Published paper
Ruderman, M.S. (2009) *Resonant magnetohydrodynamic waves in high-beta plasmas.* Physics of Plasmas, 16 (4). Art. No.042109.
http://dx.doi.org/10.1063/1.3119689

Resonant magnetohydrodynamic waves in high-beta plasmas

M. S. Ruderman
[1] *Department of Applied Mathematics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, United Kingdom (e-mail: M.S.Ruderman@sheffield.ac.uk)*
(Dated: June 29, 2009)

When a global magnetohydrodynamic (MHD) wave propagates in a weakly dissipative inhomogeneous plasma, the resonant interaction of this wave with either local Alfvén or slow MHD waves is possible. This interaction occurs at the resonant position where the phase velocity of the global wave coincides with the phase velocity of either Alfvén or slow MHD waves. As a result of this interaction a dissipative layer embracing the resonant position is formed, its thickness being proportional to $R^{-1/3}$, where $R \gg 1$ is the Reynolds number. The wave motion in the resonant layer is characterised by large amplitudes and large gradients. The presence of large gradients causes strong dissipation of the global wave even in very weakly dissipative plasmas. Very often the global wave motion is characterised by the presence of both Alfvén and slow resonances. In plasmas with small or moderate plasma beta, β, the resonance positions corresponding to the Alfvén and slow resonances are well separated, so that the wave motion in the Alfvén and slow dissipative layers embracing the Alfvén and slow resonant positions respectively, can be studied separately. However, when $\beta \gtrsim R^{1/3}$, the two resonance positions are so close that the two dissipative layers overlap. In this case, instead of two dissipative layers, there is one mixed Alfvén-slow dissipative layer. In this paper the wave motion in such a mixed dissipative layer is studied. It is shown that this motion is a linear superposition of two motions, one corresponding to the Alfvén, and the other to the slow dissipative layer. The jump of normal velocity across the mixed dissipative layer related to the energy dissipation rate is equal to the sum of two jumps, one that occurs across the Alfvén dissipative layer, and the other across the slow dissipative layer.

PACS numbers: 52.35.Bj, 52.50.Gj, 96.60.Ly

I. INTRODUCTION

In nature and in the laboratory, magnetohydrodynamic (MHD) waves propagate in nonuniform plasmas. It can occur that the phase velocity of a global MHD wave coincides with the local phase velocity of either Alfvén or slow MHD waves at a spatial position, called either Alfvén or slow resonant position. In this case the global wave resonantly interacts with either Alfvén or slow MHD waves at the resonant position. As a result the energy of the global wave is transferred to the local waves, which causes their growth. This process is called resonant absorption. In an ideal plasma the amplitude of the wave motion at the resonant position will grow unboundedly. Dissipation stops this unbounded growth. If the global wave is driven by an external driver, then this occurs when the energy dissipation rate is equal to the energy deposition rate. Hence, the driven wave motion attains a stationary state after a transitional time interval. In a weakly dissipative plasma the energy dissipation occurs only in a thin dissipative layer embracing the resonant position, its thickness being proportional to $R^{-1/3}$, where $R \gg 1$ is the Reynolds number. The amplitude of the wave motion in the dissipative layer is of the order of $R^{1/3}$ times the amplitude of the global wave far from the resonant position. Hence, the motion in the dissipative layer is characterised by large amplitudes. Since the thickness of the resonant layer is proportional to $R^{-1/3}$, it is also characterised by large gradients. Due to the presence of large gradients in the dissipative layer resonant absorption can efficiently dissipate the wave energy even in very weakly dissipative plasmas. An important property of resonant absorption is that the energy dissipation rate is independent of R when $R \gg 1$. The wave energy dissipation results in the plasma heating. Hence, resonant absorption is an efficient mechanism of heating of weakly dissipative plasmas.

The possibility of plasmas heating has made resonant absorption the subject of intense study. First it was suggested as a means for the additional heating of fusion plasmas.[1–4] Later resonant absorption was proposed as a mechanism for heating of the solar corona.[5]

Another important property of resonant MHD waves that has attracted ample attention of theorists was very efficient transfer of energy from a global MHD wave to local oscillations. This property of resonant MHD waves inspired theorists to suggest these waves for the explanation of the observed large-amplitude pulsations in the Earth magnetosphere.[6–9] Resonant absorption was also proposed as a mechanism of damping of (that time hypothetical) kink oscillations of magnetic loops in the solar corona.[10] This idea was revived after these oscillations have been observed by the Transition Region and Coronal Explorer (TRACE) spacecraft.[11,12] It was shown that resonant absorption can provide the observed damping of the coronal loop kink oscillations under very general assumptions about the equilibrium state on which these oscillations are imposed.[13,14]

Resonant absorption has been most intensively studied from two different points of view. The first point of view considers the initial value problem. In this problem a global MHD wave is excited at the initial moment of time, and then it damps due to resonant absorption. The second point of view focuses on a driven problem with an external source of energy that excites plasma oscillation. In this paper we concentrate on the driven problem. In particular, we focus on the steady state of driven MHD waves where, in the linear approximation, the perturbations of all quantities oscillate harmonically in time at the frequency of the external driver.

