector bundle

$$\begin{array}{ccc} D^*_B & \xrightarrow{p_B} & B \\ {\scriptstyle p^{\mathrm{hor}}_{C^*}}\big\downarrow & & \big\downarrow{\scriptstyle q_B} \\ C^* & \xrightarrow[q_{C^*}]{} & M, \end{array} \tag{3.10}$$

where $p_B : D^*_B \to B$ is the dual of $q^D_B : D \to B$ and, for $\xi \in (p_B)^{-1}(b_m)$,

$$\langle p^{\mathrm{hor}}_{C^*}(\xi), c_m\rangle = \langle \xi, {}^B0_{b_m} \underset{A}{+} \overline{c_m}\rangle. \tag{3.11}$$

The core bundle of D^*_B is $A^* \to M$. Similarly, the *vertical dual* is the double vector bundle

$$\begin{array}{ccc} D^*_A & \xrightarrow{p^{\mathrm{ver}}_{C^*}} & C^* \\ {\scriptstyle p_A}\big\downarrow & & \big\downarrow{\scriptstyle q_{C^*}} \\ A & \xrightarrow[q_A]{} & M \end{array} \tag{3.12}$$

with core $B^* \to M$.

In the following, we are mostly interested in the horizontal dual. For $\psi \in \Gamma(A^*)$, the corresponding core section $\widehat{\psi} \in \Gamma_c(B, D_B^*)$ is just $(q_A^D)^*\psi$. In particular,

$$\langle \widehat{\psi}, \widehat{c} \rangle = 0 \tag{3.13}$$

for $c \in \Gamma(C)$ and

$$\langle \widehat{\psi}, h(a) \rangle = \langle \psi, a \rangle \circ q_B \tag{3.14}$$

for $a \in \Gamma(A)$ and any horizontal lift $h : A \to \widehat{A}$.

Given a decomposition $\sigma : A \oplus B \oplus C \to D$, the inverse of its dual over B, $(\sigma_B^*)^{-1} : B \oplus C^* \oplus A^* \to D_B^*$, is a decomposition for D_B^*.

Example 3.9. *Let $B \to M$ be a vector bundle and consider its tangent double $(TB; TM, B; M)$. The projection of the cotangent bundle T^*B to B^* is given, for $\xi \in T^*_{b_m}B$, by*

$$\langle p^{\text{hor}}_{B^*}(\xi), c_m \rangle = \left\langle \xi, \left.\frac{d}{dt}\right|_{t=0} (b_m + tc_m) \right\rangle, \quad \text{for } c_m \in B_m,\ m \in M.$$

Given a decomposition $\sigma : TM \oplus B \oplus B \to TB$, let ∇ be the corresponding connection on B. The inverse of the dual of σ over B induces a horizontal lift $h : \Gamma(B^*) \to \Gamma_\ell(B, T^*B)$ given by

$$h(\psi)(b_m) = (\sigma_B^*)^{-1}(b_m, \psi(m), 0_m^{TM}) = d\ell_\psi(b_m) - \langle \nabla_{Tq_B(\cdot)} \psi, b_m \rangle \in T^*_{b_m}B,$$

where $\ell_\psi \in C^\infty(B)$ is the linear function corresponding to $\psi \in \Gamma(B^*)$.

4. $\mathcal{VB}$-algebroids and morphisms.

Gracia-Saz and Mehta show in [6] how representations up to homotopy of a Lie algebroid on a 2-term graded vector bundle encode the Lie algebroid structures of *$\mathcal{VB}$-algebroids*. These are double vector bundles with some additional Lie algebroid structure that is compatible with the double vector bundle structure. In this section, we recall this correspondence and show how it can be extended to morphisms. We also check in Appendix A that it behaves well under dualization.

4.1. $\mathcal{VB}$-algebroids.

We begin with the definition of $\mathcal{VB}$-algebroids. We follow [6] in our treatment of the subject.

Definition 4.1. *Let $(D; A, B; M)$ be a double vector bundle. We say that $(D \to B; A \to M)$ is a $\mathcal{VB}$-algebroid if $D \to B$ is a Lie algebroid, the anchor $\rho_D : D \to TB$ is a bundle morphism over $\rho_A : A \to TM$ and the three Lie bracket conditions below are satisfied:*

(i) $[\Gamma_\ell(B, D), \Gamma_\ell(B, D)]_D \subset \Gamma_\ell(B, D)$;
(ii) $[\Gamma_\ell(B, D), \Gamma_c(B, D)]_D \subset \Gamma_c(B, D)$;
(iii) $[\Gamma_c(B, D), \Gamma_c(B, D)]_D = 0$.

A $\mathcal{VB}$-algebroid structure on $(D; A, B; M)$ naturally induces a Lie algebroid structure on A by taking the anchor to be ρ_A and the Lie bracket $[\cdot, \cdot]_A$ defined as follows: if $\mathcal{X}, \mathcal{Y} \in \Gamma_\ell(B, D)$ cover $\mathcal{X}_0, \mathcal{Y}_0 \in \Gamma(A)$ respectively, then $[\mathcal{X}, \mathcal{Y}]_D \in \Gamma_\ell(B, D)$ covers $[\mathcal{X}_0, \mathcal{Y}_0]_A \in \Gamma(A)$. We call *$A$ the base Lie algebroid* of D.

The next result from [6] relates VB-algebroid structures on trivial double vector bundles and representations up to homotopy. Note that Arias Abad and Crainic show a related result on the relationship between representations up to homotopy and Lie algebroid extensions [1, Proposition 3.9.].

Proposition 4.2. *Let $(A, \rho_A, [\cdot,\cdot]_A)$ be a Lie algebroid over M. Let $B \to M$ and $C \to M$ be vector bundles. There is a one-to-one correspondence between $\mathcal{VB}$-algebroid structures on the trivial double vector bundle $A \oplus B \oplus C$ with core C and A as side Lie algebroid, and 2-term representations up to homotopy of A on $V = C_{[0]} \oplus B_{[1]}$.*

Let us give an explicit description of the $\mathcal{VB}$-algebroid structure on $D = A \oplus B \oplus C$ corresponding to a 2-term representation (∂, ∇, K) of A on $C_{[0]} \oplus B_{[1]}$. For $a \in \Gamma(A)$, let $h : \Gamma(A) \hookrightarrow \Gamma_\ell(B, D)$ be the canonical inclusion of Example 3.5. Define as follows the anchor of D, $\rho_D : D \to B$, on linear and core sections:

$$\rho_D(h(a)) = X_{\nabla^1_a}, \ \rho_D(\widehat{c}) = \partial(c)^\uparrow, \tag{4.1}$$

where $X_{\nabla^1_a}$, $\partial(c)^\uparrow \in \mathfrak{X}(B)$ are, respectively, the linear vector field corresponding to the derivation ${\nabla^1_a}^* : \Gamma(B^*) \to \Gamma(B^*)$ and the vertical vector field corresponding to $\partial(c) \in \Gamma(B)$ (see Example 3.7). The Lie bracket $[\cdot,\cdot]_D$ on $\Gamma(D)$ is given by the formulas below:

$$[\widehat{c}_1, \widehat{c}_2]_D = 0$$

$$[h(a), \widehat{c}]_D = \widehat{\nabla^0_a c}, \tag{4.2}$$

and

$$[h(a_1), h(a_2)]_D = h([a_1, a_2]_A) + \widehat{K}(a_1, a_2) \tag{4.3}$$

where $a, a_1, a_2 \in \Gamma(A)$ and $c, c_1, c_2 \in \Gamma(C)$ and $\widehat{K}(a_1, a_2) \in \Gamma_\ell(B, D)$ is the linear section given by (3.3).

Remark 4.3. A $\mathcal{VB}$-algebroid structure on a general double vector bundle $(D; A, B; M)$ induces a representation up to homotopy of the base Lie algebroid A on $C_{[0]} \oplus B_{[1]}$ once a decomposition $\sigma : A \oplus B \oplus C \to D$ is chosen. The structure operators (∂, ∇, K) are obtained from exactly the same formulas (4.1), (4.2) and (4.3) by taking $h : \Gamma(A) \to \Gamma_\ell(B, D)$ as the horizontal lift corresponding to σ. The isomorphism class of this representation does not depend on the choice of the decomposition. More precisely, if $\widetilde{\sigma}$ is another decomposition, then $\widetilde{\sigma} = \Phi \cdot \sigma$, for some $\Phi \in \Gamma(A^* \otimes B^* \otimes C)$ and the structure operators $(\widetilde{\partial}, \widetilde{\nabla}, \widetilde{K})$ corresponding to $\widetilde{\sigma}$ are given by

$$\widetilde{\partial} = \partial; \tag{4.4}$$

$$\widetilde{\nabla}^0_a = \nabla^0_a - \Phi_a \circ \partial \text{ and } \widetilde{\nabla}^1_a = \nabla^1_a - \partial \circ \Phi_a; \tag{4.5}$$

$$\widetilde{K}(a, b) = K(a, b) + d_{\nabla^{\mathrm{Hom}}}\Phi(a, b) + \Phi_b \circ \partial \circ \Phi_a - \Phi_a \circ \partial \circ \Phi_b, \tag{4.6}$$

for $a, b \in \Gamma(A)$. Moreover, $(\mathrm{id}_C, \mathrm{id}_B, \Phi)$ are the components of an $\Omega(A)$-linear isomorphism $\Omega(A, C_{[0]} \oplus B_{[1]}) \to \Omega(A, C_{[0]} \oplus B_{[1]})$ which intertwines $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ (see [6] for more details).

The next two Examples recall how the double and the adjoint representation arise in this way from $\mathcal{VB}$-algebroids.

