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Spectral properties of truncated Toeplitz operators by equivalence after extension

M. Cristina Câmara* and Jonathan R. Partington†

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Abstract

We study truncated Toeplitz operators in model spaces K_θ^p for $1 < p < \infty$, with essentially bounded symbols in a class including the algebra $C(\mathbb{R}_\infty) + H_\infty^+$, as well as sums of analytic and anti-analytic functions satisfying a θ-separation condition, using their equivalence after extension to Toeplitz operators with 2×2 matrix symbols. We establish Fredholmness and invertibility criteria for truncated Toeplitz operators with θ-separated symbols and, in particular, we identify a class of operators for which semi-Fredholmness is equivalent to invertibility. For symbols in $C(\mathbb{R}_\infty) + H_\infty^+$, we extend to all $p \in (1, \infty)$ the spectral mapping theorem for the essential spectrum. Stronger results are obtained in the case of operators with rational symbols, or if the underlying model space is finite-dimensional.

Keywords: Truncated Toeplitz operator, Toeplitz operator, equivalence by extension, model space.
MSC: 47B35, 30H10.

1 Introduction

This paper is concerned with truncated Toeplitz operators (TTO), a natural generalisation of finite Toeplitz matrices; these have received much attention since they were introduced by Sarason [27]: see, for instance, [2] and the recent survey [17]. They are encountered in various contexts, for example in the study of finite Toeplitz matrices and finite-time convolution operators.

*Center for Mathematical Analysis, Geometry, and Dynamical Systems, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal. ccamara@math.ist.utl.pt
†School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K. j.r.partington@leeds.ac.uk

By using the equivalence after extension of TTO to block Toeplitz operators of a particular form ([10]), the corona theorem, and the solutions to certain associated Riemann–Hilbert problems, we study here the invertibility and Fredholmness of several classes of TTO, together with their spectra and essential spectra.

Here our context is the Hardy space H_p^+ of the upper half-plane for $1 < p < \infty$, rather than simply H_2^+. Considering different values of p in $(1, \infty)$ naturally requires new approaches to the study of TTO, providing alternatives to Hilbert space methods. By doing so, we not only obtain various results that are new even for $p = 2$, but we also shed light on whether the properties that are studied, namely spectral properties of TTO, depend on the existence of an underlying Hilbert space structure, or on the value of p. In fact, properties such as Fredholmness, invertibility and the dimensions of the kernels and the cokernels of Toeplitz operators in the Hardy spaces H_p^+ may depend on the value of $p \in (1, \infty)$; it is easy to find examples of this behaviour by considering piecewise continuous symbols of the form $g_\alpha(\xi) = (\frac{\xi - i}{\xi + i})^\alpha$ ([12, 21, 23]. One would expect the same to hold for TTO defined in a model space $K_\theta^p := H_p^+ \cap \theta H_p^-$, where θ is an inner function; however, somewhat surprisingly, the results obtained for the various classes of TTO considered in this paper do not depend on p. Note however, that in general the space K_θ^p on which the TTO are defined *does* depend on p: see, for example [8, 14]. For example, this is the case for any infinite Blaschke product θ whose zeroes are not bounded away from the real axis. Thus the kernel of a TTO will in general depend on p.

We first consider here TTO with essentially bounded symbols of the form

$$g = \bar{\theta}_1 \, a_- + \theta_2 \, a_+ \quad , \quad a_\pm \in \mathcal{M}_\infty^\pm \, ,$$

where, denoting by $\mathcal{R}$ the set of all rational functions in $L_\infty(\mathbb{R})$, $\mathcal{M}_\infty^\pm := H_\infty^\pm + \mathcal{R}$ and θ_1 and θ_2 are inner functions such that θ divides $\theta_1 \, \theta_2$. An important property of this class of TTO is that it is possible to determine a solution to an associated Riemann–Hilbert problem, which makes it easier to study; in fact, the study of general TTO presents great difficulties. Moreover this class of symbols, which we call θ-separated, includes all functions in $H_\infty^+ \cup H_\infty^- \cup \mathcal{R}$, and its study reveals some remarkable properties and raises new questions.

For bounded analytic symbols we determine the spectrum of TTO on K_θ^p for each $p \in (1, \infty)$, a result previously established only for $p = 2$ (Fuhrmann's extension [16] of the Livšic-Moeller theorem [22, 24, 25]). The results obtained for symbols in $\mathcal{M}_\infty^+$ allow us to describe the essential spectra of TTO

with symbols in $C(\mathbb{R}_\infty) + H_\infty^+$, extending Bessonov's results [5] to TTO acting on K_θ^p for all $p \in (1, \infty)$.
Furthermore, for rational symbols we establish necessary and sufficient conditions for invertibility of the associated TTO, which enables us to give a more geometric description of the point spectrum and the spectrum of a TTO whose symbol R admits only one pole, and to obtain an explicit expression for the resolvent operator $(A_R^\theta - \lambda I)^{-1}$ if $\lambda \notin \sigma(A_R^\theta)$.
Finally, for TTO defined in finite-dimensional model spaces (in which case the space does not depend on p), we characterise the operator's kernel and invertibility properties, and we illustrate the results by giving a simple description of the eigenvalues and the corresponding eigenspaces of a TTO defined in a model space with dimension 2. Those results show in particular that, while the general case of TTO with discontinuous symbols of the form g_α mentioned above is yet to be fully investigated, in the particular case where the model space is defined by a finite Blaschke product the dimensions of the kernel and the cokernel of a TTO with a symbol of that type (or any other symbol in L_∞) do not depend on p. This is not the case for more general model spaces, as we show in Example 3.6.

The paper is organised as follows. The equivalence after extension of TTO to block Toeplitz operators of a particular form is explained in Section 2, along with the remaining preliminary material. In Section 3 we discuss a class of TTO with θ-separated symbols, and analyse their kernels and their Fredholm properties. Section 4 is concerned with analytic symbols, and Section 5 with $C(\mathbb{R}_\infty) + H_\infty^+$ (and, in particular, rational) symbols. Finally, in Section 6 we consider the case when the underlying model space is finite-dimensional.

2 Preliminaries

For $1 \leq p \leq \infty$ we let $H_p^\pm$ denote the Hardy spaces of the upper and lower half-planes, recalling that for $1 < p < \infty$ we have the decomposition $L_p(\mathbb{R}) = H_p^+ \oplus H_p^-$ with associated projections P_+ and P_-. In what follows we take $p \in (1, \infty)$, unless stated otherwise. For $g \in L_\infty(\mathbb{R})$ the standard Toeplitz operator T_g is defined on H_p^+ by

$$T_g = P_+(gu), \qquad u \in H_p^+,$$

and this will be extended in the obvious way to operators T_G on $(H_p^+)^2$ with essentially bounded matricial symbol $G \in (L_\infty(\mathbb{R}))^{2\times 2}$.

For an inner function $\theta \in H_\infty^+$ the *model space* K_θ^p may be defined as

$$K_\theta^p = H_p^+ \cap \theta H_p^- . \tag{2.1}$$

We will omit the p unless it is necessary for the sake of clarity. We then have

$$L_p(\mathbb{R}) = H_p^- \oplus K_\theta \oplus \theta H_p^+, \tag{2.2}$$

and we write P_θ to denote the associated projection $P_\theta : L_p(\mathbb{R}) \to K_\theta$. Then for $g \in L_\infty(\mathbb{R})$ the standard truncated Toeplitz operator (TTO) A_g^θ is defined as follows:

$$A_g^\theta : K_\theta \to K_\theta \ , \quad A_g^\theta = P_\theta (gI)_{|K_\theta} = P_\theta (gI)_{|P_\theta L_p}. \tag{2.3}$$

More generally, if α and θ are inner functions, we define the operator $A_g^{\alpha,\theta} : K_\theta \to K_\alpha$ by

$$A_g^{\alpha,\theta} := P_\alpha (gI)_{|K_\theta} = P_\alpha (gI)_{|P_\theta L_p}. \tag{2.4}$$

If α is an inner function that divides θ in H_∞^+ (we write this $\alpha \preceq \theta$), let $P_{\alpha,\theta}$ denote $P_\theta - P_\alpha$, a projection with range equal to the *shifted model space* $K_{\alpha,\theta} := \alpha K_{\overline{\alpha}\theta}$. Then we can define

$$B_g^{\alpha,\theta} := P_{\alpha,\theta} (gI)_{|K_\theta} = P_{\alpha,\theta} (gI)_{|P_\theta L_p}. \tag{2.5}$$

The operators $A_g^{\alpha,\theta}$ and $B_g^{\alpha,\theta}$ are particular cases of *general Wiener-Hopf operators* (see [28]) in $L_p(\mathbb{R})$ (abbreviated to L_p), of the form

$$P_1 A_{|P_2 L_p} \ , \tag{2.6}$$

where P_1 and P_2 are projections and A is an operator in L_p. We say that $A_g^{\alpha,\theta}$ and $B_g^{\alpha,\theta}$ are *asymmetric truncated Toeplitz operators* (ATTO) in K_θ.

One of the main tools that we shall employ in this paper is the notion of equivalence after extension. This enables to answer some questions about truncated Toeplitz operators by reducing them to analogous questions about block Toeplitz operators.

Definition 2.1. *[3, 19, 29] The operators $T : X \to \widetilde{X}$ and $S : Y \to \widetilde{Y}$ are said to be* (algebraically and topologically) equivalent *if and only if $T = ESF$ where E, F are invertible operators, and we write $T \sim S$. More generally, T and S are* equivalent after extension *if and only if there exist (possibly trivial) Banach spaces X_0, Y_0, called* extension spaces*, and* *invertible bounded linear operators* $E : \widetilde{Y} \oplus Y_0 \to \widetilde{X} \oplus X_0$ *and* $F : X \oplus X_0 \to Y \oplus Y_0$*, such that*

$$\begin{pmatrix} T & 0 \\ 0 & I_{X_0} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Y_0} \end{pmatrix} F. \tag{2.7}$$

In this case we say that $T \overset{*}{\sim} S$.