The linear ideal MHD spectrum of a nonuniform plasma consists of Alfvén and slow continua in addition to discrete eigenfrequencies. The generalised eigenfunctions corresponding to frequencies from these continua are non-square-integrable.[15] If the frequency of the global MHD wave, ω, is within one of the two continua, then the solution of the linearised MHD equations has singularities at the resonant magnetic surfaces where ω coincides either with the local Alfvén or local slow frequency. Dissipative effects such as viscosity, resistivity or thermal conduction remove these singularities. In what follows we only consider viscosity and resistivity, while we neglect thermal conduction. It is convenient to introduce the viscous and magnetic Reynolds numbers, $R_e = l_{\rm ch}v_{\rm ch}/\nu$ and $R_m = l_{\rm ch}v_{\rm ch}/\lambda$, where $l_{\rm ch}$ is the characteristic spatial scale, $v_{\rm ch}$ the characteristic speed, ν kinematic viscosity, and λ the coefficient of magnetic diffusion. Usually $v_{\rm ch}$ is taken to be of the order of the phase speed of the global MHD wave, and $l_{\rm ch}$ is taken to be equal either to the wave length or to the characteristic spatial scale of inhomogeneity. Then, to characterise the total dissipation we introduce the total Reynolds number R defined by

$$\frac{1}{R} = \frac{1}{R_e} + \frac{1}{R_m}. \tag{1}$$

As we have already mentioned, when $R \gg 1$, dissipation only operates in thin dissipative layers embracing ideal resonant magnetic surfaces. The thickness of a dissipative layer is of the order of $R^{-1/3}$. The linear theory of MHD dissipative layers has been developed by the effort of many authors. In particular, it has been shown that the wave motion both in Alfvén and slow dissipative layers can be described in term of the so-called F and G functions.[16,17]

Very often the global wave motion is characterised by the presence of both Alfvén and slow resonances. Let us introduce the plasma beta, β, as the ratio of the characteristic plasma and magnetic pressures. When $\beta \lesssim 1$ the resonance positions corresponding to the Alfvén and slow resonances are well separated, so that the wave motion in the Alfvén and slow dissipative layers can be studied separately. However, when $\beta \gtrsim R^{1/3}$, the two resonance positions are so close that the two dissipative layers overlap. In this case, instead of two dissipative layers, there is one mixed Alfvén-slow dissipative layer. When $\beta \to \infty$ we obtain the extreme case of an incompressible plasma.

This situation can be typical for the solar interior. Gough and McIntyre[18] argued that the magnetic field has to be present in the tachocline separating the radiative and convective zones of the sun. Taking the temperature $T \approx 1.5 \times 10^6$ K and density $\rho \approx 2 \times 10^2$ kg m^{-3} at the tachocline,[18] we obtain for the plasma pressure $p \approx 5 \times 10^{12}$ Pa. For the magnetic pressure we obtain $B^2/2\mu_0 \approx 4 \times 10^5 B^2$ Pa, where $\mu_0 = 4\pi \times 10^{-7}$ Tesla^2 Pa^{-1} is the magnetic permeability of empty space, and the magnetic field magnitude B is measured in Tesla. Then we have for the plasma beta $\beta \approx 10^7 B^{-2}$.

Let us now estimate R. Gough and McIntyre[18] give $\lambda \approx 0.07$ m^2 s^{-1}. It can be shown that $\nu \ll \lambda$, so that $R_m \ll R_e$ and $R \approx R_m$. Taking $v_{\rm ch} = V_A$, where $V_A = B/\sqrt{\mu_0\rho} \approx 60\,B$ m s^{-1} is the Alfvén speed, and $l_{\rm ch} = V_A\Pi \approx 60\,B\Pi$ m, where Π is the wave period, we obtain $R \approx 5 \times 10^4 B^2\Pi$. Then, eventually, we arrive at