Example 4.4. *The tangent double $(TB;TM;B;M)$ of a vector bundle $B\to M$ is canonically endowed with a $\mathcal{VB}$-algebroid structure $(TB\to B;TM\to M)$ with $\rho_{TB}=\mathrm{id}_B$, $\rho_{TM}=\mathrm{id}_{TM}$ and $[\cdot,\cdot]_{TB}$ given by the Lie bracket of vector fields. A horizontal lift $h:TM\to\widehat{TM}$ is equivalent to a connection $\nabla:\Gamma(TM)\times\Gamma(B)\to\Gamma(B)$ (see Example 3.7). Equations* (4.1) *and* (4.2) *imply the equality $\partial=\mathrm{id}_B$ and show that the connection on $B_{[0]}\oplus B_{[1]}$ is given by ∇ in degree 0 and 1. The equality $K=-R_\nabla$, with R_∇ the curvature of ∇, follows from* (4.3)*. Hence, the element on $\mathrm{Rep}^2(TM)$ associated to $(TB\to B;TM\to M)$ is the isomorphism class of the double representation of TM on $B\oplus B$ (see Example 2.4).*

Example 4.5. *Let $(A,\rho_A,[\cdot,\cdot]_A)$ be a Lie algebroid over M. The* tangent prolongation *$(TA;A,TM;M)$ of A has a $\mathcal{VB}$-algebroid structure $(TA\to TM;A\to M)$. We refer to* [10] *for more details about this. Gracia-Saz and Mehta show that the element on $\mathrm{Rep}^2(A)$ associated to such a $\mathcal{VB}$-algebroid structure is exactly the adjoint representation of A* [6]*.*

Given a $\mathcal{VB}$-algebroid $(D\to B;A\to M)$, one can prove (see [11]) that the vertical dual $(D^*_A\to C^*;A\to M)$ is a $\mathcal{VB}$-algebroid. By choosing a decomposition $\sigma\in\mathrm{Dec}(D)$, the inverse of its dual over A, $(\sigma^*_A)^{-1}$, is a decomposition for D^*_A. In Appendix A, we prove that the representations up to homotopy associated to σ and $(\sigma^*_A)^{-1}$ are dual to each other.

Example 4.6. *Let $(A,[\cdot,\cdot]_A,\rho_A)$ be a Lie algebroid over M. By Proposition A.1, the $\mathcal{VB}$-algebroid structure of $(T^*A\to A^*;A\to M)$ obtained from taking the vertical dual of the tangent prolongation $(TA\to TM;A\to M)$ gives rise to the coadjoint representation* $\mathrm{ad}^\top\in\mathbb{R}\mathrm{ep}^2(A)$*, the isomorphism class of the representation up to homotopy dual to the adjoint representation of A. We refer to* [10] *for more details concerning the cotangent Lie algebroid $T^*A\to A^*$.*

4.2. **Lie algebroid differential.** Let $(A,\rho_A,[\cdot,\cdot]_A)$ be a Lie algebroid over M. Given a 2-term representation up to homotopy of A on $V=C_{[0]}\oplus B_{[1]}$, we investigate how the Lie algebroid differential d_D of $D\to B$, where $D=A\oplus B\oplus C$, is related to the structure operators (∂,∇,K). Let $h:\Gamma(A)\hookrightarrow\Gamma_\ell(B,D)$ be the natural inclusion considered in Example 3.5.

First of all, it is straightforward to check that, for $f\in C^\infty(M)$,

$$d_D(f\circ q_B)=(\rho_A\circ q_A^D)^*(df)=q_A^{D\,*}(d_Af), \tag{4.7}$$

where d_A is the Lie algebroid differential of A.

In the following, recall the identification $\Gamma_\ell(B,D^*_B)=\Gamma(A^*\otimes B^*)\oplus\Gamma(C^*)$ (see Example 3.5).

Lemma 4.7. *Let $\ell_\psi\in C^\infty(B)$ be the linear function associated to $\psi\in\Gamma(B^*)$. The map $d_D(\ell_\psi):B\longrightarrow D^*_B$ is a linear section given by*

$$d_D(\ell_\psi)=(d_{\nabla^*}\psi,\partial^*\psi) \tag{4.8}$$

where ∇^* is the A-connection on $V^*=B^*_{[0]}\oplus C^*_{[1]}$ dual to ∇ and $d_{\nabla^*}:\Omega(A;V^*)\to\Omega(A;V^*)$ is the Koszul differential.

Proof. The result that $d_D(\ell_\psi)$ is a linear section follows from the fact that $d\ell_\psi:B\longrightarrow T^*B$ is a linear section of the cotangent bundle (covering ψ itself) and

$\rho_D : D \to TB$ is a double vector bundle morphism. Also, for $b_m \in B$, one has

$$\langle p^{\rm hor}_{C^*}(d_D\ell_\psi), c_m\rangle = d\ell_\psi(\rho_D(\tilde{0}_{b_m} \underset{A}{+} \overline{c_m})) = \left.\frac{d}{dt}\right|_{t=0} \langle \psi(m), b_m + t\partial(c_m)\rangle = \langle \partial^*\psi(m), c_m\rangle.$$

Finally, since $\rho_D(h(a))$ is the linear vector field on B which corresponds to the derivation ∇_a on $\Gamma(B)$, it follows that

$$\langle d_D(\ell_\psi), h(a)\rangle = \mathcal{L}_{\rho_D(h(a))}\ell_\psi = \ell_{\nabla^*_a\psi}.$$

Formula (4.8) follows immediately. $\square$

Now let us consider the degree one part of d_D (i.e. $d_D : \Gamma(D^*_B) \to \Gamma(\wedge^2 D^*_B)$). As $D = q^!_B(A \oplus C)$ as a vector bundle over B, one has

$$\wedge^2 D^*_B = q^!_B(\wedge^2 A^* \oplus (A^* \otimes C^*) \oplus \wedge^2 C^*). \tag{4.9}$$

Lemma 4.8. *Choose $\psi \in \Gamma(A^*)$ and consider the corresponding core section $\widehat{\psi} \in \Gamma_c(B, D)$. With respect to the decomposition* (4.9)*, one has*

$$d_D\widehat{\psi} : b_m \longmapsto (\, d_A\psi(m),\, 0^{A^*\otimes C^*}_m,\, 0^{\wedge^2 C^*}_m). \tag{4.10}$$

Proof. Choose a_1, $a_2 \in \Gamma(A)$. As $\rho_D : D \to TB$ is a vector bundle morphism over $\rho_A : A \to TM$, one has that $Tq_B \circ \rho_D(h(a_i)) = \rho_A \circ q^D_A(h(a_i)) = \rho_A(a_i)$, for $i = 1, 2$. Also, it follows from (3.14) that $\langle \widehat{\psi}, h(a_i)\rangle = \langle \psi, a_i\rangle \circ q_B$, for $i = 1, 2$. It is now straightforward to check that

$$d_D\widehat{\psi}(h(a_1), h(a_2)) = \quad (d_A\psi(a_1, a_2)) \circ q_B - \langle \widehat{\psi},\, \widehat{K}(a_1, a_2)\rangle = (d_A\psi(a_1, a_2)) \circ q_B.$$

As for the component on $A^* \otimes C^*$, we have

$$d_D\widehat{\psi}(h(a_1), \widehat{c}) = \quad \mathcal{L}_{\rho_D(h(a_1))}\langle \widehat{\psi},\, \widehat{c}\rangle - \mathcal{L}_{\rho_D(\widehat{c})}\langle \widehat{\psi},\, h(a_1)\rangle - \langle \widehat{\psi},\, [h(a_1), \widehat{c}\,]_D\rangle = 0.$$

The first and the last term on the right hand side vanish because of (3.13). Also, since $\rho_D(\widehat{c})$ is a vertical vector field, it follows with (3.14) that the second term on the right hand side vanishes. One can prove in a similar manner that the component on $\wedge^2 C^*$ is also zero. $\square$

Lemma 4.9. *Let $Q \in \Gamma(A^* \otimes B^*)$ and $\gamma \in \Gamma(C^*)$. With respect to the decomposition* (4.9)*, we have*

$$d_D(Q, 0) : b_m \longmapsto (\langle d_{\nabla^*}Q, b_m\rangle, -({\rm id}_{A^*} \otimes \partial^*)(Q),\, 0^{\wedge^2 C^*}_m), \tag{4.11}$$

and

$$d_D(0, \gamma) : b_m \longmapsto (-\langle\, K^*\gamma, b_m\,\rangle,\, d_{\nabla^*}\gamma,\, 0^{\wedge^2 C^*}_m), \tag{4.12}$$

Proof. Fix $a_1, a_2 \in \Gamma(A)$ and $c \in \Gamma(C)$. We have

$$d_D(Q, 0)(h(a_1), h(a_2)) = \mathcal{L}_{\rho_D(h(a_1))}\, \ell_{Q(a_2)} - \mathcal{L}_{\rho_D(h(a_2))}\, \ell_{Q(a_1)} - \langle (Q, 0),\, [h(a_1), h(a_2)]_D\rangle\,.$$

As $\rho_D(h(a_i)) \in \mathfrak{X}(B)$ is the linear vector field corresponding to ∇_{a_i}, we have

$$\mathcal{L}_{\rho_D(h(a_i))}\, \ell_{Q(a_j)} = \ell_{\nabla^*_{a_i}Q(a_j)},\ \text{for } 1 \le i \ne j \le 2.$$

Also, by (3.13) and (4.3), we get

$$\big\langle (Q, 0),\, [h(a_1), h(a_2)]_D\big\rangle = \big\langle (Q, 0),\, h([a_1, a_2]_A)\big\rangle = \ell_{Q([a_1, a_2]_A)}.$$

The formula for the component on $\wedge^2 A^*$ now follows by assembling the terms. Similarly, using that $\langle (Q, 0), \widehat{c}\rangle = \langle (Q, 0),\, [h(a_1), \widehat{c}\,]_D\rangle = 0$, we get

$$d_D(Q, 0)(h(a_1), \widehat{c}) = -\mathcal{L}_{\rho_D(\widehat{c})}\ell_{Q(a_1)} = \langle Q(a_1), -\partial(c)\rangle \circ q_B.$$

It is straightforward to check now that the component in $\wedge^2 C^*$ is zero. The proof of (4.12) is a similar computation that we leave to the reader. $\square$

4.3. **Morphisms.**

Definition 4.10. *Let $(D \to B; A \to M)$ and $(D' \to B'; A' \to M)$ be $\mathcal{VB}$-algebroids. A $\mathcal{VB}$-algebroid morphism from D to D' is a double vector bundle morphism $(F; F_{\rm ver}; F_{\rm hor})$ from D to D' such that F is a Lie algebroid morphism.*

Our aim is to relate $\mathcal{VB}$-algebroid morphisms with morphisms of representations up to homotopy. Using decompositions, it suffices to consider morphisms F between trivial double vector bundles $D = A \oplus B \oplus C$ and $D' = A' \oplus B' \oplus C'$. From Example 3.5, we know that a double vector bundle morphism $F : D \to D'$ is determined by vector bundle morphisms $F_{\rm ver} : A \to A'$, $F_{\rm hor} : B \to B'$, $F_c : C \to C'$ and $\Phi \in \Omega^1(A, \operatorname{Hom}(B, C'))$.