Theorem 2.2. *[3] Let* $T : X \to \widetilde{X}$*,* $S : Y \to \widetilde{Y}$ *be operators and assume that* $T \overset{*}{\sim} S$*. Then*

1. $\ker T \simeq \ker S$*;*
2. $\operatorname{Im} T$ *is closed if and only if* $\operatorname{Im} S$ *is closed and, in that case,* $\widetilde{X} / \operatorname{Im} T \simeq \widetilde{Y} / \operatorname{Im} S$*;*
3. *if one of the operators* T*,* S *is generalised (left, right) invertible, then the other is generalised (left, right) invertible too;*
4. T *is Fredholm if and only if* S *is Fredholm and in that case* $\dim \ker T = \dim \ker S$*,* $\operatorname{codim} \operatorname{Im} T = \operatorname{codim} \operatorname{Im} S$*.*

A key result for our purposes is the following, which was proved in [10].

Theorem 2.3. $A_g^{\alpha,\theta} \overset{*}{\sim} T_G$*, where* $G = \begin{pmatrix} \bar{\theta} & 0 \\ g & \alpha \end{pmatrix}$*. Here* T_G *is a block Toeplitz operator acting on* $(H_p^+)^2$*.*

Indeed, for $\alpha \preceq \theta$, the following relations hold:

$$\begin{pmatrix} A_g^{\alpha,\theta} & 0 \\ 0 & I_{\theta H_p^+} \end{pmatrix} = E_1 \begin{pmatrix} P_\alpha g P_\theta + Q_\theta & 0 \\ 0 & I_{\{0\}} \end{pmatrix} F_1, \tag{2.8}$$

where

$$F_1 : K_\theta \oplus \theta H_p^+ \to H_p^+ \oplus \{0\} \tag{2.9}$$

and

$$E_1 : (K_\alpha \oplus \theta H_p^+) \oplus \{0\} \to K_\alpha \oplus \theta H_p^+ \tag{2.10}$$

are invertible operators (defined in an obvious way), so $A_g^{\alpha,\theta} \overset{*}{\sim} P_\alpha g P_\theta + Q_\theta$, and $P_\alpha g P_\theta + Q_\theta \overset{*}{\sim} T_G$ because

$$\begin{pmatrix} P_\alpha g P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} = E_2 \, T_G \, F_2, \tag{2.11}$$

where $F_2 : (H_p^+)^2 \to (H_p^+)^2$ and $E_2 : (H_p^+)^2 \to (K_\alpha \oplus \theta H_p^+) \times H_p^+$ are invertible operators which are explicitly defined, as well as their inverses, in [10]. If $\theta \preceq \alpha$, the result of Theorem 2.3 can be obtained by considering the adjoint operators.

We have

$$\varphi_{1+} \in \ker A_g^{\alpha,\theta} \iff \varphi_{1+} \in \mathcal{P}_1(\ker T_G), \tag{2.12}$$

where $\mathcal{P}_1(x, y) = x$. Note that $\mathcal{P}_1(\ker T_G)$ uniquely defines $\ker T_G$ for G as in Theorem 2.3.

If $\alpha = \theta$, the equality (2.11) takes the form

$$\begin{pmatrix} P_\theta g P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} = \begin{pmatrix} T_\theta - P_\theta g T_\theta & P_\theta \\ -P^+ & T_{\bar{\theta}} \end{pmatrix} T_G \begin{pmatrix} P^+ & 0 \\ T_{\bar{\theta}}(P^+ - T_g) & P^+ \end{pmatrix}, \tag{2.13}$$

with

$$\begin{pmatrix} P^+ & 0 \\ T_{\bar{\theta}}(P^+ - T_g) & P^+ \end{pmatrix}^{-1} = \begin{pmatrix} P^+ & 0 \\ -T_{\bar{\theta}}(P^+ - T_g) & P^+ \end{pmatrix} \tag{2.14}$$

and

$$\begin{pmatrix} T_\theta - P_\theta g T_\theta & P_\theta \\ -P^+ & T_{\bar{\theta}} \end{pmatrix}^{-1} = \begin{pmatrix} T_{\bar{\theta}} & 0 \\ P^+ + P_\theta g Q_\theta & T_\theta \end{pmatrix}, \tag{2.15}$$

where the operators on both sides of the previous equalities are defined in $(H_p^+)^2$. Then we have the following.

Theorem 2.4. *A_g^θ is invertible if and only if T_G is invertible in $(H_p^+)^2$, with $G = \begin{pmatrix} \bar{\theta} & 0 \\ g & \theta \end{pmatrix}$, and in that case*

$$(A_g^\theta)^{-1} = P_\theta[(P_\theta g P_\theta + Q_\theta)^{-1}]_{|K_\theta} \tag{2.16}$$

where

$$(P_\theta g P_\theta + Q_\theta)^{-1}(\psi_{1+}) = \mathcal{P}_1 \left[(T_G)^{-1} \begin{pmatrix} T_{\bar{\theta}}\,\psi_{1+} \\ \psi_{1+} + P_\theta g Q_\theta\, \psi_{1+} \end{pmatrix} \right] \tag{2.17}$$

for all $\psi_{1+} \in H_p^+$.

Proof. The first part is a consequence of Theorems 2.2 and 2.3. If A_g^θ is invertible, then from (2.13), (2.14) and (2.15) we have

$$\begin{pmatrix} P_\theta g P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix}^{-1} \begin{pmatrix} \psi_{1^+} \\ \psi_{2^+} \end{pmatrix} =$$

$$\begin{pmatrix} P^+ & 0 \\ -T_{\bar\theta}(P^+ - T_g) & P^+ \end{pmatrix} T_G^{-1} \begin{pmatrix} T_{\bar\theta}\,\psi_{1^+} \\ \psi_{1^+} + P_\theta g Q_\theta\,\psi_{1^+} + \theta\psi_{2^+} \end{pmatrix} = \begin{pmatrix} \varphi_{1^+} \\ \varphi_{2^+} \end{pmatrix} \tag{2.18}$$

therefore

$$\varphi_{1^+} = \mathcal{P}_1\, T_G^{-1} \left[\begin{pmatrix} T_{\bar\theta}\,\psi_{1^+} \\ \psi_{1^+} + P_\theta g Q_\theta\,\psi_{1^+} \end{pmatrix} + \begin{pmatrix} 0 \\ \theta\,\psi_{2^+} \end{pmatrix} \right]. \tag{2.19}$$

Now, for

$$\begin{pmatrix} \eta_{1^+} \\ \eta_{2^+} \end{pmatrix} := T_G^{-1} \begin{pmatrix} 0 \\ \theta\,\psi_{2^+} \end{pmatrix}$$

we have

$$T_G \begin{pmatrix} \eta_{1^+} \\ \eta_{2^+} \end{pmatrix} = \begin{pmatrix} 0 \\ \theta\,\psi_{2^+} \end{pmatrix} \Leftrightarrow P^+ \begin{pmatrix} \bar\theta & 0 \\ g & \theta \end{pmatrix} \begin{pmatrix} \eta_{1^+} \\ \eta_{2^+} \end{pmatrix} = \begin{pmatrix} 0 \\ \theta\,\psi_{2^+} \end{pmatrix} \Leftrightarrow$$

$$P^+ \begin{pmatrix} \bar\theta & 0 \\ g & \theta \end{pmatrix} \begin{pmatrix} \eta_{1^+} \\ \eta_{2^+} - \psi_{2^+} \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} \eta_{1^+} \\ \eta_{2^+} - \psi_{2^+} \end{pmatrix} \in \ker T_G.$$

Since $\ker T_G = \{0\}$, we have $\eta_{1^+} = 0$. Thus, from (2.18) and (2.19) we have (2.17), and (2.16) follows from here. □

It is well known that T_G is invertible if and only if G admits a canonical Wiener–Hopf (or generalised) p-factorisation ([6, 23])

$$G = G_- G_+^{-1} \tag{2.20}$$

where, taking $\lambda_\pm(\xi) = \xi \pm i$ and $1/p' = 1 - 1/p$,

$$\lambda_\pm^{-1} G_\pm \in (H_p^\pm)^{2\times 2}\,,\ \lambda_\pm^{-1} G_\pm^{-1} \in (H_{p'}^\pm)^{2\times 2}\,, \tag{2.21}$$

$$\begin{array}{c} G_+ P^+ G_-^{-1} I \text{ is defined in a dense subset of } (L_p(\mathbb{R}))^2 \\ \text{and admits a bounded extension to } L_p(\mathbb{R})^2. \end{array} \tag{2.22}$$

The inverse is then given by

$$T_G^{-1} = G_+ P^+ G_-^{-1} I_+ : (H_p^+)^2 \to (H_p^+)^2. \tag{2.23}$$

3 Truncated Toeplitz operators with θ-separated symbols

We study here a class of truncated Toeplitz operators A_g^θ with symbol g of the form

$$g = \bar{\theta}_1 \, a_- + \theta_2 \, a_+ \quad , \quad a_\pm \in H_\infty^\pm \, , \tag{3.1}$$

where θ_1 and θ_2 are inner functions such that $\theta \preceq \theta_1 \, \theta_2$; by changing a_- and a_+ if necessary, we can assume without loss of generality that

$$\theta_1 \, \theta_2 = \theta. \tag{3.2}$$

This class of symbols, which we call *θ-separated*, includes all analytic symbols $g \in H_\infty^+$ (take, for instance, $a_- = 0, \theta_2 = 1$) as well as the anti-analytic symbols $g \in H_\infty^-$ ($a_+ = 0, \theta_1 = 1$). Later in this section we also study more general symbols.

We first address the question of describing $\ker A_g^\theta$.
It is clear from Theorems 2.2 and 2.3 and from (2.12) that $\varphi_{1+} \in \ker A_g^{\alpha,\theta}$, where α, θ are inner functions, if and only if there are $\varphi_{2+} \in H_p^+$, $\varphi_{1-}, \varphi_{2-} \in H_p^-$ such that $G\varphi_+ = \varphi_-$ with $\varphi_\pm = (\varphi_{1\pm}, \varphi_{2\pm})$ and G defined as in Theorem 2.3. Having this in mind, and considering the form of the symbol g in (3.1), we start with the following result.