$$\beta R^{-1/3} \approx 3 \times 10^5 B^{-8/3}\Pi^{-1/3}. \tag{2}$$

In what follows we take $\Pi = 10^3$ s. Gough and McIntyre[18] estimated that the magnetic field magnitude needed to provide the transition from the differential rotation observed in the convective zone of the sum to the rigid body rotation below the tachocline is 10^{-4} Tesla. Substituting this value in Eq. (2) with $\Pi = 10^3$ s we obtain $\beta R^{-1/3} \approx 1.4\times10^{15}$. In this case the slow and Alfvén resonant positions practically coincide, and the motion in the dissipative layer can be considered as incompressible. However, Gough and McIntyre[18] estimated only the poloidal magnetic field. Many contemporary theories of the solar magnetic dynamo assume that a strong toroidal magnetic field is generated near the tachocline. We obtain $\beta R^{-1/3} \approx 1$ if we take $B \approx 50$ Tesla, which is quite realistic value of the toroidal magnetic field magnitude near the tachocline. Hence, for the magnetic field magnitude of the order of a few tens of Teslas the motion in the mixed dissipative layer cannot be described in the incompressible plasma approximation.

The limit of incompressible plasmas has been considered by many authors (e.g., Ref. [19]). In this paper we aim to study the wave motion in a mixed dissipative layer when β is large but finite. The paper is organised as follows. In the next section we formulate the problem and present the equilibrium state and the governing equations. In Sect. III we obtain the solution describing the wave motion in a mixed dissipative layer. Sect. IV contains the summary of the obtained results and our conclusions.

II. PROBLEM FORMULATION

We consider the wave motion superimposed on a static one-dimensional equilibrium. In this equilibrium all quantities depend on the x coordinate of Cartesian coordinates x, y, z only. The equilibrium pressure, density, and magnetic field magnitude are $p_0(x)$, $\rho_0(x)$, and $B_0(x)$

respectively. The equilibrium magnetic field, $\boldsymbol{B}_0$, is in the yz-plane, and the angle between $\boldsymbol{B}_0$ and the z-axis is $\alpha = \text{const}$. Hence,

$$\boldsymbol{B}_0 = B_0(0, \sin\alpha, \cos\alpha). \tag{3}$$

The equilibrium quantities satisfy the equilibrium condition

$$p_0 + \frac{B_0^2}{\mu_0} = \text{const}. \tag{4}$$

The linearised system of MHD equations is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}) = 0, \tag{5}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla P + \frac{1}{\mu_0}(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{b} + \frac{b_x}{\mu_0}\frac{d\boldsymbol{B}_0}{dx} + \rho_0 \nu \frac{\partial^2 \boldsymbol{v}}{\partial x^2}, \tag{6}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{v} - u\frac{d\boldsymbol{B}_0}{dx} - \boldsymbol{B}_0 \nabla \cdot \boldsymbol{v} + \lambda \frac{\partial^2 \boldsymbol{b}}{\partial x^2}, \tag{7}$$

$$\frac{\partial}{\partial t}(p - c_S^2 \rho) + u\left(\frac{dp_0}{dx} - c_S^2 \frac{d\rho_0}{dx}\right) = 0, \tag{8}$$

$$P = p + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0}. \tag{9}$$

Here $\boldsymbol{v} = (u, v, w)$ is the velocity, p, ρ and $\boldsymbol{b} = (b_x, b_y, b_z)$ are the perturbations of the plasma pressure, density and magnetic field, and P is the perturbation of the total pressure (plasma plus magnetic); c_S is the sound speed determined by $c_S^2 = \gamma p_0/\rho_0$, and γ is the ratio of specific heats. Since $R \gg 1$, dissipation is only important in narrow resonant layers, where there are large gradients in the x-direction, while gradients in the y and z-direction are relatively small. This observation enabled us to write the dissipative terms, which are the terms proportional to ν and λ in Eqs. (6) and (7), in a simplified form. Equation (8) is the linearised adiabatic equation. It expresses the entropy conservation. Dissipation causes the entropy to increase. However, when the equilibrium is static, the dissipative terms in the entropy equation that describe the entropy increase due to viscosity and resistivity are nonlinear with respect to perturbations. Hence they do not appear in the linearised entropy equation, so that the entropy is conserved in the linear approximation.