Theorem 4.11. *$F : D \to D'$ is a $\mathcal{VB}$-algebroid morphism if and only if $F_{\rm ver} : A \to A'$ is a Lie algebroid morphism and $(F_c, F_{\rm hor}, \Phi)$ are the components of a morphism $(A, V) \Rightarrow (A', W)$ over $F_{\rm ver}$ between the associated representations up to homotopy $V = C_{[0]} \oplus B_{[1]} \in \mathbb{R}{\rm ep}^2(A)$ and $W = C'_{[0]} \oplus B'_{[1]} \in \mathbb{R}{\rm ep}^2(A')$.*

Remark 4.12. *Combining the results in* [6] *with Theorem 4.11 we conclude that the category of* 2*-term representations up to homotopy of a Lie algebroid A is equivalent to the category of $\mathcal{VB}$-algebroids with side algebroid A.*

In the following, let d_D and $d_{D'}$ be the Lie algebroid differentials of $D \to B$ and $D' \to B'$, respectively. Recall that $F : D \to D'$ is a Lie algebroid morphism if and only if the associated exterior algebra morphism, $F^* : \Gamma(\wedge^\bullet {D'}^*_{B'}) \to \Gamma(\wedge^\bullet D^*_B)$, intertwines d_D and $d_{D'}$[1]. Theorem 4.11 will follow from the thorough study of the relation $F^* \circ d_{D'} = d_D \circ F^*$, which we carry on in Lemmas 4.14 and 4.15 below. First, we shall need a Lemma which gives useful formulas for F^* in degree 1, $F^* : \Gamma({D'}^*_{B'}) \to \Gamma(D^*_B)$.

Lemma 4.13. *Choose $Q \in \Gamma(B'^* \otimes A'^*)$, $\gamma \in \Gamma(C'^*)$ and $\psi \in \Gamma(A'^*)$. Then $F^* \widehat{\psi} = \widehat{F^*_{\rm ver}\psi}$ and $F^*(Q, \gamma) = (F^*_{\rm ver} \otimes F^*_{\rm hor}(Q) + \langle \Phi, \gamma \rangle,\ F^*_c \gamma)$.*

Proof. The result follows directly from Example 3.5. We leave the details to the reader. $\square$

Let now $(\partial_W, \nabla^W, K_W)$ and $(\partial_V, \nabla^V, K_V)$ be the structure operators of the representations up to homotopy of A on $V = C_{[0]} \oplus B_{[1]}$ and of A' on $W = C'_{[0]} \oplus B'_{[1]}$, respectively, and let $\nabla^{\rm Hom}$ be the A-connection on $\underline{\operatorname{Hom}}(V, W)$ obtained from ∇^V and $(\nabla^W)^{F_{\rm ver}}$ (see (2.7)).

Lemma 4.14. *$F^* \circ d_{D'} = d_D \circ F^*$ holds on $\Gamma(\wedge^0 {D'}^*_{B'}) = C^\infty(B')$ if and only if*

$$\rho_{A'} \circ F_{\rm ver} = \rho_A, \tag{4.13}$$

$$F_{\rm hor} \circ \partial_V = \partial_W \circ F_c \tag{4.14}$$

and

$$\nabla^{\rm Hom}_a F_{\rm hor} = \partial_W \circ \Phi_a,\ \forall\, a \in \Gamma(A). \tag{4.15}$$

[1]Recall that F^* is defined by $(F^*\omega)(b)(d_1(b), \ldots, d_n(b)) = \omega_{F_{\rm hor}(b)}(F(d_1(b)), \ldots, F(d_n(b)))$ for $\omega \in \Gamma(\wedge^n {D'}^*_{B'})$, $b \in B$ and $d_1, \ldots, d_n \in \Gamma(B, D)$. In particular, we have $F^*(g) = g \circ F_{\rm hor}$, for $g \in C^\infty(B) = \Gamma(\wedge^0 {D'}^*_{B'})$.

Proof. It suffices to consider $f \circ q_{B'}$, for $f \in C^\infty(M)$ and linear functions ℓ_β, for $\beta \in \Gamma(B'^*)$. Now, (4.13) follows directly from (4.7). For the other two equations, first observe that $F^*(\ell_\beta) = \ell_{F^*_{\rm hor}\beta}$. The identity

$$d_D(\ell_{F^*_{\rm hor}\beta}) = (d_{\nabla^{V^*}} F^*_{\rm hor}\,\beta, (F_{\rm hor} \circ \partial_V)^*\beta) \in \Gamma(A^* \otimes B^*) \oplus \Gamma(C^*)$$

follows from (4.8). On the other hand, due to (4.8) and Lemma 4.13, we have $F^*(d_D(\ell_\beta)) = (Q, \gamma)$, where

$$Q = F^*_{\rm ver} \otimes F^*_{\rm hor}\,(d_{\nabla^{W^*}}\beta) + \langle \Phi, \partial^*_W \beta\rangle$$
$$\gamma = (\partial_W \circ F_c)^*\beta.$$

By comparing the components in $\Gamma(C^*)$ and $\Gamma(A^* \otimes B^*)$, one finds equations which are dual to (4.14) and (4.15), respectively. This proves the lemma. □

Lemma 4.15. *$F^* \circ d_{D'} = d_D \circ F^*$ holds on $\Gamma(\wedge^1 D'^*_{B'})$ if and only if*

$$d_A \circ F^*_{\rm ver} = F^*_{\rm ver} \circ d_{A'} \ \text{ in } \Gamma(A'^*), \tag{4.16}$$

$$\nabla^{\rm Hom}_a F_c = \Phi_a \circ \partial_V,\ \forall\, a \in \Gamma(A) \tag{4.17}$$

and

$$d_{\nabla^{\rm Hom}}\Phi = F_c \circ K_V - (F^*_{\rm ver} K_W) \circ F_{\rm hor}. \tag{4.18}$$

Proof. It suffices to consider core sections $\widehat{\psi}$ and linear sections of the type $(0, \gamma)$, where $\gamma \in \Gamma(C'^*)$ and $\psi \in \Gamma(A'^*)$. Equation (4.16) is equivalent to $F^* \circ d_{D'}(\widehat{\psi}) = d_D \circ F^*(\widehat{\psi})$. Now, according to the decomposition (4.9), we find $F^* \circ d_{D'}(0, \gamma) = (\Lambda_1, \Lambda_2, \Lambda_3)$, where Λ_3 is the zero section of $\wedge^2 q^!_B C^*$,

$$\Lambda_1(b_m) = -\langle\, (F^*_{\rm ver} K_W)^*\gamma, F_{\rm hor}(b_m)\,\rangle + (F^*_{\rm ver} \wedge \Phi^*(b_m))(d_{\nabla^{W^*}}\gamma(m)),$$

and

$$\Lambda_2(b_m) = F^*_{\rm ver} \otimes F^*_c\,(d_{\nabla^{W^*}}\gamma(m)),$$

where $m \in M$, $b_m \in B_m$ and $\Phi(b_m)$ is seen as a map from A to C' with dual $\Phi^*(b_m) : C'^* \to A^*$. Similarly, by Lemma 4.13 and formulas (4.11), (4.12), it follows that $d_D(F^*(0, \gamma)) = d_D(\langle \Phi, \gamma\rangle,\, F^*_c\gamma) = (\Theta_1, \Theta_2, \Theta_3)$, where Θ_3 is again the zero section of $\wedge^2 q^!_B C^*$,

$$\Theta_1(b_m) = \langle d_{\nabla^{V^*}}\langle \Phi, \gamma\rangle,\, b_m\rangle - \langle K^*_V(F^*_c\gamma), b_m\rangle.$$

and

$$\Theta_2(b_m) = -({\rm id}^*_A \otimes \partial^*_V)\langle \Phi, \gamma(m)\rangle + d_{\nabla^{V^*}} F^*_c \gamma(m).$$

The equalities $\Lambda_2 = \Theta_2$ and $\Lambda_1 = \Theta_1$ are equivalent to the equations dual to (4.17) an (4.18), respectively. □

Proof of Theorem 4.11. Equations (4.13) and (4.16) are equivalent to $F_{\rm ver}$ being a Lie algebroid morphism. Similarly, equations (4.14), (4.15), (4.17) and (4.18) are equivalent to $(F_c, F_{\rm hor}, \Phi)$ being the components of a morphism $(A, V) \Rightarrow (A', W)$. This proves the Theorem. □

Example 4.16. *Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over M. An IM-2-form* [4] *on A is a pair (μ, ν) where $\mu : A \to T^*M$ and $\nu : A \to \wedge^2 T^*M$ such that*