Theorem 3.1. *If $g_1 \in L_\infty$, $a_+ \in H_\infty^+$ and θ, θ_2 are inner functions with $\theta_2 \preceq \theta$, then for every $\varphi_{1+} \in H_p^+$ the following propositions are equivalent:*
(i) there exist $\varphi_{2+} \in H_p^+$, $\varphi_{1-}, \varphi_{2-} \in H_p^-$ such that

$$\begin{pmatrix} \bar{\theta} & 0 \\ g_1 + \theta_2 a_+ & \theta \end{pmatrix} \begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix} = \begin{pmatrix} \varphi_{1-} \\ \varphi_{2-} \end{pmatrix} ; \tag{3.3}$$

(ii) there exist $\psi_{2+} \in H_p^+$, $\psi_{1-}, \psi_{2-} \in H_p^-$ such that

$$\begin{pmatrix} \bar{\theta} & 0 \\ g_1 & \theta_2 \end{pmatrix} \begin{pmatrix} \varphi_{1+} \\ \psi_{2+} \end{pmatrix} = \begin{pmatrix} \psi_{1-} \\ \psi_{2-} \end{pmatrix} \tag{3.4}$$

and

$$\psi_{2+} - a_+ \varphi_{1+} \in \theta \bar{\theta}_2 H_p^+. \tag{3.5}$$

If (i) and (ii) hold, then $\varphi_{2+} = \bar{\theta}\theta_2 \, (\psi_{2+} - a_+ \varphi_{1+})$, $\varphi_{1-} = \psi_{1-}$ and $\varphi_{2-} = \psi_{2-}$.

Proof. We have

$$\begin{pmatrix} \bar{\theta} & 0 \\ g_1 + \theta_2 a_+ & \theta \end{pmatrix} = \begin{pmatrix} \bar{\theta} & 0 \\ g_1 & \theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_+ & \theta\bar{\theta}_2 \end{pmatrix}. \tag{3.6}$$

Thus, if (i) holds, then $\psi_{2+} = a_+\varphi_{1+} + \theta\bar{\theta}_2\varphi_{2+} \in H_p^+$ and from (3.6) it follows that (3.4) is satisfied with $\psi_{1-} = \varphi_{1-}$ and $\psi_{2-} = \varphi_{2-}$.
Conversely, if (ii) holds, then it follows from (3.5) that $\varphi_{2+} = \bar{\theta}\theta_2\,(\psi_{2+} - a_+\varphi_{1+}) \in H_p^+$ and, by (3.6), the equality (3.3) is satisfied with $\varphi_{1-} = \psi_{1-}$ and $\varphi_{2-} = \psi_{2-}$. □

Theorem 3.2. *Let θ_1, θ_2 and θ be inner functions with $\theta_1\theta_2 = \theta$ and let $a_- \in H_\infty^-$. Then*

$$\ker A^{\theta_2,\theta}_{\bar{\theta}_1 a_-} = K_{\beta\theta_1} \tag{3.7}$$

with

$$\beta = GCD(\bar{a}^i_-, \theta_2) \text{ if } a_- \neq 0, \quad \beta = \theta_2 \text{ if } a_- = 0, \tag{3.8}$$

where $\bar{a}^i_-$ is the inner factor of the inner-outer factorisation $\overline{a}_- = \bar{a}^i_-\bar{a}^o_-$.

Proof. Taking (2.12) into account, we study the solutions of the Riemann–Hilbert problem (3.4) with $g_1 = \bar{\theta}_1 a_-$. We have

$$\begin{cases} \bar{\theta}\varphi_{1+} = \varphi_{1-} \\ \bar{\theta}_1 a_-\varphi_{1+} + \theta_2\psi_{2+} = \psi_{2-}\,, \end{cases}$$

which is equivalent to

$$\begin{cases} \bar{\theta}\varphi_{1+} = \varphi_{1-} \\ a_-\varphi_{1-} + \psi_{2+} = \bar{\theta}_2\psi_{2-}\,, \end{cases} \tag{3.9}$$

and the second equation in (3.9) implies that

$$\psi_{2+} = -a_-\varphi_{1-} + \bar{\theta}_2\psi_{2-} = 0. \tag{3.10}$$

It is easy to see, from Lemma 3.3 below and the first equation in (3.9), that the solutions of (3.9) are defined by $\overline{\varphi_{1-}} \in K_{\bar{\beta}\theta_2,\theta}$ with β given by (3.8), i.e., taking the first equation of (3.9) into account, $\varphi_{1+} \in K_{\beta\theta_1}$. □

Lemma 3.3. *Suppose that $g_+ \in H_\infty^+$ and θ is inner. Then for $\varphi_+ \in H_p^+$ we have*

$$g_+\varphi_+ \in \theta H_p^+ \Leftrightarrow \varphi_+ \in \bar{\beta}\theta H_p^+$$

with $\beta = GCD(g^i_+, \theta)$, where g^i_+ is the inner factor of the inner-outer factorization $g_+ = g^i_+\,g^o_+$, if $g_+ \neq 0$, and $\beta = \theta$ if $g_+ = 0$.

Theorem 3.4. *Let g be given by* (3.1)–(3.2) *with $a_\pm \in H_\infty^\pm$. Then*

$$\ker A_g^\theta = K_{\theta_1\bar\beta_1,\,\theta_1\beta} = \theta_1\bar\beta_1\, K_{\beta\beta_1}, \tag{3.11}$$

where β is defined by (3.8) *and*

$$\beta_1 = GCD(a_+^i, \theta_1) \text{ if } a_+ \neq 0, \quad \beta_1 = \theta_1 \text{ if } a_+ = 0. \tag{3.12}$$

Proof. From Theorem 3.1, with $g_1 = \bar\theta_1 a_-$, we conclude that $\varphi_{1+} \in \ker A_g^\theta$ if and only if $\varphi_{1+} \in \ker A_{\bar\theta_1 a_-}^{\theta_2,\theta}$ and (3.5) is satisfied with $\psi_{2+} = 0$, taking (3.10) into account. Therefore $\varphi_{1+} \in \ker A_g^\theta$ if and only if $\varphi_{1+} \in K_{\theta_1\beta}$ with β defined by (3.8), by Theorem 3.2, and moreover

$$a_+\varphi_{1+} = \theta_1\varphi_{2+} \tag{3.13}$$

with $\varphi_{2+} \in H_p^+$. By Lemma 3.3, (3.13) holds if and only if $\varphi_{1+} \in \theta_1\bar\beta_1 H_p^+$ with β_1 defined by (3.12).
Thus $\varphi_1^+ \in \ker A_g^\theta$ if and only if $\varphi_1^+ \in K_{\theta_1\beta} \cap \theta_1\bar\beta_1 H_p^+ = K_{\theta_1\bar\beta_1,\theta_1\beta}$. □

Corollary 3.5. *With the same assumptions as in Theorem 3.4,* $\ker A_g^\theta$ *is finite dimensional if and only if β and β_1 are finite Blaschke products, and the operator A_g^θ is injective if and only if β and β_1 are constant. In particular, if $a_\pm \neq 0$, A_g^θ is injective if and only if $(\bar a_-^i,\, \theta_2)$ and $(a_+^i,\, \theta_1)$ are pairs of relatively prime inner functions.*

Example 3.6. For general inner functions θ the question whether a truncated Toeplitz operator A_g^θ is injective on K_θ^p can depend on p, as the following example shows.
Let $2 < p_1 < p_2 < \infty$, and suppose that $1/p_1 + 1/p_2 = 1/r$, where $r > 1$. Let $\theta \in H_\infty^+$ be the Blaschke product with zero set $\{i/k^2 : k = 1, 2, \ldots\}$. We may choose a positive sequence (a_k) such that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{\xi + i/k^2}$$

converges in $H_{p_1}^+$ to a function $f \in K_\theta^{p_1}$ that is outer (consider its imaginary part) and not in $H_{p_2}^+$. Let $g = \overline{f_+}/f_+$, and consider A_g^θ. Regarded as an operator on $K_\theta^{p_1}$, it has f_+ in its kernel.
Now, if $\varphi_+ \in K_\theta^p$ lies in $\ker A_g^\theta$ (for $p = p_1$ or p_2), then $g\varphi_+ = \varphi_- + \theta\psi_+$ for some $\varphi_- \in H_p^-$ and $\psi_+ \in H_p^+$ and so $\overline{f_+}\varphi_+ = f_+\varphi_- + f_+\theta\psi_+$.
We see that $\psi_+ = 0$ and so $\varphi_+ \in \ker T_g$. However, it follows easily from [9, Thm 5.3] (with $M = 0$) that, with $p = p_1$ or p_2, all functions in $\ker T_g$ are constant multiples of f. Hence the $K_\theta^{p_1}$ kernel of A_g^θ is one-dimensional, while the $K_\theta^{p_2}$ kernel is trivial.

Truncated Toeplitz operators associated to a singular inner function are particularly interesting, given their close connection with finite interval convolution equations when θ is of the form $\theta(\xi) = e^{i\mu\xi}, \mu \in \mathbb{R}$. We have the following:

Corollary 3.7. *If θ is a singular inner function and g is given by* (3.1)-(3.2) *with $a_\pm \in H_\infty^\pm$, then $\ker A_g^\theta$ and $\ker A_{\bar{g}}^\theta$ are either both equal to $\{0\}$ or infinite dimensional, and A_g^θ is Fredholm if and only if it is invertible.*

Proof. It is clear that, in this case, β and β_1 are either constant or singular inner functions, so the first part follows from (3.11). Since T_G, with

$$G = \begin{pmatrix} \bar{\theta} & 0 \\ g & \theta \end{pmatrix}, \tag{3.14}$$

has Fredholm index 0 whenever T_G is Fredholm, the same happens with A_g^θ; thus it must be invertible if it is Fredholm. □

From Theorem 3.4 we see in particular that, if $g = a_+ \in H_\infty^+ \setminus \{0\}$, we have

$$\ker A_{a_+}^\theta = \theta\bar{\beta}\, K_\beta, \quad \text{with} \quad \beta = GCD(a_+^i, \theta) \tag{3.15}$$

and, if $g = a_- \in H_\infty^- \setminus \{0\}$,

$$\ker A_{a_-}^\theta = K_\beta, \quad \text{with} \quad \beta = GCD(\bar{a}_-^i, \theta). \tag{3.16}$$

Since $(A_g^\theta)^* = A_{\bar{g}}^\theta : K_\theta^q \to K_\theta^q$ with $1/p + 1/q = 1$ and $\bar{g} = \bar{\theta}_2\, \bar{a}_+ + \theta_1\, \bar{a}_-$, it also follows from Theorem 3.4 that

$$\ker A_g^\theta = \theta_1\bar{\beta}_1\, K_{\beta\beta_1}^p\,, \quad \ker(A_g^\theta)^* = \theta_2\bar{\beta}\, K_{\beta\beta_1}^q. \tag{3.17}$$

In the case that $K_{\beta\beta_1}^p = K_{\beta\beta_1}^q$ as vector spaces, we see from the closed graph theorem that the L_p and L_q norms are equivalent on this space. Thus we have:

Theorem 3.8. *Let g take the form* (3.1) *with $a_\pm \in H_\infty^\pm$ and θ_1, θ_2 satisfying* (3.2)*. Then $\ker A_g^\theta$ and $\ker(A_g^\theta)^*$ are isomorphic whenever $K_{\beta\beta_1}^p = K_{\beta\beta_1}^q$.*

Apart from the obvious cases that $p = 2$ or $\beta\beta_1$ is a finite Blaschke product, necessary and sufficient conditions for the property $K_{\beta\beta_1}^p = K_{\beta\beta_1}^q$ are given by Dyakonov [13] (see also [14, 15]) and some further equivalent conditions are given in [8]. Under these circumstances, $\bar{\theta}_1\beta_1 \ker A_g^\theta = \bar{\theta}_2\beta \ker(A_g^\theta)^*$.