III. SOLUTION IN DISSIPATIVE LAYERS

Since the equilibrium quantities are independent of time and y and z-coordinate, we can Fourier-analyse Eqs. (5)–(9) with respect to y, z and time. We also can take the wave vector in the z-direction. After that we take the perturbations of all quantities proportional to $\exp(-i\omega t + ikz)$. We assume that the plasma is only weakly dissipative and take $R^{-1/3} = \epsilon \ll 1$. Then $R = \epsilon^{-3}$, which inspires us to introduce the scaled kinematic viscosity and the coefficient of magnetic diffusion,

$$\bar{\nu} = \epsilon^{-3}\nu, \quad \bar{\lambda} = \epsilon^{-3}\lambda. \tag{10}$$

The ideal Alfvén resonant position x_A is defined by

$$V_A(x_A)\cos\alpha = \frac{\omega}{k}, \quad V_A^2 = \frac{B_0^2}{\mu_0 \rho_0}, \tag{11}$$

where V_A is the Alfvén speed. Let us introduce the tube speed with the square given by

$$c_T^2 = \frac{c_S^2 V_A^2}{c_S^2 + V_A^2}. \tag{12}$$

The ideal slow resonant position x_c is defined by

$$c_T(x_c)\cos\alpha = \frac{\omega}{k}. \tag{13}$$

When $\beta \gg 1$ we have $c_S^2/V_A^2 = \mathcal{O}(\beta)$. Then Eq. (13) can be rewritten as

$$V_A(x_c)\cos\alpha\left(1 - \frac{V_A^2(x_c)}{2c_S^2(x_c)} + \mathcal{O}(\beta^{-2})\right) = \frac{\omega}{k}. \tag{14}$$

It immediately follows from Eqs. (12) and (14) that

$$x_c = x_A + \frac{V_A^4(x_A)\cos^2\alpha}{\Delta c_s^2(x_A)} + \mathcal{O}(\beta^{-2}), \tag{15}$$

where

$$\Delta = \cos^2\alpha \left.\frac{dV_A^2}{dx}\right|_{x=x_A}. \tag{16}$$

Since $\Delta \sim V_A^2/l_{\rm ch}$, where $l_{\rm ch}$ is the characteristic scale of inhomogeneity, it follows from Eq. (15) that $|x_c - x_A| = \mathcal{O}(\beta^{-1} l_{\rm ch})$. Since the thicknesses of the dissipative layers are of the order of $R^{-1/3} l_{\rm ch} = \epsilon l_{\rm ch}$, the two dissipative layers overlap when $\beta \gtrsim \epsilon^{-1}$. In what follows we assume that this inequality is held. Then $c_S^2 = \mathcal{O}(\epsilon^{-1} V_A^2)$, and it is convenient to introduce the scaled sound speed C_S defined by $C_S^2 = \epsilon c_S^2$. When $\beta \gtrsim \epsilon^{-1}$, the two dissipative layers are situated in a spatial region defined by $|x - x_A| \lesssim \epsilon l_{\rm ch}$. This observation inspires us to introduce the stretching spatial variable $\xi = \epsilon^{-1}(x - x_A)$.

It is well-known that the wave motion in an Alfvén dissipative layer mainly resides in the components of the velocity and magnetic field perturbation that are perpendicular both to the equilibrium magnetic field and the inhomogeneity direction (see, e.g., Ref. [16,20]). In a slow dissipative layer it mainly resides in the components of the velocity and magnetic field perturbation that are parallel to the equilibrium magnetic field. This implies that it is convenient to introduce the parallel and perpendicular components of the velocity and magnetic field perturbation,

$$v_\parallel = v\sin\alpha + w\cos\alpha, \quad v_\perp = v\cos\alpha - w\sin\alpha,$$
$$b_\parallel = b_y\sin\alpha + b_z\cos\alpha, \;\; b_\perp = b_y\cos\alpha - b_z\sin\alpha. \tag{17}$$

Now, using the new variables, we rewrite Eqs. (5)–(9) as

$$\epsilon^{-1}\rho_0\frac{du}{d\xi} = i\omega\rho - i\rho_0 k(v_\parallel\cos\alpha - v_\perp\sin\alpha) - u\frac{d\rho_0}{dx}, \tag{18}$$

$$\epsilon^{-1}\frac{dP}{d\xi} = i\omega\rho_0 u + \frac{ik}{\mu_0}B_0 b_x\cos\alpha + \epsilon\rho_0\bar{\nu}\frac{d^2u}{d\xi^2}, \tag{19}$$

$$\begin{aligned} i\omega\rho_0 v_\parallel \;&+\; \frac{ik}{\mu_0}B_0 b_\parallel\cos\alpha \\ &=\; ikP\cos\alpha - \frac{b_x}{\mu_0}\frac{dB_0}{dx} - \epsilon\rho_0\bar{\nu}\frac{d^2v_\parallel}{d\xi^2}, \end{aligned} \tag{20}$$