(1) $\langle \mu(a), \rho_A(b)\rangle = -\langle \mu(b), \rho_A(a)\rangle$;
(2) $\mu([a, b]) = \mathcal{L}_{\rho_A(a)}\mu(b) - i_{\rho_A(b)}(d\mu(a) + \nu(a))$;
(3) $\nu([a, b]) = \mathcal{L}_{\rho_A(a)}\nu(b) - i_{\rho_A(b)}d\nu(a)$,

for $a, b \in \Gamma(A)$. In [2] *(see also* [3] *for the case where $\nu = 0$), it is shown that every IM-2-form is associated to a 2-form $\Lambda \in \Omega^2(A)$ whose associated map $\Lambda_\sharp : TA \to T^*A$ is a $\mathcal{VB}$-algebroid morphism from $(TA \to TM; A \to M)$ to $(T^*A \to A^*; A \to M)$ inducing $\mu : A \to T^*M$ on the core bundles and $-\mu^* : TM \to A^*$ on the side bundles. Let $\sigma \in \mathrm{Dec}(TA)$ and σ_A^* be its dual over the side bundle A. From* [2] *(see Lemma 3.6 there), it follows that $F = \sigma_A^* \circ \Lambda_\sharp \circ \sigma : A \oplus TM \oplus A \to A \oplus A^* \oplus T^*M$ has components given by $F_{\mathrm{ver}} = \mathrm{id}_A$, $F_{\mathrm{hor}} = -\mu^*$, $F_c = \mu$ and*

$$\Phi = \nu + d_{\nabla^*}\mu^* \in \Omega^1(A; \mathrm{Hom}(TM, T^*M)).$$

where ∇ is the connection on A associated to σ and $d_{\nabla^*} : \Omega(TM; A^*) \to \Omega(TM; A^*)$ the Koszul differential associated to the dual connection. Note that we are identifying $\Omega^2(TM, A^*)$ with $\Omega^1(A; \wedge^2 T^*M)$ and seeing $\wedge^2 T^*M$ as a subset of $\mathrm{Hom}(TM, T^*M)$. So, as a result of Theorem 4.11, one has that (μ, ν) is an IM-2-form if and only if $(\mu, -\mu^*, \nu + d_{\nabla^*}\mu^*)$ are the components of a morphism from the adjoint representation $ad_\nabla(A)$ to the coadjoint representation $ad_\nabla^\top(A)$.

Example 4.17. *Let $(A, [\cdot,\cdot], \rho_A)$ be a Lie algebroid such that its dual A^* has also a Lie algebroid structure $(A^*, [\cdot,\cdot]_{A^*}, \rho_{A^*})$. It is shown in* [12] *that (A, A^*) is a Lie bialgebroid if and only if $\pi_A^\sharp : T^*A \to TA$ is a $\mathcal{VB}$-algebroid morphism from the cotangent Lie algebroid $(T^*A \to A^*, A \to M)$ to the tangent prolongation $(TA \to TM; A \to M)$, where $\pi_A \in \Gamma(\wedge^2 TA)$ is the linear Poisson bivector corresponding to the Lie algebroid A^*. For any decomposition $\sigma \in \mathrm{Dec}(TA)$, it follows from* [12] *(see Corollary 6.5 there) that $\sigma^{-1} \circ \pi_A^\sharp \circ (\sigma_A^*)^{-1} : A \oplus A^* \oplus T^*M \to A \oplus TM \oplus A$ has components given by $F_{\mathrm{ver}} = \mathrm{id}_A$, $F_{\mathrm{hor}} = \rho_{A^*}$, $F_c = -\rho_{A^*}^*$ and $\Phi \in \Omega^1(A; \mathrm{Hom}(A^*, A))$ defined by*

$$\langle \Phi_a(\alpha), \beta\rangle = -d_{A^*}a(\alpha, \beta) + \langle \beta, \nabla_{\rho_{A^*}(\alpha)} a\rangle - \langle \alpha, \nabla_{\rho_{A^*}(\beta)} a\rangle, \ (\alpha, \beta) \in A^* \times_M A^*,$$

where $d_{A^*} : \Gamma(\wedge A) \to \Gamma(\wedge A)$ is the Lie algebroid differential of A^*, $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ is the connection corresponding to σ and σ_A^* is the dual of σ over A. Note that

$$\langle \Phi_a(\alpha), \beta)\rangle = \langle a, T^{\mathrm{bas}}(\alpha, \beta)\rangle,$$

where T^{bas} is the torsion of the basic connection

$$\begin{array}{ccc} \Gamma(A^*) \times \Gamma(A^*) & \longrightarrow & \Gamma(A^*) \\ (\alpha, \beta) & \longmapsto & [\alpha, \beta]_{A^*} + \nabla^*_{\rho_{A^*}(\beta)}\alpha. \end{array}$$

As a result of Theorem 4.11, we have that (A, A^*) is a Lie bialgebroid if and only if $(-\rho_{A^*}^*, \rho_{A^*}, T^{\mathrm{bas}})$ are the components of a morphism from the coadjoint representation $\mathrm{ad}_\nabla^\top$ to the adjoint representation ad_∇.

5. Distributions and foliations.

Let $q_B : B \to M$ be a vector bundle. A *linear distribution* on B is a subbundle $\Delta \subset TB$ such that

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, "Tq_B"'] & B \arrow[d, "q_B"] \\ \Delta_M \arrow[r] & M \end{tikzcd} \tag{5.1}$$

is a double vector bundle. It is called a *linear foliation* if Δ is integrable (or, equivalently, $\Delta \to B$ is a Lie subalgebroid of $TB \to B$). Linear distributions and foliations are particular examples of double vector subbundles and $\mathcal{VB}$-subalgebroids, respectively. In this section we develop the general theory of these objects. Our goal is to identify invariants of distributions and foliations on Lie algebroids.

5.1. **Double vector subbundles and adapted decompositions.**

Definition 5.1. *Let $(D'; A', B'; M)$ be a double vector bundle. We say that $(D; A, B; M)$ is a double vector subbundle of D' if*

(1) $(D; A; B; M)$ is a double vector bundle;
(2) $D \subset D'$; $A \subset A'$ and $B \subset B'$ are subbundles;
(3) the inclusion $i : D \hookrightarrow D'$ is a morphism of double vector bundles inducing the inclusions $i_A : A \hookrightarrow A'$ and $i_B : B \hookrightarrow B'$ on the side bundles.

Note that the core C' of D' is a subbundle of C and the map $i : D \hookrightarrow D'$ induces the inclusion $i_C : C' \hookrightarrow C$ on the core bundles.

Example 5.2. *Let $D' = A' \oplus B' \oplus C'$ be the trivial double vector bundle with core C'. Given vector subbundles $A \subset A'$, $B \subset B'$ and $C \subset C'$, the trivial double vector bundle $D = A \oplus B \oplus C$ with core C is canonically a double vector subbundle of D'.*

The inclusion of trivial double vector bundles of Example 5.2 should be seen as a model for general double vector subbundles. Let us be more precise.

Definition 5.3. *Let $(D; A, B; M)$ be a double vector sub-bundle of $(D'; A', B'; M)$. We say that a decomposition $\sigma' : A' \oplus B' \oplus C' \to D'$ is adapted to D if $\sigma'(A \oplus B \oplus C) = D$. In this case, the induced decomposition $\sigma := \sigma'|_{A\oplus B\oplus C}$ of D makes the diagram below commutative*

$$
\begin{tikzcd}
A' \oplus B' \oplus C' \arrow[r, "\sigma'"] & D' \\
A \oplus B \oplus C \arrow[r, "\sigma"'] \arrow[u, hook] & D \arrow[u, hook, "i"']
\end{tikzcd}
\tag{5.2}
$$

where the left vertical arrow is the canonical inclusion of Example 5.2

A horizontal lift $h : A' \to \widehat{A'}$ is adapted to D if its corresponding decomposition σ_h (3.7) is adapted to D. Equivalently, h is adapted to D if and only if, for $a \in \Gamma(A)$, the linear section $h(a) : B' \to D'$ satisfies $h(a)(B) \subset D$. In this case, $h|_A$ is a horizontal lift for D.

Example 5.4. *A connection $\nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)$ is adapted to a linear distribution Δ if for every $x \in \Gamma(TM)$ the linear vector field $X_{\nabla_x} : B \to TB$ corresponding to the derivation $\nabla_x : \Gamma(B) \to \Gamma(B)$ is a section of the distribution Δ.*

Recall that $\mathrm{Dec}(D')$, the space of decompositions of D', is affine modelled over $\Gamma(A'^* \otimes B'^* \otimes C')$. Define

$$\Gamma_{A,B,C} = \{\Phi \in \Gamma(A'^* \otimes B'^* \otimes C') \mid \Phi_a(B) \subset C, \forall\, a \in A\}. \tag{5.3}$$

Proposition 5.5. *Let $(D'; A', B'; M)$ be a double vector bundle and A, B and C vector subbundles of A', B' and C', respectively. There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Double vector subbundles}\,(D; A, B; M) \textit{ of } D' \\ \textit{having } C \textit{ as core bundle.} \end{array} \right\} \xleftrightarrow{1-1} \frac{\mathrm{Dec}(D')}{\Gamma_{A,B,C}}.$$

More precisely, for a double vector subbundle $(D; A, B; M)$, the set of decompositions adapted to D is an orbit for $\Gamma_{A,B,C}$. Reciprocally, given a decomposition $\sigma' \in \mathrm{Dec}(D')$, the double vector subbundle $D = \sigma'(A \oplus B \oplus C)$ only depends on the $\Gamma_{A,B,C}$-orbit of σ' and any decomposition in this orbit is adapted to D.