Using the same notation, an immediate consequence of Theorem 3.8 and Corollary 3.7 is the following.

Corollary 3.9. *With the same assumptions as in Theorem 3.8, A_g^θ is Fredholm if and only if it is semi-Fredholm; if θ is a singular inner function, A_g^θ is invertible if and only if it is semi-Fredholm .*

Note that, for all $g \in L_\infty(\mathbb{R})$, we also have that A_g^θ is Fredholm if and only if it is semi-Fredholm when $p = 2$. In fact, on the one hand, the equivalence between Fredholmness and semi-Fredholmness for Toeplitz operators defined in $(H_2^+)^{2\times 2}$, with symbols whose determinants admit a bounded factorisation, was proved in [1], Corollary 3.13; on the other hand, it is easy to see from (2.12) that the conjugate-linear operator $\mathcal{C}_\theta$ defined by

$$\mathcal{C}_\theta(\varphi_+) = \theta\, \overline{P_\theta\, \varphi_+} \quad , \quad \varphi_+ \in H_p^+, \tag{3.18}$$

which maps K_θ onto K_θ isometrically, also maps $\ker A_g^\theta$ onto $\ker(A_g^\theta)^* = \ker A_{\bar{g}}^\theta$ isometrically when $p = 2$. Whether Fredholmness and semi-Fredholmness are equivalent for TTO in all H_p settings is an open question, to the authors' knowledge.

By Theorems 2.2 and 2.3, we can obtain conditions for Fredholmness and invertibility of A_g^θ by using the relations between the corresponding properties for Toeplitz operators with matrix symbols and the solutions of certain associated Riemann–Hilbert problems ([4],[6]).
We define

$$CP_\pm := \{(f_{1\pm}, f_{2\pm}) \in (H_\infty^\pm)^2 : \inf_{z\in\mathbb{C}^\pm} (|f_{1\pm}(z)| + |f_{2\pm}(z)|\,) > 0\}. \tag{3.19}$$

By the corona theorem, $(f_{1\pm}, f_{2\pm}) \in CP_\pm$ if and only if there exists a pair $(h_{1\pm}, h_{2\pm}) \in (H_\infty^\pm)^2$ such that

$$f_{1\pm}(z)\, h_{1\pm}(z) + f_{2\pm}(z)\, h_{2\pm}(z) = 1 \quad \text{for all } z \in \mathbb{C}^\pm. \tag{3.20}$$

Now let

$$\mathcal{M}_\infty^\pm := H_\infty^\pm + \mathcal{R} \tag{3.21}$$

where $\mathcal{R}$ denotes the set of all rational functions in $L_\infty(\mathbb{R})$. We have

$$a_\pm \in \mathcal{M}_\infty^\pm \Leftrightarrow a_\pm = sA_\pm \quad \text{with} \quad s \in \mathcal{GR}\,,\, A_\pm \in H_\infty^\pm$$

where $\mathcal{GR}$ denotes the group of invertible elements of $\mathcal{R}$.
We denote by $CP_\pm^M$ the set of all pairs $(\varphi_{1\pm}^M, \varphi_{2\pm}^M) \in (\mathcal{M}_\infty^\pm)^2$ such that $\varphi_j^\pm = r_j f_j^\pm, j = 1, 2$, with $r_j^{\pm 1} \in \mathcal{R}$ and $(f_1^\pm, f_2^\pm) \in CP_\pm$.

Identifying a pair of the form (f_1, f_2) with $[f_1 \quad f_2]^T$, we have the following, which is a direct consequence of Theorems 4.1 and 4.5 in [6]:

Theorem 3.10. *Let $G \in (L_\infty(\mathbb{R}))^{2\times 2}$ with $\det G$ admitting a canonical p-factorisation and assume that (f_+, f_-) is a solution to the Riemann–Hilbert problem $Gf_+ = f_-$, $f_\pm \in \mathcal{M}_\infty^\pm$. Then T_G is Fredholm if $f_\pm \in CP_\pm^M$, and $Ind\, T_G = 0$; moreover, T_G is invertible if $f_\pm \in CP_\pm$.*

It is clear that the determinant of any G of the form (3.14) with $g \in L_\infty(\mathbb{R})$ admits a canonical p-factorisation, since $\det G = 1$.
We will also need the following result.

Theorem 3.11. *Let $G \in (L_\infty(\mathbb{R}))^{2\times 2}$ with $\det G$ admitting a canonical p-factorisation, and let $f_\pm \in (H_\infty^\pm)^2$ satisfy $G\, f_+ = f_-$. If $f_+ \in CP_+$, then T_G is invertible if and only if $f_- \in CP_-$; analogously, if $f_- \in CP_-$, then T_G is invertible if and only if $f_+ \in CP_+$.*

Proof. Assume that $f_+ = (f_{1+}, f_{2+}) \in CP_+$. Then, by Theorem 3.10, $f_- \in CP_-$ is a sufficient condition for T_G to be invertible; it is left to show that $f_- = (f_{1-}, f_{2-}) \in CP_-$ is a necessary condition for the invertibility of T_G, i.e., for the existence of a canonical p-factorisation of the symbol G. Let $h_{1+}, h_{2+} \in H_\infty^+$ satisfy (3.20); then

$$H_+ = \begin{pmatrix} h_{1+} & h_{2+} \\ -f_{2+} & f_{1+} \end{pmatrix} \in \mathcal{G}(H_\infty^+)^{2\times 2}.$$

and, if G admits a canonical p-factorisation, GH_+^{-1} also admits a canonical p-factorisation. We have

$$GH_+^{-1}(H_+ f_+) = f_- \Leftrightarrow GH_+^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f_- \, ,$$

thus f_- is equal to the first column in GH_+^{-1}. If $f_- \notin CP_-$, then for every $\epsilon > 0$ there exists $z_0 \in \mathbb{C}^-$ such that $|f_{1-}(z_0)| + |f_{2-}(z_0)| < \epsilon$. Let G_{z_0} be the matrix function obtained by subtracting $f_-(z_0)$ from the first column of GH_+^{-1}; for sufficiently small ϵ, by the stability of the canonical p-factorization, G_{z_0} also admits a canonical p-factorization, i.e., $T_{G_{z_0}}$ is invertible. On the other hand, we have

$$G_{z_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f_- - f_-(z_0) \Leftrightarrow G_{z_0} \frac{1}{z - z_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{f_- - f_-(z_0)}{z - z_0}$$

and since

$$\frac{1}{z - z_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in (H_p^+)^2 \, , \; \frac{f_- - f_-(z_0)}{z - z_0} \in (H_p^-)^2 \, ,$$

we conclude that $\ker T_{G_{z_0}} \neq \{0\}$, which is impossible. Thus we must have $f_- \in CP_-$.
Regarding the second part of the theorem, it is enough to apply the first part to $\overline{G^{-1}}$ instead of G. □

We now apply these results to truncated Toeplitz operators.

Theorem 3.12. *The operator A_g^θ, with g of the form* (3.1) *and $a_\pm \in \mathcal{M}_\infty^\pm$, is Fredholm if*

$$(\bar{\theta}_2, a_-) \in CP_-^M\,, \quad (\theta_1, a_+) \in CP_+^M. \tag{3.22}$$

Moreover, A_g^θ is invertible if

$$(\bar{\theta}_2, a_-) \in CP_-\,, \quad (\theta_1, a_+) \in CP_+. \tag{3.23}$$

Proof. Let

$$G = \begin{pmatrix} \bar{\theta} & 0 \\ g & \theta \end{pmatrix}. \tag{3.24}$$

We have $G\varphi_+ = \varphi_-$, where $\varphi_+ = (\theta_1, -a_+), \varphi_- = (\bar{\theta}_2, a_-)$. If (3.22) is satisfied then, by Theorem 3.10, T_G is Fredholm; consequently, the same is true for A_g^θ by Theorems 2.2 and 2.3. If (3.23) holds then $\beta, \beta_1 \in \mathbb{C}$ and, by Corollary 3.7, A_g^θ is injective and, therefore, invertible. □

Note that, by Theorem 2.8 in [6], condition (3.22) is equivalent to having $\varphi_\pm^M = s_\pm h_\pm$ with $h_\pm \in CP_\pm$ and $s_\pm \in \mathcal{GR} \cap H_\infty^\pm$, where $\mathcal{GR}$ denotes the group of invertible elements in $\mathcal{R}$. Since θ_1 and θ_2 are inner functions, we must then have $s_+ = \beta_1$, $s_- = \bar{\beta}$.

Corollary 3.13. *A_g^θ is Fredholm if one of the elements in each pair $(\bar{\theta}_2, a_-)$, (θ_1, a_+) belongs to $\mathcal{GR}$.*

Proof. In this case condition (3.22) is satisfied because the meromorphic corona problems with data $(\bar{\theta}_2, a_-)$ and (θ_1, a_+) (see [6]) obviously have a solution and therefore $(\bar{\theta}_2, a_-) \in CP_-^M$ and $(\theta_1, a_+) \in CP_+^M$. □

Moreover, we have the following.