$$i\omega\rho_0 v_\perp + \frac{ik}{\mu_0}B_0 b_\perp\cos\alpha = -ikP\sin\alpha - \epsilon\rho_0\bar{\nu}\frac{d^2v_\perp}{d\xi^2}, \tag{21}$$

$$i\epsilon^{-1}(\omega b_x + kB_0 u\cos\alpha) = -\bar{\lambda}\frac{d^2b_x}{d\xi^2}, \tag{22}$$

$$i(\omega b_\parallel + kB_0 v_\perp\sin\alpha) - \epsilon^{-1}B_0\frac{du}{d\xi} = u\frac{dB_0}{dx} - \epsilon\bar{\lambda}\frac{d^2b_\parallel}{d\xi^2}, \tag{23}$$

$$i(\omega b_\perp + kB_0 v_\perp\cos\alpha) = -\epsilon\bar{\lambda}\frac{d^2b_\perp}{d\xi^2}, \tag{24}$$

$$i\omega(p - \epsilon^{-1}C_S^2\rho) = u\left(\frac{dp_0}{dx} - \epsilon^{-1}C_S^2\frac{d\rho_0}{dx}\right), \tag{25}$$

$$P = p + \frac{1}{\mu_0}B_0 b_\parallel. \tag{26}$$

Note that, in accordance with Eq. (4), dp_0/dx is of the order of magnetic pressure divided by $l_{\rm ch}$, and not of the order of $p_0/l_{\rm ch}$.

The system of Eqs. (17)–(26) contains the small parameter ϵ. This implies that we can look for a solution to this system in the form of asymptotic expansions with respect to ϵ. We write these expansions for the parallel and perpendicular components of the velocity and magnetic field perturbations as

$$v_\parallel = v_{\parallel 1} + \epsilon v_{\parallel 2} + \ldots, \qquad v_\perp = v_{\perp 1} + \epsilon v_{\perp 2} + \ldots,$$
$$b_\parallel = b_{\parallel 1} + \epsilon b_{\parallel 2} + \ldots, \qquad b_\perp = b_{\perp 1} + \epsilon b_{\perp 2} + \ldots. \tag{27}$$

Then it follows from Eqs. (18), (19), (22), (25) and (26) that the expansion for p has to start from a term of the order of unity, while the expansions for u, b_x, ρ and P have to start from terms of the order of ϵ. Hence, we write the expansions for these quantities as

$$\begin{aligned} &u = \epsilon u_1 + \epsilon^2 u_2 + \ldots, \qquad &&\rho = \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots, \\ &b_x = \epsilon b_{x1} + \epsilon^2 b_{x2} + \ldots, \qquad &&p = p_1 + \epsilon p_2 + \ldots, \\ &P = \epsilon P_1 + \epsilon^2 P_2 + \ldots. && \end{aligned} \tag{28}$$

Substituting expansions (27) and (28) in Eqs. (17)–(26) and collecting terms of the order of unity we obtain

$$\frac{du_1}{d\xi} = -ik(v_{\parallel 1}\cos\alpha - v_{\perp 1}\sin\alpha), \tag{29}$$

$$\frac{dP_1}{d\xi} = 0, \tag{30}$$

$$\omega\rho_0 v_{\parallel 1} + \frac{k}{\mu_0}B_0 b_{\parallel 1}\cos\alpha = 0, \tag{31}$$

$$\omega\rho_0 v_{\perp 1} + \frac{k}{\mu_0}B_0 b_{\perp 1}\cos\alpha = 0, \tag{32}$$

$$\omega b_{x1} + kB_0 u_1\cos\alpha = 0, \tag{33}$$

$$B_0\frac{du_1}{d\xi} = i(\omega b_{\parallel 1} + kB_0 v_{\perp 1}\sin\alpha), \tag{34}$$

$$\omega b_{\perp 1} + kB_0 v_{\perp 1}\cos\alpha = 0, \tag{35}$$

$$\omega(p_1 - C_S^2\rho_1) = -u_1 C_S^2\frac{d\rho_0}{dx}, \tag{36}$$

$$p_1 + \frac{1}{\mu_0}B_0 b_{\parallel 1} = 0. \tag{37}$$

All the coefficient in this system of equations are calculated at $x = x_A$.