Proof. For a double vector subbundle $(D; A, B; M)$, we shall first prove that decompositions adapted to D always exist and then that they form an orbit for $\Gamma_{A,B,C}$. Begin with two arbitrary decompositions $\sigma : A \oplus B \oplus C \to D$ and $\sigma' : A' \oplus B' \oplus C' \to D'$ and consider $\sigma'^{-1} \circ i \circ \sigma : A \oplus B \oplus C \to A' \oplus B' \oplus C'$. It is a morphism between trivial double vector bundles inducing the inclusions on A, B and C. Hence, there exists $\Phi \in \Gamma(A^* \otimes B^* \otimes C')$ such that

$$\sigma'^{-1} \circ i \circ \sigma(a, b, c) = (a, b, c + \Phi_a(b)).$$

For any $\Phi' \in \Gamma(A'^* \otimes B'^* \otimes C')$ extending Φ (i.e. $\Phi'_a(b) = \Phi_a(b)$, $\forall\, a \in A, b \in B$), the decomposition $\Phi' \cdot \sigma'$ is adapted to D. Now, if σ_1, σ_2 are arbitrary decompositions, there exists an unique $\Phi \in \Gamma(A'^* \otimes B'^* \otimes C')$ such that $\sigma_1 = \Phi \cdot \sigma_2$. It is straightforward to check that they lie in the same orbit if and only if $I_\Phi = \sigma_2^{-1} \circ \sigma_1$ (see (3.5)) preserves $A \oplus B \oplus C$. This implies that the decompositions adapted to D is an $\Gamma_{A,B,C}$-orbit and that the map

$$\frac{\mathrm{Dec}(D')}{\Gamma_{A,B,C}} \ni [\sigma'] \mapsto \sigma'(A \oplus B \oplus C)$$

is well-defined. □

Now we will use Proposition 5.5 to classify linear distributions. Let $(\Delta; \Delta_M, B; M)$ be a linear distribution with core $C \subset B$ and consider the quotient map $\pi : B \to B/C$. In the following, we identify $\mathrm{Dec}(TB)$ with the space of connections $\nabla : \Gamma(B) \to \Gamma(T^*M \otimes B)$.

Lemma 5.6. *The map*

$$\begin{array}{ccc} \dfrac{\mathrm{Dec}(TB)}{\Gamma_{\Delta_M,B,C}} & \longrightarrow & \left\{ \mathbb{D} : \Gamma(B) \to \Gamma(\Delta_M^* \otimes (B/C)) \;\middle|\; \begin{array}{l} \mathbb{D}_x(fb) = f\mathbb{D}_x(b) + (\mathcal{L}_x f)\pi(b), \\ \forall\, f \in C^\infty(M),\, b \in \Gamma(B),\, x \in \Gamma(\Delta_M) \end{array} \right\} \\ [\nabla] & \longmapsto & (r \otimes \pi) \circ \nabla \end{array}$$

is a bijection, where $r : T^*M \to \Delta_M^*$ is the restriction map.

Proof. The fact that the map is well-defined follows directly from the definition of the affine action (3.6) on the space of connections. Let us now prove that given $\mathbb{D} : \Gamma(B) \to \Gamma(\Delta_M^* \otimes (B/C))$ satisfying the Leibniz equation

$$\mathbb{D}_x(fb) = f\mathbb{D}_x(b) + (\mathcal{L}_x f)\pi(b), \tag{5.4}$$

there exists a connection $\nabla \in \mathrm{Dec}(TB)$ such that $\mathbb{D} = (r \otimes \pi) \circ \nabla$. For this, let $s : B/C \to B$ be a linear section for the quotient projection $\pi : B \to B/C$ and identify B with $(B/C) \oplus C$ using s. Also, choose a connection $\widetilde{\nabla} : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)$ which preserves both B and B/C.

First, note that formula (5.4) implies that, for $x \in \Gamma(\Delta_M)$, $\mathbb{D}_x : \Gamma(B) \to \Gamma(B/C)$ is actually linear when restricted to $\Gamma(C)$. Second, note that (5.4) implies that the map

$$\begin{array}{rccc} \mathbb{D}_x \circ s : & \Gamma(B/C) & \longrightarrow & \Gamma(B/C) \\ & \gamma & \longmapsto & \mathbb{D}_x(s(\gamma)) \end{array}$$

is a derivation on B/C having $x \in \Gamma(\Delta_M)$ as symbol. So, define $\widetilde{\Phi} \in \Gamma(T^*M \otimes B^* \otimes B)$

$$\widetilde{\Phi}_x(\gamma) = \begin{cases} \begin{cases} s \circ \mathbb{D}_x(\gamma), & \text{if } \gamma \in C; \\ (\mathbb{D}_x \circ s - \widetilde{\nabla}_x)(\gamma), & \text{if } \gamma \in B/C; \end{cases} & \text{if } x \in \Gamma(\Delta_M); \\ 0, & \text{otherwise.} \end{cases}$$

It is now straightforward to check that the connection $\nabla : \Gamma(B) \to \Gamma(T^*M \otimes B)$ given by $\nabla = \widetilde{\nabla} + \widetilde{\Phi}$ satisfies $\mathbb{D} = (r \otimes \pi) \circ \nabla$. □

For a linear distribution $(\Delta; \Delta_M, B; M)$ with core C, one can canonically construct a map $\mathbb{D}^\Delta : \Gamma(B) \to \Gamma(\Delta_M^* \otimes (B/C))$ satisfying (5.4) as follows:

$$\mathbb{D}^\Delta_x(b) = \pi \circ L_X(b),\ x \in \Gamma(\Delta_M),\ b \in \Gamma(B), \tag{5.5}$$

where $X : B \to \Delta$ is any linear section covering x, $L_X : \Gamma(B) \to \Gamma(B)$ is the derivation defined by

$$L_X(b)^\uparrow = [X, b^\uparrow],\ \text{for } b \in \Gamma(B). \tag{5.6}$$

It is straightforward to check that $\pi \circ L_X$ depends only on x and not on the particular choice of $X : B \to \Delta$.

Theorem 5.7. *A connection $\nabla : \Gamma(B) \to \Gamma(T^*M \otimes B)$ is adapted to Δ if and only if*

$$(r \otimes \pi) \circ \nabla = \mathbb{D}^\Delta. \tag{5.7}$$

In particular, the map $\Delta \mapsto \mathbb{D}^\Delta$ establishes a one-to-one correspondence between linear distributions $(\Delta; \Delta_M, B; M)$ with core C and $\mathbb{R}$-linear maps $\mathbb{D} : \Gamma(B) \to \Gamma(\Delta_M^* \otimes (B/C))$ satisfying the Leibniz equation (5.4).

Proof. If ∇ is adapted to Δ, then the linear vector field $X_{\nabla_x} : B \to TB$ corresponding to the derivation $\nabla_x : \Gamma(B) \to \Gamma(B)$ is a linear section of Δ. Hence, it follows from the definition (5.5) that

$$\mathbb{D}^\Delta_x = \pi \circ L_{X_{\nabla_x}} = \pi \circ \nabla_x,$$

for $x \in \Gamma(\Delta_M)$. On the other hand, if (5.7) holds, then, for every $x \in \Gamma(\Delta_M)$, there exists a linear section $X : B \to \Delta$ covering x such that $\delta := \nabla_x - L_X \in \mathrm{Hom}(B, C)$, where L_X is the derivation defined by (5.6). In terms of sections,

$$X_{\nabla_x} = X + \delta^\uparrow.$$

As C is the core bundle of Δ, one gets that X_{∇_x} is a section of Δ and, therefore, ∇ is adapted to Δ. The last statement follows from Lemma (5.6). □

Remark 5.8. If one considers the Lie groupoid $B \rightrightarrows M$, where the source and the target are the projection $q_B : B \to M$ and the multiplication is addition on the fibers, then a linear distribution is just a multiplicative distribution in the sense of [5, 9]. In this situation, the one-to-one correspondence above was also obtained in [5] for $\Delta_M = TM$. The map $\mathbb{D}^\Delta$ is called by the authors the *Spencer operator relative to π*.

5.2. **Infinitesimal ideal systems, distributions on Lie algebroid and subrepresentations.** Let us recall the definition of an infinitesimal ideal system [7, 9] on a Lie algebroid $(A, [\cdot,\cdot]_A, \rho_A)$.

Definition 5.9. *An infinitesimal ideal system on A is a triple $(\Delta_M, C, \widetilde{\nabla})$, where $C \subset A$ is a subalgebroid, $\Delta_M \subset TM$ is a integrable distribution and $\widetilde{\nabla} : \Gamma(\Delta_M) \times \Gamma(A/C) \to \Gamma(A/C)$ is a flat connection satisfying the following properties:*

(1) if $\pi(a)$ is parallel, then $[a, \Gamma(C)]_A \subset \Gamma(C)$.
(2) if $\pi(a), \pi(b)$ are parallel, then $\pi([a, b]_A)$ is parallel;
(3) if $\pi(a)$ is parallel, then $[\rho(a), \Gamma(\Delta_M)] \subset \Gamma(\Delta_M)$;

where $a, b \in \Gamma(A)$ and $\pi : A \to A/C$ is the quotient map.

Given an infinitesimal ideal system $(\Delta_M, C, \widetilde{\nabla})$ on A, it follows from Theorem 5.7 that there exists an associated linear distribution $(\Delta; \Delta_M, A; M)$ on A having core C corresponding to an operator $\mathbb{D} : \Gamma(A) \to \Gamma(\Delta_M^* \otimes (A/C))$ defined as zero on C and equal to $\widetilde{\nabla}$ on the quotient A/C. The paper [9] shows that the properties of an infinitesimal ideal system are equivalent to $\Delta \to \Delta_M$ and $\Delta \to A$ being Lie subalgebroids of $TA \to TM$ and $TA \to A$, respectively. In this section, we prove this in an alternative manner, using representations up to homotopy. Along the way, we shall understand necessary and sufficient conditions on $\mathbb{D}$ for $\Delta \to \Delta_M$ to be a Lie subalgebroid of $TA \to TM$ and for and $\Delta \to A$ to be a Lie subalgebroid of $TA \to A$.

Let us start with the definition of $\mathcal{VB}$-subalgebroids.