Theorem 3.14. *Let one of the following conditions hold:*
(i) $(\theta_1\,,\, a_+) \in CP_+^M$ *;*
(ii) $(\bar{\theta}_2\,,\, a_-) \in CP_-^M$*.*
Then condition (3.22) *is necessary and sufficient for Fredholmness of A_g^θ.*

Proof. Taking Theorem 3.12 into account, it is left to show that, under these assumptions, Fredholmness of A_g^θ implies that (3.22) holds.
Let us first consider the case where $a_\pm \in H_\infty^\pm$, and let β, β_1 be defined by (3.8) and (3.12), respectively. Assume for instance that (i) holds. If A_g^θ is Fredholm, so is T_G with G given by (3.24), and β, β_1 are finite Blaschke products. Let $\tilde{G} = M_- G M_+$, with

$$M_- = \begin{pmatrix} \beta & 0 \\ a_-\theta_2(\beta - \bar{\beta}) & \bar{\beta} \end{pmatrix} \in \mathcal{GM}_\infty^-,$$

$$M_+ = \begin{pmatrix} \beta_1 & 0 \\ a_+\bar{\theta}_1(\bar{\beta}_1 - \beta_1) & \bar{\beta}_1 \end{pmatrix} \in \mathcal{GM}_\infty^+,$$

i.e.,

$$\tilde{G} = \begin{pmatrix} \overline{(\frac{\theta_1}{\beta_1})(\frac{\theta_2}{\beta})} & 0 \\ a_-\beta\overline{(\frac{\theta_1}{\beta_1})} + a_+\bar{\beta}_1\frac{\theta_2}{\beta} & \frac{\theta_1}{\beta_1}\frac{\theta_2}{\beta} \end{pmatrix}.$$

By Theorem 3.10 in [21], $T_{\tilde{G}}$ is also Fredholm. Moreover, by Corollary 3.7, $T_{\tilde{G}}$ is injective; so it is invertible. Since

$$\tilde{G}\begin{pmatrix} \frac{\theta_1}{\beta_1} \\ -\bar{\beta}_1 a_+ \end{pmatrix} = \begin{pmatrix} \overline{(\frac{\theta_2}{\beta})} \\ \beta a_- \end{pmatrix}$$

and (i) is equivalent to $\bar{\beta}_1(\theta_1, a_+) \in CP^+$, then by Theorem 3.11 we must have $\beta(\bar{\theta}_2, a_-) \in CP^-$ and thus (ii) must hold.
Assume now that $a_\pm \in \mathcal{M}_\infty^\pm$, and (i) holds. Then, by Theorem 2.6 in [6], there exists $R \in \mathcal{GR}$ such that $R(\theta_1, a_+) = (\gamma_1, \tilde{a}_+) \in CP^+$. On the other hand there exists a Blaschke product B such that $\bar{B}(\bar{\theta}_2, a_-) = (\bar{\gamma}_2, \tilde{a}_-) \in (H_\infty^-)^2$. Thus, if we replace β by $\bar{B}$ and β_1 by $\bar{R}$ in the expressions of $M_\pm$ above, then $\tilde{G} = M_- G M_+$ is of the form (3.24) with g satisfying (3.1), and we can conclude by the previous reasoning that $(\bar{\gamma}_2, \tilde{a}_-) \in CP_-^M$, and thus $(\theta_2, a_-) \in CP_-^M$. □

A simple example where at least one of the conditions (i) and (ii) of Theorem 3.14 is satisfied is the case where a_+ or a_- are rational functions in $\mathcal{GR}$. Another case will be considered in the next section.
Analogously, we have the following.

Theorem 3.15. *Let one of the following conditions hold:*
(i) $(\theta_1, a_+) \in CP^+$*;*
(ii) $(\bar{\theta}_2, a_-) \in CP^-$*;*
then (3.23) *is a necessary and sufficient condition for invertibility of* A_g^θ.

4 Fredholmness, invertibility and spectra for TTO with analytic symbols

We now apply the results of the previous section to study truncated Toeplitz operators with analytic symbols $g_+ \in H_\infty^+$ and, in particular, the restricted shift A_r^θ. For any $g \in L_\infty(\mathbb{R})$, we use the notation

$$G_g = \begin{pmatrix} \bar{\theta} & 0 \\ g & \theta \end{pmatrix}. \tag{4.1}$$

Recall that for $p = 2$, the classical Livšic–Moeller theorem [22, 24, 25] describes the spectrum of A_r^θ in terms of the spectrum $\Sigma(\theta)$, which may be defined by

$$\Sigma(\theta) := \{\xi \in \mathbb{C}^+ \cup \mathbb{R}_\infty : \liminf_{z \to \xi, z \in \mathbb{C}^+} |\theta(z)| = 0 \}, \tag{4.2}$$

where $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. A generalization to $A_{g_+}^\theta$ for $g_+ \in H_\infty^+$ was given by Fuhrmann [16], using Hilbert-space methods. We start by generalising this result to arbitrary p.

Theorem 4.1. *The operator $A_{g_+}^\theta$ is invertible if and only if $(\theta, g_+) \in CP^+$. The spectrum of $A_{g_+}^\theta$ is*

$$\sigma(A_{g_+}^\theta) = \{\lambda \in \mathbb{C} : \inf_{z \in \mathbb{C}^+} (|\theta(z)| + |g_+(z) - \lambda|) = 0\}.$$

Proof. The invertibility condition is a direct consequence of Theorem 3.15, taking $\theta_1 = \theta$ and $\theta_2 = 1$. In fact since, for $\lambda \in \mathbb{C}$, we have $A_{g_+ - \lambda}^\theta \overset{*}{\sim} T_{G_{g_+ - \lambda}}$, then by Theorem 3.15 (since in this case $(\bar{\theta}_2, a_-) = (1, 0) \in CP_-$) the operator $A_{g_+ - \lambda}^\theta$ is invertible if and only if $(\theta, g_+ - \lambda) \in CP^+$, i.e., $\inf_{z \in \mathbb{C}^+} (|\theta(z)| + |g_+(z) - \lambda|) \neq 0$. ☐

For $f \in H_\infty^+$ let

$$f_{ess}(\Sigma(\theta)) := \{\lambda \in \mathbb{C} : \inf_{z \in \mathbb{C}^+} (|\theta(z)| + |f(z) - \lambda|) = 0\} \tag{4.3}$$

$$f_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty) := \{\lambda \in \mathbb{C} : \liminf_{z \to \xi, z \in \mathbb{C}^+} (|\theta(z)| + |f(z) - \lambda|) = 0 \text{ for some } \xi \in \mathbb{R}_\infty\}. \tag{4.4}$$

If f is continuous in $\mathbb{C}^+ \cup \mathbb{R}_\infty$, then $f_{ess}(\Sigma(\theta))$ defined by (4.3) coincides with the image of $\Sigma(\theta)$ by f, and analogously for $f_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$. With these definitions, we have:

Corollary 4.2. *If* $g_+ \in H_\infty^+$*, then*

$$\sigma(A^\theta_{g_+}) = (g_+)_{ess}(\Sigma(\theta)).$$

To describe the point spectrum and the essential spectrum of $A^\theta_{g_+}$, we define

$$\beta_\lambda := GCD(\,\theta, (g_+ - \lambda)^i) \tag{4.5}$$

where $(g_+ - \lambda)^i$ denotes the inner factor in an inner-outer factorisation of $g_+ - \lambda$ if the latter is not the zero function, and $(g_+ - \lambda)^i = \theta$ otherwise.

Theorem 4.3. *The point spectrum of* $A^\theta_{g_+}$ *is the set*

$$\sigma_P(A^\theta_{g_+}) = \{\lambda \in \mathbb{C} : \beta_\lambda \notin \mathbb{C}\}$$

and, for each $\lambda \in \sigma_p(A^\theta_{g_+})$*, the corresponding eigenspace is the shifted model space*

$$E_\lambda = K_{\bar{\beta}_\lambda \theta, \theta} = \bar{\beta}_\lambda \theta \, K_{\beta_\lambda}.$$

Proof. It is clear from Theorem 3.4 that a necessary and sufficient condition for the kernel of the operator $A^\theta_{g_+ - \lambda}$ to be non-zero is that β_λ is a non-constant inner function; on the other hand, from (3.11), we have $E_\lambda = \ker A^\theta_{g_+ - \lambda}$ given as above. □

Theorem 4.4. *The operator* $A^\theta_{g_+}$ *is Fredholm if and only if*

$$\beta \in FBP \quad \textit{and} \quad \bar{\beta}(\theta, g_+) \in CP^+, \tag{4.6}$$

where $\beta = GCD(\theta, g_+^i)$ *and FBP denotes the set of all finite Blaschke products. The essential spectrum of* $A^\theta_{g_+}$ *is*

$$\sigma_{ess}(A^\theta_{g_+}) = (g_+)_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty).$$

Proof. Taking $\theta_2 = 1, a_- = 0$ and $f_- = 1, h_- = 0$ as in the proof of Theorem 4.1, it is clear that condition (ii) in Theorem 3.14 is satisfied, so $A^\theta_{g_+}$ is Fredholm if and only if $(\theta_1, a_+) = (\theta, g_+) \in CP_+^M$, which is equivalent to (4.6). Replacing g_+ by $g_+ - \lambda$ with $\lambda \in \mathbb{C}$, we conclude that the essential spectrum of $A^\theta_{g_+}$ is the union of the sets

$$S_1 = \{\lambda \in \mathbb{C} : \beta_\lambda \notin FBP\}$$

and

$$S_2 = \{\lambda \in \mathbb{C} : \beta_\lambda \in FBP\ ,\ \inf_{z \in \mathbb{C}^+} \big(|(\bar{\beta}_\lambda \theta)(z)| + |(\bar{\beta}_\lambda (g_+ - \lambda))(z)| \big) = 0\}.$$

If $\lambda \in S_1$, i. e., $\beta_\lambda \notin FBP$, then $\Sigma(\beta_\lambda) \cap \mathbb{R}_\infty$ is not empty and, for some $\xi \in \mathbb{R}_\infty$, we must have $\liminf_{z\to\xi, z\in\mathbb{C}^+} (|\theta(z)| + |g_+(z) - \lambda|) = 0$; it follows that $\lambda \in (g_+)_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$. If $\lambda \in S_2$, then $\lambda \in (g_+)_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$ because, when $\beta_\lambda \in FBP$,

$$\inf_{z\in\mathbb{C}^+} \big(|(\bar{\beta}_\lambda \theta)(z)| + |(\bar{\beta}_\lambda (g_+ - \lambda))(z)|\big) = 0 \quad \Leftrightarrow \tag{4.7}$$

$$\liminf_{z\to\xi, z\in\mathbb{C}^+} (|\theta(z)| + |g_+(z) - \lambda|) = 0 \ \text{ for some } \xi \in \mathbb{R}_\infty.$$