First of all, it follows from Eq. (30) that $P_1 = $ const. Equations (32) and (34) constitute the system of two linear algebraic equations for $v_{\perp 1}$ and $b_{\perp 1}$. This system has a non-trivial solution only when its determinant is zero. This condition gives $\omega^2 = V_A^2 k^2\cos^2\alpha$. If we consider waves propagating in the positive z-direction, then it reduces to Eq. (11). Now we can easily obtain the following relations:

$$b_{x1} = -\frac{B_0}{V_A}u_1, \quad b_{\parallel 1} = -\frac{B_0}{V_A}v_{\parallel 1}, \quad b_{\perp 1} = -\frac{B_0}{V_A}v_{\perp 1}, \tag{38}$$

$$\rho_1 = \frac{\rho_0 V_A}{C_S^2}v_{\parallel 1} - \frac{iu_1}{\omega}\frac{d\rho_0}{dx}. \tag{39}$$

Collecting terms of the order of ϵ in Eqs. (18), (20), (21), (23) and (24), and using Eqs. (29), (38) and (39), we obtain the system of equations

$$\frac{du_2}{d\xi} + ik(v_{\|2}\cos\alpha - v_{\perp 2}\sin\alpha) = \frac{i\omega V_A}{C_S^2}\, v_{\|1}, \tag{40}$$

$$\begin{aligned} i\omega\left(v_{\|2} + \frac{V_A}{B_0}\, b_{\|2}\right) &= \frac{ik}{\rho_0}\, P_1\cos\alpha + \frac{V_A u_1}{B_0}\frac{dB_0}{dx} \\ &- i\omega\left(\frac{1}{\rho_0}\frac{d\rho_0}{dx} - \frac{1}{B_0}\frac{dB_0}{dx}\right)\xi v_{\|1} - \bar\nu\frac{d^2 v_{\|1}}{d\xi^2}, \end{aligned} \tag{41}$$

$$\begin{aligned} i\omega\left(v_{\perp 2} + \frac{V_A}{B_0}\, b_{\perp 2}\right) &= -\frac{ik}{\rho_0}\, P_1\sin\alpha \\ &- i\omega\left(\frac{1}{\rho_0}\frac{d\rho_0}{dx} - \frac{1}{B_0}\frac{dB_0}{dx}\right)\xi v_{\perp 1} - \bar\nu\frac{d^2 v_{\perp 1}}{d\xi^2}, \end{aligned} \tag{42}$$

$$\begin{aligned} i(\omega b_{\|2} \;+\; kB_0 v_{\perp 2}\sin\alpha) - B_0\frac{du_2}{d\xi} &= u_1\frac{dB_0}{dx} \\ &- ik\frac{dB_0}{dx}\,\xi v_{\|1}\cos\alpha + \frac{B_0\bar\lambda}{V_A}\frac{d^2 v_{\|1}}{d\xi^2}, \end{aligned} \tag{43}$$

$$i\omega\left(v_{\perp 2} + \frac{V_A}{B_0}\, b_{\perp 2}\right) = -\frac{i\omega}{B_0}\frac{dB_0}{dx}\,\xi v_{\perp 1} + \bar\lambda\frac{d^2 v_{\perp 1}}{d\xi^2}. \tag{44}$$

Once again all the coefficient in this system of equations are calculated at $x = x_A$. Eliminating u_2 from Eqs. (40) and (43) yields

$$\begin{aligned} i\omega\left(v_{\|2} + \frac{V_A}{B_0}\, b_{\|2}\right) &= \frac{i\omega V_A^2}{C_S^2}\, v_{\|1} + \frac{V_A u_1}{B_0}\frac{dB_0}{dx} \\ &- \frac{i\omega}{B_0}\frac{dB_0}{dx}\,\xi v_{\|1} + \bar\lambda\frac{d^2 v_{\|1}}{d\xi^2}. \end{aligned} \tag{45}$$

Equations (42) and (44) constitute a system of two linear inhomogeneous algebraic equations for $v_{\perp 2}$ and $b_{\perp 2}$. The determinant of this system is zero, so that the system has solutions only when the right-hand sides of Eqs. (42) and (44) satisfy the compatibility condition. To obtain this condition we subtract Eq. (44) from Eq. (42). As a result we obtain

$$\Delta\xi v_{\perp 1} + \frac{i\omega(\bar\nu + \bar\lambda)}{k^2}\frac{d^2 v_{\perp 1}}{d\xi^2} = \frac{\omega\sin\alpha}{k\rho_0}\, P_1. \tag{46}$$

In a similar way we obtain the compatibility condition for the system of equations (41) and (45),

$$\begin{aligned} \Delta\left(\xi - \frac{V_A^4\cos^2\alpha}{\Delta C_S^2}\right) v_{\|1} \;&+\; \frac{i\omega(\bar\nu + \bar\lambda)}{k^2}\frac{d^2 v_{\|1}}{d\xi^2} \\ &= -\frac{\omega\cos\alpha}{k\rho_0}\, P_1. \end{aligned} \tag{47}$$