Definition 5.10. *Let $(D' \to B'; A' \to M)$ be a $\mathcal{VB}$-algebroid. We say that a double vector subbundle $(D; A, B; M)$ is a $\mathcal{VB}$-subalgebroid of D' if $D \to B$ is a Lie subalgebroid of $D' \to B'$.*

Proposition 5.11. *A double vector subbundle $(D; A, B; M)$ is a $\mathcal{VB}$-subalgebroid of D' if and only if*

(1) $(D \to B; A \to M)$ is a $\mathcal{VB}$-algebroid;
(2) the inclusion map $i : (D \to B; A \to M) \hookrightarrow (D' \to B'; A' \to M)$ is a $\mathcal{VB}$-algebroid morphism.

Proof. It is straightforward to see that conditions (1) and (2) imply that D is a $\mathcal{VB}$-subalgebroid of D'. Conversely, assume that $D \to B$ is a Lie subalgebroid of $D' \to B'$. The fact that the inclusion $i : D \to D'$ is a bundle morphism over $i_A : A \to A'$ implies that the anchor of D, $\rho_D = \rho_{D'} \circ i$, is a bundle morphism over $\rho_A = \rho_{A'} \circ i_A$. To prove that $(D \to B; A \to M)$ is a $\mathcal{VB}$-algebroid, we still have to check conditions (i), (ii) and (iii) of Definition 4.1. These will follow from exactly the same conditions on D' if we prove that core (respectively linear) sections of D can be extended to core (respectively linear) sections of D'. Now, (3.1) implies that

$$\Gamma_c(B, D) = \{\widehat{c}|_B \mid c \in \Gamma(C) \text{ and } \widehat{c} \in \Gamma_c(B', D')\}.$$

Also, for $\mathcal{X} : B \to D$, a linear section of D covering $a \in \Gamma(A)$, choose any horizontal lift $h' : A' \to \widehat{A'}$ adapted to D (the existence of which is guaranteed by Proposition 5.5). For $b \in B$,

$$\mathcal{X}(b) \underset{B}{-} h(a)(b) = \widetilde{0}_b^B \underset{A}{+} \overline{\Phi_a(b)},$$

for some $\Phi \in \Omega^1(A, \mathrm{Hom}(B, C))$. Extend Φ to $\Phi' \in \Omega(A', \mathrm{Hom}(B', C'))$. The horizontal lift $h'' = h' + \Phi'$ is still adapted to D and $\mathcal{X} = h''(a)|_B$. $\square$

Definition 5.12. *Let $W \in \mathbb{R}\mathrm{ep}^2(A)$ be a 2-term representation of a Lie algebroid A and let (∂, ∇, K) be its structure operators. We say that a (graded) subbundle $V \subset W$ is a subrepresentation if $V \in \mathbb{R}\mathrm{ep}^2(A)$ and $(i_{V_0}, i_{V_1}, 0)$ are the components of a morphism $(A, V) \Rightarrow (A, W)$, where $i_{V_0} : V_0 \hookrightarrow W_0$ and $i_{V_1} : V_1 \hookrightarrow W_1$ are the inclusions and $0 \in \Gamma(A^* \otimes V_1^* \otimes W_0)$.*

Remark 5.13. If (∂, ∇, K) are the structure operators for $W \in \mathbb{R}\mathrm{ep}^2(A)$, then $V \subset W$ is a subrepresentation if and only if

$$\partial(V_0) \subset V_1; \tag{5.8}$$

$$\nabla_a \text{ preserves } V, \forall\, a \in \Gamma(A) \tag{5.9}$$

$$K(a_1, a_2)(V_1) \subset V_0, \forall\, a_1,\, a_2 \in A. \tag{5.10}$$

In this case, the restrictions $(\partial|_{V_0}, \nabla|_V, K|_{V_1})$ are the structure operators for the representation up to homotopy of A on V. This follows directly from equations (2.2), (2.3) and (2.4).

Let us give an example.

Example 5.14. *Let $\nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)$ be a connection on B and consider the double representation $\mathcal{D}_\nabla(B) \in \mathbb{R}\mathrm{ep}^2(TM)$ (see Example 2.4). Given a vector subbundle $C \subset B$, the graded vector bundle $C_{[0]} \oplus B_{[1]}$ is a subrepresentation if and only if ∇ preserves C and the induced connection on the quotient $\widehat{\nabla} : \Gamma(TM) \times \Gamma(B/C) \to \Gamma(B/C)$ is flat.*

The following result shows how $\mathcal{VB}$-subalgebroids and subrepresentations are related.

Theorem 5.15. *Let $W = C'_{[0]} \oplus B'_{[1]} \in \mathbb{R}\mathrm{ep}^2(A')$ be the representation up to homotopy corresponding to a $\mathcal{VB}$-algebroid structure on $(A' \oplus B' \oplus C' \to B'; A' \to M)$. Then $(A \oplus B \oplus C \to B; A \to M)$ is a $\mathcal{VB}$-subalgebroid if and only if $A \subset A'$ is a subalgebroid and $C_{[0]} \oplus B_{[1]}$ is a subrepresentation of $i_A^! W \in \mathbb{R}\mathrm{ep}^2(A)$, where $i_A : A \hookrightarrow A'$ is the inclusion.*

Proof. Assume $A \oplus B \oplus C$ is a $\mathcal{VB}$-subalgebroid of D'. By Proposition (5.11), it follows that $(A \oplus B \oplus C \to B; A \to M)$ is a $\mathcal{VB}$-algebroid, so there is a corresponding Lie algebroid structure on A and $V = C_{[0]} \oplus B_{[1]} \in \mathbb{R}\mathrm{ep}^2(A)$. As the inclusion $i : A \oplus B \oplus C \hookrightarrow A' \oplus B' \oplus C'$ is a $\mathcal{VB}$-morphism, it follows from Theorem 4.11 that the inclusion $i_A : A \to A'$ is a Lie algebroid morphism and $(i_C, i_B, 0)$ are the components of a morphism $(A, V) \Rightarrow (A', W)$ over the inclusion i_A. Conversely, assume that $A \subset A'$ is a subalgebroid and that V is a subrepresentation of $i_A^! W$. The representation up to homotopy of A on V give $(A \oplus B \oplus C \to B; A \to M)$ a $\mathcal{VB}$-algebroid structure and one can use Theorem (4.11) once again to prove that the inclusion $i : A \oplus B \oplus C \to A' \oplus B' \oplus C'$ is a $\mathcal{VB}$-morphism. This concludes the proof. □

Corollary 5.16. [5, 9] *A linear distribution $(\Delta; \Delta_M, B; M)$ on B with core bundle C is involutive if and only if Δ_M is involutive and the associated map $\mathbb{D}^\Delta : \Gamma(B) \to \Gamma(\Delta_M^* \otimes (B/C))$ satisfies*

1. $\mathbb{D}^\Delta|_{\Gamma(C)} = 0$;
2. *the map induced on the quotient $\Gamma(B/C) \to \Gamma(\Delta_M^* \otimes (B/C))$ is a flat Δ_M-connection on B/C.*

Proof. Δ is involutive if and only if $(\Delta; \Delta_M; A; M)$ is a $\mathcal{VB}$-subalgebroid of the double $(TA \to A; TM \to M)$. Choose any connection $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ with

$$\pi \circ \nabla_x = \mathbb{D}_x^{\Delta},\ \forall\, x \in \Gamma(\Delta_M)$$

and consider the double representation $\mathcal{D}_\nabla(B) \in \mathbb{R}\mathrm{ep}^2(TM)$. It is the representation up to homotopy of TM associated to the $\mathcal{VB}$-algebroid $(TB \to B; TM \to M)$ decomposed by the choice of ∇. As ∇ is adapted to Δ (see Theorem 5.7), it follows from Theorem 5.15 that Δ is involutive if and only if Δ_M is involutive and $C_{[0]} \oplus B_{[1]}$ is a subrepresentation of $i^!_{\Delta_M}\mathcal{D}_\nabla \in \mathbb{R}\mathrm{ep}^2(\Delta_M)$, where $i_{\Delta_m} : \Delta_M \hookrightarrow TM$ is the inclusion. The result now follows from Example 5.14. $\square$

Compatibility with the Lie algebroid structure. Let $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ be a connection on the Lie algebroid A and consider the adjoint representation ad_∇. The graded subbundle $C_{[0]} \oplus \Delta_{M,[1]} \subset A_{[0]} \oplus TM_{[1]}$ is a subrepresentation of ad_∇ if and only if $\rho_A(C) \subset \Delta_M$ and

$$[a, c] + \nabla_{\rho(c)}\, a \in \Gamma(C); \tag{5.11}$$

$$[\rho_A(a), x] + \rho_A(\nabla_x a) \in \Gamma(\Delta_M); \tag{5.12}$$

$$R_{bas}(a, b)(\Delta_M) \subset C; \tag{5.13}$$

for $a, b \in \Gamma(A)$, $c \in \Gamma(C)$ and $x \in \Gamma(\Delta_M)$. In the case ∇ is a connection adapted to a linear distribution $(\Delta; A, \Delta_M; M)$, equations (5.11), (5.12) and (5.13) can be reinterpreted in terms of the $\mathbb{D}^\Delta$ (5.5) to give conditions for $\Delta \to \Delta_M$ to be a Lie subalgebroid of $TA \to TM$. In the following, we shall need the quotient maps $\pi : A \to A/C$ and $\pi_{TM} : TM \to TM/\Delta_M$.