Therefore $S_1 \cup S_2 \subset (g_+)_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$. Conversely, if $\lambda \in (g_+)_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$, then either $\beta_\lambda \notin FBP$, or $\beta_\lambda \in FBP$ and in this case $\lambda \in S_2$ by (4.7). $\square$

For the restricted shift A_r^θ defined in K_θ^p, the previous results yield, for all $p \in (1, \infty)$:

Corollary 4.5.

$$\sigma(A_r^\theta) = r(\Sigma(\theta)),$$
$$\sigma_P(A_r^\theta) = r(\Sigma(\theta)) \cap \mathbb{D},$$
$$\sigma_{ess}(A_r^\theta) = r(\Sigma(\theta)) \cap \mathbb{T}.$$

5 Truncated Toeplitz operators with $C(\mathbb{R}_\infty) + H_\infty^+$ symbols

We start by generalising (4.4) for $f \in C(\mathbb{R}_\infty) + H_\infty^+$. Let $f = f_1 + f_2$ with $f_1 \in C(\mathbb{R}_\infty)$ and $f_2 \in H_\infty^+$; then we define $f_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$ as

$$\{\lambda \in \mathbb{C} : \liminf_{z\to\xi, z\in\mathbb{C}^+} (|\theta(z)| + |f_1(\xi) + f_2(z) - \lambda|) = 0 \text{ for some } \xi \in \mathbb{R}_\infty\}. \tag{5.1}$$

It is clear that $f_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$ coincides with the image of $\Sigma(\theta) \cap \mathbb{R}_\infty$ by f if $f \in C(\mathbb{R}_\infty)$, and with the set defined in (4.4) if $f \in H_\infty^+$.
Let us first consider $g \in \mathcal{M}_\infty^+$, with $g = s_1 + h_+$ where $s_1 \in \mathcal{R}$ and $h_+ \in H_\infty^+$ (see (3.21)). We can write

$$g = sg_+ \text{ with } s \in \mathcal{GR}\,,\ g_+ \in H_\infty^+$$

([6], Proposition 2.3). Thus, with G_g as defined in (4.1), we have

$$G_g = \operatorname{diag}(1, s)\, G_{g_+}\ \operatorname{diag}(1, s^{-1}), \tag{5.2}$$

where $\operatorname{diag}(1, s^{\pm 1}) \in \mathcal{GR}^{2\times 2}$. Therefore, T_{G_g} is Fredholm if and only if $T_{G_{g_+}}$ is Fredholm and, by Theorem 4.4 and (4.4), this is equivalent to

$$\liminf_{z\to\xi, z\in\mathbb{C}^+} (\,|\theta(z)| + |g_+(z)|\,) > 0 \quad \text{for all } \xi \in \mathbb{R}_\infty. \tag{5.3}$$

Since $s \in \mathcal{GR}$, there exists $\epsilon > 0$ such that $s^{\pm 1}$ are analytic and bounded in the strip $\mathcal{S}$ defined by $0 < \Im z < \epsilon$, and (5.3) is equivalent to

$$\liminf_{z\to\xi, z\in\mathcal{S}} (\,|\theta(z)| + |s(z)g_+(z)|\,) > 0 \quad \text{for all } \xi \in \mathbb{R}_\infty$$

$$\Leftrightarrow \liminf_{z\to\xi, z\in\mathbb{C}^+} (\,|\theta(z)| + |s_1(\xi) + h_+(z)|\,) > 0 \quad \text{for all } \xi \in \mathbb{R}_\infty.$$

Therefore we conclude that $A_{g-\lambda}$ is not Fredholm if and only if

$$\liminf_{z\to\xi, z\in\mathbb{C}^+} (\,|\theta(z)| + |s_1(\xi) + h_+(z) - \lambda|\,) = 0 \quad \text{for some } \xi \in \mathbb{R}_\infty.$$

We have thus proved the following.

Theorem 5.1. *If $g \in \mathcal{M}_\infty^+$ then $\sigma_{ess}(A_g^\theta) = g_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$, for all $p \in (1, \infty)$.*

Corollary 5.2. *Let $R \in \mathcal{R}$. A_R^θ is Fredholm if and only if $R(\xi) \neq 0$ for all $\xi \in \Sigma(\theta) \cap \mathbb{R}_\infty$, and $\sigma_{ess}(A_R^\theta) = R\,(\Sigma(\theta) \cap \mathbb{R}_\infty)$.*

In particular we see that $\sigma_{ess}(A_R^\theta) = \varnothing$ if $\Sigma(\theta) \cap \mathbb{R}_\infty = \varnothing$ or $R \in \mathcal{GR}$.

We are now ready to calculate the essential spectrum of A_g^θ where g is a symbol in $C(\mathbb{R}_\infty) + H_\infty^+$. The H_2 version of the following result (formulated on the disc) may be found in [5]; the special case $g \in C(\mathbb{R}_\infty)$ is much older and appears in [25, Cor. V.4.1].

Theorem 5.3. *For all $p \in (1, \infty)$ and for $g \in C(\mathbb{R}_\infty) + H_\infty^+$ we have $\sigma_{ess}(A_g^\theta) = g_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$.*

Proof. We prove that $\sigma_{ess}(A_g^\theta) \supseteq g_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$ by an approximation argument. For if $g = g_c + h$ with $g_c \in C(\mathbb{R}_\infty)$ and $h \in H_\infty^+$, then we may take rational functions $f_n \in C(\mathbb{R}_\infty)$ tending to g_c uniformly, so that $A_n := A_{f_n+h}^\theta$ tends to A_g^θ in norm. We write $g_n = f_n + h$, with $g_n \in \mathcal{M}_\infty^+$. Now if $w \notin \sigma_{ess}(A_g^\theta)$, then since the complement of the essential spectrum is open we see that there is a disc $D(w, \epsilon)$ which is disjoint from $\sigma_{ess}(A_n)$ for sufficiently large n. This is a contradiction if $w \in g_{ess}(\Sigma(\theta) \cap \mathbb{R}_\infty)$, since

then $(g_n)_{ess}(\Sigma(\theta)\cap\mathbb{R}_\infty)$ meets this disc for large n, and by Theorem 5.1, $(g_n)_{ess}(\Sigma(\theta)\cap\mathbb{R}_\infty)=\sigma_{ess}(A_n)$.
For the reverse inclusion $\sigma_{ess}(A_g^\theta)\subseteq g_{ess}(\Sigma(\theta)\cap\mathbb{R}_\infty)$, one may adapt Bessonov's argument from [5, Lem. 2.3]; namely, for $w\in\mathbb{C}\backslash g_{ess}(\Sigma(\theta)\cap\mathbb{R}_\infty)$ one can use the corona theorem in $H_\infty^+ + C(\mathbb{R}_\infty)$ to find functions $h_1,h_2\in H_\infty^+ + C(\mathbb{R}_\infty)$ with

$$(g-w)h_1+\theta h_2=1 \qquad \text{a.e. on } \mathbb{R}_\infty.$$

We now have

$$A_{h_1}^\theta(A_g^\theta-wI)=I+K_L, \qquad (A_g^\theta-wI)A_{h_1}^\theta=I+K_R,$$

where K_L and K_R are compact; for the results needed for this calculation, that

- $A_\theta^\theta=0$, and
- for $g\in C(\mathbb{R}_\infty)$ and $h\in L^\infty(\mathbb{R})$ the semi-commutators $A_g^\theta A_h^\theta - A_{gh}^\theta$ and $A_h^\theta A_g^\theta - A_{gh}^\theta$ are compact (see [18]),

hold for H_p^+ as well.

□

For rational symbols, we can establish invertibility conditions and thus say more about the spectrum of A_R^θ, with $R\in\mathcal{R}$. This leads to the question of characterising the kernel of T_{G_R} bearing in mind that, if A_R^θ is Fredholm, then it is invertible if and only if $\ker A_R^\theta=\{0\}$, which is equivalent to $\ker T_{G_R}=\{0\}$.

Theorem 5.4. *Let $R=\frac{P_N}{P_{\mathbb{C}^-}P_{\mathbb{C}^+}}\in L_\infty$, where P_N is a polynomial of degree N and $P_{\mathbb{C}^\pm}$ are polynomials with zeroes in $\mathbb{C}^\pm$, at most, with degrees $N^\pm$, respectively ($N\le N^+ + N^-$). We have $\ker T_{G_R}\neq\{0\}$ if and only if there are polynomials Q_1 and Q_2, with $\deg Q_1<N^+$ and $\deg Q_2<N^-$ such that*

$$\frac{Q_1P_{\mathbb{C}^-}+Q_2P_{\mathbb{C}^+}\theta}{P_N}\in H_p^+\setminus\{0\}\,,\ \frac{Q_1P_{\mathbb{C}^-}\bar\theta+Q_2P_{\mathbb{C}^+}}{P_N}\in H_p^-\setminus\{0\}. \tag{5.4}$$

Proof. We have $\varphi_+\in\ker T_{G_R}$ if and only if $\varphi_+\in(H_p^+)^2$ is a solution to the Riemann–Hilbert problem

$$G_R\,\varphi_+=\varphi_-\,,\quad \varphi_\pm\in(H_p^\pm)^2. \tag{5.5}$$

Taking $\varphi_\pm=(\varphi_{1\pm},\varphi_{2\pm})$, (5.5) is equivalent to

$$\begin{cases} \bar{\theta}\varphi_{1+} = \varphi_{1-} \\ R\varphi_{1+} + \theta\varphi_{2+} = \varphi_{2-}\,. \end{cases} \tag{5.6}$$