Following Ref. [16] we introduce the thickness of the Alfvén dissipative layer (note that, in high-beta plasmas, it coincides with the thickness of the slow dissipative layer)

$$\delta_A = \left(\frac{\omega(\nu + \lambda)}{k^2|\Delta|}\right)^{1/3}, \tag{48}$$

and the dimensionless variable $\tau = (x - x_A)/\delta_A = \epsilon\xi/\delta_A$. Then, taking $v_\| \approx v_{\|1}$, $v_\perp \approx v_{\perp 1}$ and $P \approx \epsilon P_1$, we rewrite Eqs. (46) and (47) as

$$\frac{d^2 v_\perp}{d\tau^2} - i\tau v_\perp \operatorname{sgn}\Delta = -\frac{i\omega\sin\alpha}{k\delta_A|\Delta|\rho_0}\, P, \tag{49}$$

$$\frac{d^2 v_\|}{d\tau^2} - i(\tau - d)v_\| \operatorname{sgn}\Delta = \frac{i\omega\cos\alpha}{k\delta_A|\Delta|\rho_0}\, P, \tag{50}$$

where

$$d = \frac{V_A^4\cos^2\alpha}{\Delta c_S^2\delta_A} = x_c - x_A + \mathcal{O}(\beta^{-2}) \tag{51}$$

is the approximate distance between the slow and Alfvén resonant positions (taken with the minus sign when $x_c < x_A$). Equation (49) coincides with the equation obtained for the normal component of the plasmas displacement, $\xi_\perp$, obtained in Ref. [16] (see Eq. (56) in Ref. [16]), while Eq. (50) reduces to this equation by the substitution $\tau = \tilde\tau + d$ (note that Δ has the different sign in Ref. [16]). Then we can immediately write down the solutions to Eqs. (49) and (50),

$$v_\perp = \frac{i\omega P\sin\alpha}{k\delta_A|\Delta|\rho_0}\, F(\tau), \tag{52}$$

$$v_\| = -\frac{i\omega P\cos\alpha}{k\delta_A|\Delta|\rho_0}\, F(\tau - d), \tag{53}$$

where

$$F(\tau) = \int_0^\infty \exp(-i\eta\tau\operatorname{sgn}\Delta - \eta^3/3)\, d\eta. \tag{54}$$

Taking $u \approx \epsilon u_1$ and substituting Eqs. (52) and (53) in Eq. (29) we obtain

$$\frac{du}{d\tau} = -\frac{\omega P}{\rho_0|\Delta|}\{F(\tau)\sin^2\alpha + F(\tau - d)\cos^2\alpha\}. \tag{55}$$

The simple integration of this equation yields

$$u = -\frac{i\omega P}{\rho_0\Delta}\{G(\tau)\sin^2\alpha + G(\tau - d)\cos^2\alpha\} + C, \tag{56}$$

where C is a constant that has to be determined by matching the solution in the dissipative layer with the external solution, and the function $G(\tau)$ is given by

$$G(\tau) = \int_0^\infty \frac{e^{-\eta^3/3}}{\eta}\{\exp(-i\eta\tau\operatorname{sgn}\Delta) - 1\}\, d\eta. \tag{57}$$

In Ref. [20] the concept of connection formulae was introduced. The connection formulae give the jump of the total pressure and velocity component normal to the dissipative layer across the dissipative layer. The plasma motion in the external region, which is the region outside the dissipative layer, can be described by the system of linearised ideal MHD equations. This system can be reduced to a system of two first-order differential equations for P and u.[15] The connection formulae connect the solutions of ideal MHD equations at the left and the right of the dissipative layer. If we know the jumps of P and u across the dissipative layer, we can consider the dissipative layer as a surface of discontinuity when solving the ideal MHD equations. This approach is similar to one used in gas dynamics when studying gas motions with shocks. The Rankine-Hugoniot relations at the shock are used to connect the solutions at the left and the right of the shock.