Theorem 5.17. *Let $(\Delta; \Delta_M, A; M)$ be a linear distribution on A with core C and choose any connection $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ adapted to Δ. Then $\Delta \to \Delta_M$ is a Lie subalgebroid of $TA \to TM$ if and only if $\rho_A(C) \subset \Delta_M$,*

$$\mathbb{D}_{\rho_A(c)}(a) = -\pi([a, c]); \tag{5.14}$$

$$\widetilde{\rho_A}(\mathbb{D}_x(a)) = -\pi_{TM}([\rho_A(a), x]) \tag{5.15}$$

$$\mathbb{D}_x([a, b]_A) = \widehat{\nabla}^{\mathrm{bas}}_a \mathbb{D}_x(b) - \widehat{\nabla}^{\mathrm{bas}}_b \mathbb{D}_x(a) + \pi(\nabla_{[\rho_A(b), x]} a - \nabla_{[\rho_A(a), x]} b) \tag{5.16}$$

where $\widetilde{\rho_A} : A/C \to TM/\Delta_M$ is the quotient map (i.e. $\widetilde{\rho_A} \circ \pi = \pi_{TM} \circ \rho_A$) and where $\widehat{\nabla}^{\mathrm{bas}}$ is the A-connection on the quotient A/C given by

$$\widehat{\nabla}^{\mathrm{bas}}_a \pi(b) = \pi([a, b] + \nabla_{\rho(b)} a) = \pi(\nabla^{\mathrm{bas}}_a b)$$

for $a, b \in \Gamma(A)$, $c \in \Gamma(C)$ and $x \in \Gamma(\Delta_M)$.

Proof. As ∇ is adapted to Δ, Theorem 5.15 assures that Δ is compatible with the Lie algebroid structure if and only if $V = C_{[0]} \oplus \Delta_{M[1]}$ is a subrepresentation of $\mathrm{ad}_\nabla \in \mathbb{R}\mathrm{ep}^2(A)$. So, one is left to prove that (5.11), (5.12) and (5.13) correspond to (5.14), (5.15) and (5.16). Now, by applying the quotient maps π and π_{TM} to (5.11) and (5.12), one has

$$\left\{\begin{array}{l} [a, c] + \nabla_{\rho_A(c)}\, a \in \Gamma(C) \\ [\rho_A(a), x] + \rho_A(\nabla_x a) \in \Gamma(\Delta_M) \end{array}\right. \iff \left\{\begin{array}{l} \pi(\nabla_{\rho_A(c)} a) = -\pi([a, c]) \\ \pi_{TM}(\rho_A(\nabla_x a)) = -\pi_{TM}([\rho_A(a), x]) \end{array}\right.$$

for every $c \in \Gamma(C)$, $a \in \Gamma(A)$ and $x \in \Gamma(\Delta_M)$. The result now follows from the fact that, for any adapted connection ∇, $\mathbb{D}_x = \pi \circ \nabla_x$, $\forall\, x \in \Gamma(\Delta_M)$ and $\pi_{TM} \circ \rho_A = \widetilde{\rho}_A \circ \pi$.

At last, using the explicit expression for R^{bas}, one has that (5.13) holds if and only if

$$\pi(\nabla_x[a,b] - [\nabla_x a, b] - [a, \nabla_x b] - \nabla_{\nabla_b^{\text{bas}} x}\, a + \nabla_{\nabla_a^{\text{bas}} x}\, b) = 0, \tag{5.17}$$

for $a, b \in \Gamma(A)$ and $x \in \Gamma(\Delta_M)$. Now, note that

$$-[\nabla_x a, b] + \nabla_{\nabla_a^{\text{bas}} x} b = [b, \nabla_x a] + \nabla_{\rho_A(\nabla_x a)} b + \nabla_{[\rho_A(a), x]} b = \nabla_b^{\text{bas}} \nabla_x a + \nabla_{[\rho_A(a),x]} b$$

and similarly

$$[a, \nabla_x b] + \nabla_{\nabla_b^{\text{bas}} x} a = \nabla_a^{\text{bas}} \nabla_x b + \nabla_{[\rho_A(b),x]} a.$$

So, by definition of $\widehat{\nabla}^{\text{bas}}$, one has that (5.17) is equivalent to

$$\mathbb{D}_x([a,b]) + \widehat{\nabla}_b^{\text{bas}} \mathbb{D}_x(a) - \widehat{\nabla}_a^{\text{bas}} \mathbb{D}_x(b) + \pi(\nabla_{[\rho(a),x]} b - \nabla_{[\rho_A(b),x]} a) = 0,$$

as required. $\square$

Remark 5.18. In the particular case where $\Delta_M = TM$, one can get rid of the choice of an adapted connection. Indeed, note that $\widehat{\nabla}^{\text{bas}}$ can be alternatively given by

$$\widehat{\nabla}_a^{\text{bas}} b = \pi([a,b]) + \mathbb{D}_{\rho(b)}(a)$$

and (5.16) becomes

$$\mathbb{D}_x([a,b]_A) = \widehat{\nabla}_a^{\text{bas}} \mathbb{D}_x(b) - \widehat{\nabla}_b^{\text{bas}} \mathbb{D}_x(a) + \mathbb{D}_{[\rho_A(b),x]}(a) - \mathbb{D}_{[\rho_A(a),x]}(b).$$

In this form, Theorem 5.17 gives the infinitesimal counterpart of a result from [5] characterizing (wide) multiplicative distributions in terms of Spencer operators relative to π.

Our last result explains how infinitesimal ideal systems and representations up to homotopy are related.

Theorem 5.19. *Let A be a Lie algebroid over M. A triple $(\Delta_M, C, \widetilde{\nabla})$ is an infinitesimal ideal system if and only if*

(1) $\Delta_M \subset TM$ is integrable;
(2) $C_{[0]} \oplus \Delta_{M[1]}$ is a subrepresentation of $\mathrm{ad}_\nabla(A)$ and
(3) $C_{[0]} \oplus A_{[1]}$ is a subrepresentation of $i^!_{\Delta_M} \mathcal{D}_\nabla(A)$,

$i_{\Delta_M} : \Delta_M \hookrightarrow TM$ is the inclusion and $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ is any connection preserving C and inducing $\widetilde{\nabla}$ on the quotient.

Proof. Define $\mathbb{D} : \Gamma(A) \to \Gamma(\Delta_M^* \otimes (A/C))$ by

$$\mathbb{D}_x(a) = \widetilde{\nabla}_x \pi(a),\ a \in \Gamma(A),\ x \in \Gamma(\Delta_M).$$

By Theorem (5.7), there exists a linear distribution $(\Delta; \Delta_M, A; M)$ with core C such that $\mathbb{D} = \mathbb{D}^\Delta$ (5.5). We already know (see [9]) that $(\Delta_M, C, \widetilde{\nabla})$ is an infinitesimal ideal system if and only $\Delta \to A$ and $\Delta \to \Delta_M$ are simultaneously Lie subalgebroids of $TA \to A$ and $TA \to TM$, respectively. The result now follows directly from Theorem (5.15). $\square$

Appendix A. Dualization of $\mathcal{VB}$-algebroids and representations up to homotopy.

Let $(D;A,B;M)$ be a double vector bundle and consider its horizontal (3.10) and vertical (3.12) duals. The vector bundles $p^{\rm ver}_{C^*}\colon D^*_A\to C^*$ and $p^{\rm hor}_{C^*}\colon D^*_B\to C^*$ are dual to each other via the nondegenerate pairing $\|\cdot,\cdot\|\colon D^*_A\times_{C^*}D^*_B\to\mathbb{R}$ given by

$$\|\Theta,\Psi\| := \langle\Psi,d\rangle_B-\langle\Theta,d\rangle_A \tag{A.1}$$

where $d\in D$ is any element with $q^D_A(d)=p_A(\Theta)$ and $q^D_B(d)=p_B(\Psi)$ [10]. The pairings on the right-hand side of (A.1) are defined with respect to the fibers over B and over A, respectively. Henceforth, we identify the dual of $D^*_B\to C^*$ with $D^*_A\to C^*$ via the pairing in (A.1). Given a section $\Theta\colon C^*\to D^*_A$, we denote by $\ell^{C^*}_\Theta\in C^\infty(D^*_B)$ the function which is linear with respect to the vector bundle structure $D^*_B\to C^*$ and given by

$$\ell^{C^*}_\Theta(\Psi)=\|\Theta(p^{\rm hor}_{C^*}(\Psi)),\Psi\|.$$

In particular, for the core section $\widehat{\psi}\in\Gamma(C^*,D^*_A)$ associated to some $\psi\in\Gamma(B^*)$, one gets by choosing $d=0^D_{p_B(\Psi)}$ in (A.1):

$$\ell^{C^*}_{\widehat{\psi}}=-\ell_\psi\circ p_B. \tag{A.2}$$

Also, for $T\in\Gamma(B^*\otimes C)$,

$$\ell^{C^*}_{\widehat{T^*}}=-\ell_{\widehat{T}}, \tag{A.3}$$

where $\widehat{T}\in\Gamma_\ell(B,D)$ and $\widehat{T^*}\in\Gamma_\ell(C^*,D^*_A)$ are the linear sections (3.3) corresponding to T and its dual $T^*\in\Gamma(C^*\otimes B)$, respectively.

We shall need one more formula (which follows directly from (3.11)) for $c\in\Gamma(C)$ and the corresponding core section $\widehat{c}\in\Gamma(B,D)$, namely

$$\ell_{\widehat{c}}=\ell_c\circ p^{\rm hor}_{C^*}. \tag{A.4}$$

Assume now that $(D\to B;A\to M)$ is a $\mathcal{VB}$-algebroid. Then $(D^*_A\to C^*;A\to M)$ has a natural $\mathcal{VB}$-algebroid structure. The Lie algebroid structure on $D^*_A\to C^*$ is obtained by noticing that the linear Poisson structure on $D^*_B\to B$ associated to the Lie algebroid $D\to B$ is also linear with respect to the vector bundle structure $D^*_B\to C^*$ [2]. In particular, besides the usual formulas

$$\ell_{[\chi_1,\chi_2]_D}=\{\ell_{\chi_1},\ell_{\chi_2}\}_{D^*_B},\ \mathcal{L}_{\rho_D(\chi)}(f)\circ p_B=\{\ell_\chi,f\circ p_B\}_{D^*_B},$$

defining the Lie bracket $[\cdot,\cdot]_D$ and the anchor ρ_D on $D\to B$, for $\chi,\chi_1,\chi_2\in\Gamma(B,D)$ and $f\in C^\infty(B)$, we have

$$\ell^{C^*}_{[\Theta_1,\Theta_2]_{D^*_A}}=\{\ell^{C^*}_{\Theta_1},\ell^{C^*}_{\Theta_2}\}_{D^*_B},\ \mathcal{L}_{\rho_{D^*_A}(\Theta)}(g)\circ p^{\rm hor}_{C^*}=\{\ell^{C^*}_\Theta,g\circ p^{\rm hor}_{C^*}\}_{D^*_B},$$

defining the Lie bracket $[\cdot,\cdot]_{D^*_A}$ and the anchor $\rho_{D^*_A}$ on $D^*_A\to C^*$, for $\Theta,\Theta_1,\Theta_2\in\Gamma(C^*,D^*_A)$ and $g\in C^\infty(C^*)$. We refer to [10] (see also [6]) for more details.