From the second equation in (5.6) we have

$$R\varphi_{1+} + \theta\varphi_{2+} = \varphi_{2-} = \frac{Q_1}{P_{\mathbb{C}^+}},$$

where Q_1 is a polynomial with $\deg Q_1 < N^+$, and taking into account the first equation in(5.6), we also obtain

$$R\varphi_{1-} - \frac{Q_1}{P_{\mathbb{C}^+}}\bar{\theta} = -\varphi_{2+} = \frac{Q_2}{P_{\mathbb{C}^-}},$$

where Q_2 is a polynomial with $\deg Q_2 < N^-$. It follows that we must have

$$\varphi_{1+} = \frac{Q_1 P_{\mathbb{C}^-} + Q_2 P_{\mathbb{C}^+}\theta}{P_N} \in H_p^+ \tag{5.7}$$

$$\varphi_{1-} = \frac{Q_1 P_{\mathbb{C}^-}\bar{\theta} + Q_2 P_{\mathbb{C}^+}}{P_N} \in H_p^-, \tag{5.8}$$

and it is clear that a necessary and sufficient condition for the kernel of T_{G_R} (or, equivalently, A_R^θ) to be nontrivial is that, for some polynomials Q_1 and Q_2, with $\deg Q_1 < N^+$ and $\deg Q_2 < N^-$, the conditions in (5.4) are satisfied. □

It follows that $\lambda \in \sigma(A_R^\theta)$ if and only if either $\lambda \in R\,(\Sigma(\theta) \cap \mathbb{R}_\infty)$, or there are polynomials Q_1 and Q_2 such that

$$\frac{Q_1 P_{\mathbb{C}^-} + Q_2 P_{\mathbb{C}^+}\theta}{P_N - \lambda P_{\mathbb{C}^+} P_{\mathbb{C}^-}} \in H_p^+ \setminus \{0\}\,, \; \frac{Q_1 P_{\mathbb{C}^-}\bar{\theta} + Q_2 P_{\mathbb{C}^+}}{P_N - \lambda P_{\mathbb{C}^+} P_{\mathbb{C}^-}} \in H_p^- \setminus \{0\}. \tag{5.9}$$

Remark 5.5. *Although* (5.9) *does not immediately provide a clear geometric description of the spectrum of A_R^θ for rational symbols with more than one pole, it nevertheless provides a simple criterion to know whether a particular value of $\lambda \in \mathbb{C}$ belongs to $\sigma(A_R^\theta)$. Thus, for instance, if $\theta(\xi) = e^{i\xi}$ and $R(\xi) = \frac{(\xi-i)(\xi+2i)}{(\xi+i)(\xi-2i)}$, we easily see that $0 \notin \sigma(A_R^\theta)$, i.e., A_R^θ is invertible.*

From these conditions we easily obtain a simple geometric description of the spectrum $\sigma(A_R^\theta)$ when R is a rational function with just one pole, as in Corollary 4.5 for the restricted shift A_r^θ . Assuming that

$$R(\xi) = \frac{A\xi + B}{\xi - z_0}, \tag{5.10}$$

with $Az_0 + B \neq 0, z_0 \in \mathbb{C} \setminus \mathbb{R}$, the function

$$F : \mathbb{C}_\infty \to \mathbb{C}_\infty\,,\; F(\lambda) = \frac{z_0\lambda + B}{\lambda - A}$$

is a bijection and we have

$$\lambda = R(\xi) \Leftrightarrow \xi = F(\lambda).$$

Let Γ_R denote the closed contour defined by $w = R(\xi), \xi \in \mathbb{R}$, and let Γ_R^* be its image in the complex plane, i.e., $\Gamma_R^* = R(\mathbb{R}_\infty)$. Note that

$$\lambda \in \Gamma_R^* \Leftrightarrow F(\lambda) \in \mathbb{R}_\infty$$

and, if $\lambda \notin \Gamma_R^*$, we have, for $z_0 \in \mathbb{C}^\mp$,

$$\lambda \in \operatorname{Int}\Gamma_R \Leftrightarrow \oint_{\Gamma_R} \frac{1}{w-\lambda}\,dw \neq 0 \Leftrightarrow F(\lambda) \in \mathbb{C}^\pm.$$

Theorem 5.6. *For all $p \in (1,\infty)$ and for R given by* (5.10)*, we have $\sigma(A_R^\theta) = \sigma_{ess}(A_R^\theta) \cup \sigma_P(A_R^\theta) = R(\Sigma(\theta))$ with*

$$\sigma_{ess}(A_R^\theta) = R(\Sigma(\theta) \cap \mathbb{R}_\infty) = R(\Sigma(\theta)) \cap \Gamma_R^*, \tag{5.11}$$

$$\sigma_P(A_R^\theta) = R(\Sigma(\theta) \cap \mathbb{C}^+) = R(\Sigma(\theta)) \cap \operatorname{Int}\Gamma_R. \tag{5.12}$$

Proof. The equality in (5.11) is an immediate consequence of the previous results. Now let, for example, $z_0 \in \mathbb{C}^-$ in (5.10). From (5.9) it follows that $\ker A_{R-\lambda}^\theta \neq \{0\}$ if and only if there exists $Q_2 \in \mathbb{C} \setminus \{0\}$ such that

$$\varphi_{1_+} = \frac{Q_2\,\theta}{(A-\lambda)\xi + B + \lambda z_0} \in H_p^+\,, \tag{5.13}$$

$$\varphi_{1_-} = \frac{Q_2}{(A-\lambda)\xi + B + \lambda z_0} \in H_p^-\,. \tag{5.14}$$

If $\lambda \in \mathbb{C} \setminus \operatorname{Int}\Gamma_R$, then the denominator in (5.14) vanishes for $\xi = F(\lambda) \in \mathbb{R}_\infty \cup \mathbb{C}^-$ and thus (5.14) is satisfied only if $Q_2 = 0$. If $\lambda \in \operatorname{Int}\Gamma_R$, then $F(\lambda) \in \mathbb{C}^+$ and (5.14) is satisfied for any $Q_2 \in \mathbb{C}$, but (5.13) implies that we must have $\theta(F(\lambda)) = 0$. Therefore $\ker A_{R-\lambda} \neq \{0\}$ if and only if $F(\lambda) \in \mathbb{C}^+$ and $\theta(F(\lambda)) = 0$, so that (5.12) holds.
The case $z_0 \in \mathbb{C}^+$ is similar, or can be deduced from the above by considering adjoints. $\square$

If λ belongs to the resolvent of A_R^θ, an analogous approach allows us, moreover, to determine $(A_R^\theta - \lambda I)^{-1}$ from $(T_{G_{R-\lambda}})^{-1}$ by means of (2.8) and (2.11). For those values of λ, $G_{R-\lambda}$ admits a canonical Wiener–Hopf (or generalised) p-factorisation (see (2.20)-(2.22))

$$G_{R-\lambda} = G_- \, G_+^{-1} \tag{5.15}$$

and the inverse of $T_{G_{R-\lambda}}$ is given by

$$(T_{G_{R-\lambda}})^{-1} = G_+ P^+ G_-^{-1} I_+ : (H_p^+)^2 \to (H_p^+)^2. \tag{5.16}$$

The factors $G_\pm$ can be explicitly determined by solving the Riemann–Hilbert problem

$$G_{R-\lambda} f_+ = f_- \, , \; f_\pm \in (\mathcal{H}_p^\pm)^2, \tag{5.17}$$

where we assume that $\lambda \notin \sigma(A_R^\theta) = R(\Sigma(\theta))$. In this case the Riemann–Hilbert problem (5.17) admits two linearly independent solutions (f_{1+}, f_{1-}) and (f_{2+}, f_{2-}), and we can take (f_{1+}, f_{2+}) (respectively, (f_{1-}, f_{2-})) as the two columns of G_+^{-1} (respectively, G_-), according to the following result, which was proved in [11] for $p = 2$, but is equally valid for any $p \in (1, \infty)$.

Theorem 5.7. *Let G possess a canonical generalised p-factorisation. Then, if (φ_+, φ_-) and (ψ_+, ψ_-) are two solutions to the equation*

$$G\varphi_+ = r\varphi_- \quad , \quad \varphi_\pm \in (H_p^\pm)^2$$

such that $\det[\varphi_+, \psi_+]\,(\xi) \neq 0$ for some $\xi \in \mathbb{C}^+$, then we can choose the factors in (2.20) *to be $G_\pm = [\varphi_\pm, \psi_\pm]$.*

As an illustration we consider the case of the truncated shift, with $R = r$. Using Theorem 5.7, we obtain, for $G_\pm$ in (5.15), assuming that $\lambda \notin \sigma(A_r^\theta) = r(\Sigma(\theta))$:

$$G_- = [g_{jk}^-] \quad , \quad G_+ = [g_{jk}^+] \quad , \quad j, k \in \{1, 2\},$$

where, defining $\xi_\lambda := i\frac{1+\lambda}{1-\lambda}$ and $\theta_\lambda = \theta(\xi_\lambda)$ if $|\lambda| \leq 1$, $\theta_\lambda = \theta(\overline{\xi_\lambda})$ if $|\lambda| > 1$:
(i) for $\lambda \neq 1$

$$\begin{aligned}
g_{11}^+ &= \frac{\theta(\xi_\lambda + i) - \theta_\lambda(\xi + i)}{(\xi_\lambda + i)(\xi - \xi_\lambda)} \, , \qquad g_{21}^+ = -\frac{1-\lambda}{\xi + i} \, , \\
g_{12}^+ &= \frac{\theta\xi(\xi_\lambda + i) - \theta_\lambda \xi_\lambda(\xi + i)}{(\xi_\lambda + i)(\xi - \xi_\lambda)} \, , \qquad g_{22}^+ = -\frac{(1-\lambda)\xi}{\xi + i} \, , \\
g_{11}^- &= \frac{(\xi_\lambda + i) - \theta_\lambda(\xi + i)\bar{\theta}}{(\xi_\lambda + i)(\xi - \xi_\lambda)} \, , \qquad g_{21}^- = -\frac{(1-\lambda)\theta_\lambda}{\xi_\lambda + i} \, , \\
g_{12}^- &= \frac{\xi(\xi_\lambda + i) - \theta_\lambda \xi_\lambda(\xi + i)\bar{\theta}}{(\xi_\lambda + i)(\xi - \xi_\lambda)} \, , \qquad g_{22}^- = -\frac{(1-\lambda)\theta_\lambda \xi_\lambda}{\xi_\lambda + i} \, ;
\end{aligned}$$

(ii) for $\lambda = 1$

$$g_{11}^{+} = \theta\,,\; g_{21}^{+} = \frac{2i}{\xi + i}\,,\; g_{12}^{+} = [\theta - \theta(\infty)]\xi - i\theta(\infty)\,,\; g_{22}^{+} = \frac{2i\xi}{\xi + i}\,,$$
$$g_{11}^{-} = 1\,,\; g_{21}^{-} = 0\,,\; g_{12}^{-} = [1 - \bar{\theta}\theta(\infty)]\xi - i\bar{\theta}\theta(\infty)\,,\; g_{22}^{-} = 2i\theta(\infty).$$

We remark that $G_\pm \in \mathcal{G}(H_\infty^\pm)^{2\times 2}$, i.e., the canonical factorisation is bounded and does not depend on p. Thus the operator $G_+P^+G_-^{-1}I_+$ is defined in $(H_p^+)^2$ and $(A_r^\theta)^{-1}$ is given by (2.16), (2.17) and (2.23), with $G_\pm$ defined as above, for all $p \in (1, \infty)$.