[2]Let $q_E\colon E\to M$ be a vector bundle. A Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(E)$ is linear if for all $\xi,\xi'\in\Gamma(E^*)$ and $f,f'\in C^\infty(M)$, the bracket $\{\ell_\xi,\ell_{\xi'}\}$ is again linear, the bracket $\{q_E^*f,q_E^*f'\}$ vanishes and $\{\ell_\xi,q_E^*f\}$ is the pullback under q_E of a function on M. This defines a Lie algebroid $(E^*\to M,\rho,[\cdot,\cdot])$ by setting $\ell_{[\xi,\xi']}:=\{\ell_\xi,\ell_{\xi'}\}$ and $q_E^*(\mathcal{L}_{\rho(\xi)}f):=\{\ell_\xi,q_E^*f\}$. Conversely, a Lie algebroid structure on E^* defines a linear Poisson structure on E.

Our aim here is to prove that the representations up to homotopy associated to $D \to B$ and $D^*_A \to C^*$ are dual to each other. Let us first consider how horizontal lifts for $(D; A, B; M)$ and $(D^*_A; A, C^*; M)$ are related. Let $h : \Gamma(A) \to \Gamma_\ell(B, D)$ be a horizontal lift for D. There exists a corresponding horizontal lift $h^\top : \Gamma(A) \longrightarrow \Gamma_\ell(C^*, D^*_A)$ given as follows: take the decomposition $\sigma_h \in \mathrm{Dec}(D)$ associated to h by (3.7) and consider the inverse of its dual over A, $(\sigma_h)^{*^{-1}}_A \in \mathrm{Dec}(D^*_A)$. Set $h^\top$ to be the horizontal lift corresponding to $(\sigma_h)^{*^{-1}}_A$ via (3.8). It is straightforward to check that

$$\ell_{h(a)} = \ell^{C^*}_{h^\top(a)} \in C^\infty(D^*_B) \tag{A.5}$$

for every $a \in \Gamma(A)$.

Let (∂, ∇, K) be the structure operators of the representation up to homotopy $C_{[0]} \oplus B_{[1]} \in \mathbb{R}\mathrm{ep}^2(A)$ associated to (D, h) and $(\partial^{\mathrm{ver}}, \nabla^{\mathrm{ver}}, K^{\mathrm{ver}})$ be the structure operators of the representation up to homotopy $B^*_{[0]} \oplus C^*_{[1]} \in \mathbb{R}\mathrm{ep}^2(A)$ associated to $(D^*_A, h^\top)$. The next result relates the two representations.

Proposition A.1. *The structure operators* $(\partial^{\mathrm{ver}}, \nabla^{\mathrm{ver}}, K^{\mathrm{ver}})$ *coincide with the structure operators* (2.5) *of the representation* $(C_{[0]} \oplus B_{[1]})^\top \in \mathbb{R}\mathrm{ep}^2(A)$ *dual to* (∂, ∇, K).

Proof. Let $c \in \Gamma(C)$ and $\psi \in \Gamma(B^*)$. By (4.1), we have

$$\begin{aligned} \langle \partial^{\mathrm{ver}}(\psi), c\rangle \circ q_{C^*} \circ p^{\mathrm{hor}}_{C^*} &= \mathcal{L}_{\rho_{D^*_A}(\widehat{\psi})}(\ell_c) \circ p^{\mathrm{hor}}_{C^*} = \{\ell^{C^*}_{\widehat{\psi}}, \ell_c \circ p^{\mathrm{hor}}_{C^*}\}_{D^*_B} \\ &= -\{\ell_\psi \circ p_B,\, \ell_{\widehat{c}}\}_{D^*_B} = \mathcal{L}_{\rho_D(\widehat{c})}(\ell_\psi) \circ p_B \\ &= \langle \psi, \partial(c)\rangle \circ q_B \circ p_B. \end{aligned}$$

Since $q^*_C \circ p^{\mathrm{hor}}_{C^*} = q_B \circ p_B$, the equality $\partial^{\mathrm{ver}} = \partial^*$ follows.

Let us prove the relation between the A-connections. By (4.1) and (4.2) together with (A.2) and (A.5), one has that

$$\begin{aligned} \ell_{\nabla^{\mathrm{ver}}_a \psi} \circ p_B &= -\ell^{C^*}_{\widehat{\nabla^{\mathrm{ver}}_a \psi}} = -\ell^{C^*}_{[h^\top(a), \widehat{\psi}]_{D^*_A}} = -\{\ell^{C^*}_{h^\top(a)}, \ell^{C^*}_{\widehat{\psi}}\}_{D^*_B} \\ &= \{\ell_{h(a)}, \ell_\psi \circ p_B\}_{D^*_B} = \mathcal{L}_{\rho_D(h(a))}(\ell_\psi) \circ p_B = \ell_{\nabla^*_a \psi} \circ p_B. \end{aligned}$$

Similarly,

$$\begin{aligned} \ell_{\nabla^{\mathrm{ver}*}_a c} \circ p^{\mathrm{hor}}_{C^*} &= \mathcal{L}_{\rho_{D^*_A}(h^\top(a))}(\ell_c) \circ p^{\mathrm{hor}}_{C^*} = \{\ell^{C^*}_{h^\top(a)}, \ell_c \circ p^{\mathrm{hor}}_{C^*}\}_{D^*_B} = \{\ell_{h(a)}, \ell_{\widehat{c}}\}_{D^*_B} \\ &= \ell_{[h(a), \widehat{c}]_D} = \ell_{\widehat{\nabla_a c}} = \ell_{\nabla_a c} \circ p^{\mathrm{hor}}_{C^*}. \end{aligned}$$

Hence, we have verified the equality $\nabla^{\mathrm{ver}} = \nabla^*$.

It remains to compare the curvatures. For that, choose $a_1, a_2 \in \Gamma(A)$ and let $\widehat{K} \in \Gamma_\ell(B, D)$ and $\widehat{K^{\mathrm{ver}}} \in \Gamma_\ell(C^*, D^*_A)$ be the linear sections (3.3) corresponding to $K(a_1, a_2) \in \Gamma(B^* \otimes C)$ and $K^{\mathrm{ver}}(a_1, a_2) \in \Gamma(C^* \otimes B)$ respectively. First note that, by (A.5),

$$\ell^{C^*}_{[h^\top(a_1),\, h^\top(a_2)]_{D^*_A}} = \left\{\ell^{C^*}_{h^\top(a_1)}, \ell^{C^*}_{h^\top(a_2)}\right\}_{D^*_B} = \left\{\ell_{h(a_1)}, \ell_{h(a_2)}\right\}_{D^*_B} = \ell_{[h(a_1), h(a_2)]_D}$$

Therefore, by (4.3) and (A.3), we find

$$\ell^{C^*}_{\widehat{K^{\mathrm{ver}}}} = \ell^{C^*}_{[h^\top(a_1), h^\top(a_2)]_{D^*_A} - h^\top([a_1, a_2]_A)} = \ell_{[h(a_1), h(a_2)]_D - h([a_1, a_2]_A)} = \ell_{\widehat{K}} = -\ell^{C^*}_{\widehat{K^*}}$$

This proves that $K^{\mathrm{ver}} = -K^*$. $\square$

References

[1] C. Arias Abad and M. Crainic. Representations up to homotopy of Lie algebroids. *J. Reine Angew. Math.*, 663:91–126, 2012.
[2] H. Bursztyn and A. Cabrera. Multiplicative forms at the infinitesimal level. *Math. Ann.*, 353(3):663–705, 2012.
[3] H. Bursztyn, A. Cabrera, and C. Ortiz. Linear and multiplicative 2-forms. *Lett. Math. Phys.*, 90(1-3):59–83, 2009.
[4] H. Bursztyn, M. Crainic, A. Weinstein, and C. Zhu. Integration of twisted Dirac brackets. *Duke Math. J.*, 123(3):549–607, 2004.
[5] M. Crainic, M.A. Salazar, and I. Struchiner. Linearization of multiplicative forms. *arXiv:1210.2277*, 2012.
[6] A. Gracia-Saz and R. A. Mehta. Lie algebroid structures on double vector bundles and representation theory of Lie algebroids. *Adv. Math.*, 223(4):1236–1275, 2010.
[7] E. Hawkins. A groupoid approach to quantization. *J. Symplectic Geom.*, 6(1):61–125, 2008.
[8] M. Jotz. Dirac Lie groups, Dirac homogeneous spaces and the theorem of Drinfeld. *Indiana Univ. Math. J.*, 60:319–366, 2011.
[9] M. Jotz Lean and C. Ortiz. Foliated groupoids and infinitesimal ideal systems. *Indag. Math. (N.S.)*, 25(5):1019–1053, 2014.
[10] K. C. H. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
[11] K. C. H. Mackenzie. Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids. *J. Reine Angew. Math.*, 658:193–245, 2011.
[12] K. C. H. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.
[13] C. Ortiz. Multiplicative Dirac structures on Lie groups. *C. R., Math., Acad. Sci. Paris*, 346(23-24):1279–1282, 2008.

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