6 Truncated Toeplitz operators on finite-dimensional model spaces

Let B be a finite Blaschke product

$$\prod_{j=1}^{N} \left(\frac{\xi - z_j}{\xi - \bar{z}_j}\right)^{m_j}, \qquad z_j \in \mathbb{C}^+\,,\; \sum_{j=1}^{N} m_j = n\,, \tag{6.1}$$

and let A_g^B be a TTO with symbol $g \in L_\infty$ defined in K_B. By Theorem 2.3

$$A_g^B \overset{*}{\sim} T_G \quad \text{where} \quad \mathrm{G} = \begin{pmatrix} \bar{B} & 0 \\ g & B \end{pmatrix}. \tag{6.2}$$

It is clear that A_g^B is Fredholm with index zero for any $g \in L_\infty$, thus it is invertible if and only if $\ker A_g^B = \{0\}$, i.e., $\ker T_G = \{0\}$. Now, characterising $\ker T_G$ is equivalent to solving the Riemann–Hilbert problem

$$G\varphi_+ = \varphi_-, \quad \varphi_\pm \in (H_p^\pm)^2, \tag{6.3}$$

which, taking $\varphi_\pm = (\varphi_{1\pm}, \varphi_{2\pm})$, can be written as

$$\begin{cases} \bar{B}\varphi_{1+} = \varphi_{1-} \\ g\varphi_{1+} + B\varphi_{2+} = \varphi_{2-}\,. \end{cases} \tag{6.4}$$

From the first equation we have

$$\varphi_{1+} = \frac{P_{n-1}}{P_{\bar{z}_1,\dots,\bar{z}_N}} \quad \text{with} \quad P_{n-1} \in \mathcal{P}_{n-1}\;,\; P_{\bar{z}_1,\dots,\bar{z}_N} = \prod_{j=1}^{N} (z - \bar{z}_j)^{m_j} \tag{6.5}$$

and, substituting in the second equation of (6.4), we get

$$P^+\left(g\frac{P_{n-1}}{P_{\bar z_1,...,\bar z_N}}\right)+B\varphi_{2+}=-P^-\left(g\frac{P_{n-1}}{P_{\bar z_1,...,\bar z_N}}\right)+\varphi_{2-}=0\,.$$

Therefore,

$$B\varphi_{2+}=P^+\left(g\frac{P_{n-1}}{P_{\bar z_1,...,\bar z_N}}\right) \tag{6.6}$$

and it follows that (6.3) has a nonzero solution if and only if the function on the right-hand side of (6.6) has a zero of order m_j at each point z_j , $j = 1, 2, ..., N$. Writing

$$P_{n-1}=C_0+C_1\xi+...+C_{n-1}\xi$$

where $C_0, C_1, ..., C_{n-1} \in \mathbb{C}$, that condition is equivalent to the existence of a nontrivial solution to the linear system

$$[M_{k,l}]C=0, \quad C=[C_0\ C_1,\ \dots\ ,C_{n-1}]^T \tag{6.7}$$

with

$$M_{k,l}=\left[\frac{ds_k}{d\xi^{s_k}}P^+(\xi^l g)\right]_{(w_k)} \quad , \quad k,l=0,1,...,n-1, \tag{6.8}$$

where s_k and w_k are defined by

$$\begin{cases} s_k=k\,,\ w_k=z_1\,, \quad \text{if} \quad k=0,...,m_1-1\,,\\ s_k=k-m_1\,,\ w_k=z_2\,, \quad \text{if} \quad k=m_1,...,m_1+m_2-1\,,\\ \quad\quad ...\\ s_k=k-(m_1+...+m_{N-1})\,,\ w_k=z_N\,, \quad \text{if} \quad k=m_1+...+m_{N-1}\,. \end{cases} \tag{6.9}$$

We have thus proved the following.

Theorem 6.1. *The operator A_g^B is invertible if and only if*

$$\det\,[M_{k,l}]_{k,l=0,...,n-1}\neq 0 \tag{6.10}$$

where the entries $M_{k,l}$ are defined by (6.8) *and* (6.9).

Using the factorisation

$$B=h_-r^n h_+ \tag{6.11}$$

with $n \in \mathbb{N}, h_\pm \in \mathcal{G}(\mathcal{R}\cap H_\infty^\pm)$ and $h_-^{-1}=\overline{h_+}$, we also have:

Theorem 6.2. *The operator A_g^B is invertible in K_B if and only if $A_{\tilde{g}}^{r^n}$ is invertible in K_{r^n}, where*

$$\tilde{g} = h_-^{-1} g h_+. \tag{6.12}$$

Proof. From (6.11) it follows that G can be factorised as

$$G = \begin{pmatrix} \overline{h_+} & 0 \\ 0 & h_- \end{pmatrix} \begin{pmatrix} r^{-n} & 0 \\ \tilde{g} & r^n \end{pmatrix} \begin{pmatrix} \overline{h_-} & 0 \\ 0 & h_+ \end{pmatrix}. \tag{6.13}$$

Denoting by $\tilde{G}$ the middle factor on the right-hand side of (6.13), and taking into account that the left-hand side factor is invertible in $(H_\infty^-)^{2\times 2}$, while the right-hand side factor is invertible in $(H_\infty^+)^{2\times 2}$, we have

$$A_g^B \text{ is invertible} \Leftrightarrow T_G \text{ is invertible} \Leftrightarrow T_{\tilde{G}} \text{ is invertible} \Leftrightarrow A_{\tilde{g}}^B \text{ is invertible.}$$

□

Corollary 6.3. *The operator A_g^B is invertible in K_B if and only if*

$$\det\,[g_{k,l}]_{k,l=0,\ldots,n-1} \neq 0 \tag{6.14}$$

where

$$g_{k,l} = (\tilde{g}_l^+)_{(i)}^{(k)} \quad \text{with} \quad \tilde{g}_l^+ = P^+(\frac{\tilde{g}}{\xi + i}\, r^l)\ ,\ l == 0, ..., n-1.$$

Proof. Following the proof of Theorem 6.1 with $\tilde{g}$ and r^n instead of g and B, respectively, equation (6.6) becomes

$$r^n \varphi_{2+} = -P^+\left(\tilde{g}\frac{P_{n-1}}{(\xi + i)^n}\right). \tag{6.15}$$

Using the equality

$$\frac{P_{n-1}}{(\xi+i)^n} = \frac{A_0 + A_1 r + ... + A_{n-1} r^{n-1}}{\xi + i}$$

where $A_0, A_1, ..., A_{n-1} \in \mathbb{C}$, the matrix equation (6.7) can be replaced by

$$[g_{k,l}]A = 0 \quad , \quad A = [A_0\ A_1,\ \ldots\ , A_{n-1}]^T$$

which has a nontrivial solution if and only if (6.14) holds. □

Note that, using the relation

$$P^{\pm}(rf) = rf^{\pm} \mp \frac{2i}{\xi + i} f^{-}_{(-i)} \tag{6.16}$$

where $f \in L_p$ and $f^{\pm} = P^{\pm} f$, all the elements $g_{k,l}$ in (6.14) can be expressed in terms of $\tilde{g}_0^{\pm} := P^{\pm}\left(\frac{\tilde{g}}{\xi + i}\right)$ and their derivatives at $\pm i$, respectively.

The invertibility criteria of Theorem 6.2 and Corollary 6.3 enable us to determine the n eigenvalues (counting their multiplicity) of A_g^B and to characterise the corresponding eigenspaces, as illustrated in the following example.

Example Let $B = r^2$, $g \in L_\infty$. By Corollary 6.3, and using (6.16), for any $\lambda \in \mathbb{C}$ the operator $A_{g-\lambda}^{r^2}$ is invertible if and only if

$$\det \begin{pmatrix} (g_0^+)_{(i)} - \frac{\lambda}{2i} & -(g_0^-)_{(-i)} \\ (g_0^+)'_{(i)} + \frac{\lambda}{(2i)^2} & \frac{1}{2i}[((g_0^+)_{(i)} - \frac{\lambda}{2i}) + (g_0^-)_{(-i)}] \end{pmatrix} \neq 0.$$

Thus, the eigenvalues of $A_g^{r^2}$ are the zeroes of the second degree polynomial in λ

$$D(\lambda) = [(g_0^+)_{(i)} - \frac{\lambda}{2i}]^2 + (g_0^-)_{(-i)}[(g_0^+)_{(i)} + 2i(g_0^+)'_{(i)}].$$

If

$$(g_0^+)_{(i)} + 2i(g_0^+)'_{(i)} = 0\,, \tag{6.17}$$

then we have a double zero

$$\lambda_0 = 2i(g_0^+)_{(i)}. \tag{6.18}$$

The corresponding eigenspace $\ker A_{g-\lambda_0}^{r^2}$ is determined by the solutions of the equation

$$\begin{pmatrix} (g_0^+)_{(i)} - \frac{\lambda_0}{2i} & -(g_0^-)_{(-i)} \\ (g_0^+)'_{(i)} + \frac{\lambda_0}{(2i)^2} & \frac{1}{2i}[((g_0^+)_{(i)} - \frac{\lambda_0}{2i}) + (g_0^-)_{(-i)}] \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.19}$$

It is easy to see that

$$\ker A_{g-\lambda_0}^{r^2} = \operatorname{span}\{\frac{1}{\xi + i}\}\,, \quad \text{if } (g_0^-)_{(-i)} \neq 0 \tag{6.20}$$

$$\ker A_{g-\lambda_0}^{r^2} = K_{r^2}\,, \quad \text{if } (g_0^-)_{(-i)} = 0. \tag{6.21}$$

If (6.17) is not satisfied, then $A_g^{r^2}$ has two simple eigenvalues, and the corresponding eigenspaces can be determined analogously from (6.19).

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