The jump of function $f(\tau)$ across the dissipative layer is defined by

$$[f] = \lim_{\tau\to\infty}\{f(\tau) - f(-\tau)\}. \tag{58}$$

Since $P \approx \epsilon P_1$ is independent of τ, it immediately follows that $[P] = 0$. It is straightforward to calculate the jump of $G(\tau)$ across the dissipative layer:

$$\begin{aligned}[G] &= \lim_{\tau\to\infty}\int_0^\infty \frac{e^{-\eta^3/3}}{\eta} \\ &\times \{\exp(-i\eta\tau\,\mathrm{sgn}\,\Delta) - \exp(i\eta\tau\,\mathrm{sgn}\,\Delta)\}\,d\eta \\ &= -2i\lim_{\tau\to\infty}\int_0^\infty \frac{e^{-(\zeta/\tau)^3/3}}{\zeta}\sin(\zeta\,\mathrm{sgn}\,\Delta)\,d\zeta \\ &= -2i\,\mathrm{sgn}\,\Delta\int_0^\infty \frac{\sin\zeta}{\zeta}\,d\zeta = -\pi i\,\mathrm{sgn}\,\Delta. \end{aligned} \tag{59}$$

Using this result we obtain from Eq. (56)

$$[u] = -\frac{\pi\omega P}{\rho_0|\Delta|}. \tag{60}$$

It is straightforward to see that this jump is equal to the sum of two jumps, one across the Alfvén and the other across the slow dissipative layer.

If we denote the dimensionless amplitude of the wave motion far from the dissipative layer as a, then it follows from the matching of the external ideal solution and the solution in the dissipative layer that P and u are of the order of a. Then we conclude that $v_\parallel$, $v_\perp$, $b_\parallel$, and $v_\perp$ are of the order of $\epsilon^{-1}a = R^{1/3}a$. These variables are called the large variables. Recall that, in an Alfvén dissipative layer, the large variables are $v_\perp$ and $b_\perp$,[16,20] while in a slow dissipative layer the large variable are $v_\parallel$ and $b_\parallel$.[17,20] Hence, the fact that $v_\parallel$, $v_\perp$, $b_\parallel$, and $v_\perp$ are large variable is in the complete agreement with the fact that, in high-beta plasmas, dissipative layers are mixed. In general, our analysis shows that the wave motion in a mixed dissipative layer is a linear superposition of two motions, one in the Alfvén dissipative layer embracing the ideal Alfvén resonant position $x = x_A$, and the other in the slow dissipative layer embracing the ideal slow resonant position $x = x_c$, x_c being very close to x_A.

We obtain the extreme case of incompressible plasma if we take $\beta \to \infty$. In this case $x_c = x_A$ and it follows from Eqs. (52) and (53) that $v = v_\perp\cos\alpha + v_\parallel\sin\alpha = 0$ and $b_y = b_\perp\cos\alpha + b_\parallel\sin\alpha = 0$. Hence, in incompressible plasmas the only large variables are $w = v_\parallel\cos\alpha - v_\perp\sin\alpha$ and $b_z = b_\parallel\cos\alpha - b_\perp\sin\alpha$, which are the components of the velocity and magnetic field perturbation in the direction of wave propagation.

IV. SUMMARY AND CONCLUSIONS

In this paper we have studied in the linear approximation the wave motion in a dissipative layer in a weakly dissipative plasma with the large plasma beta. We have considered a one-dimensional planar equilibrium. We have shown that, in this case, the ideal Alfvén and slow resonant positions are so close that the Alfvén and slow dissipative layers overlap. As a result there is one mixed dissipative layer embracing the both ideal resonant positions. The wave motion in this mixed layer is a linear superposition of two wave motions, one in the Alfvén dissipative layer and the other in the slow dissipative layer. The jump of the component of the velocity normal to the dissipative layer across the dissipative layer is equal to the sum of its jumps across the Alfvén and slow dissipative layers. The large variables in a mixed dissipative layer are the two components of the velocity and magnetic field perturbation parallel to the dissipative layer.

In the extreme case of incompressible plasmas the only large variables are the components of the velocity and magnetic field perturbation in the direction of wave propagation.

1. J. Tataronis and W. Grossmann, Z. Phys. **261**, 203 (1973).
2. W. Grossmann and J. Tataronis, Z. Phys. **261**, 217 (1973).
3. L. Chen and A. Hasegawa, Phys. Fluids **17**, 1399 (1974).
4. A. Hasegawa and L. Chen, Phys. Fluids **19**, 1924 (1976).
5. J. A. Ionson, Astrophys. J. **226**, 650 (1978).
6. L. J. Lanzerotti, H. Fukumishi, A. Hasegawa, and L. Chen, Phys. Rev. Lett. **31**, 624 (1973).
7. D. J. Southwood, Planet, Space, Sci. **22**, 483 (1974).
8. L. Chen and A. Hasegawa, J. Geophys. Res. **79**, 1024 (1974).
9. L. Chen and A. Hasegawa, J. Geophys. Res. **79**, 1033 (1974).
10. J. V. Hollweg and G. Yang, J. Geophys. Res. **93**, 5423 (1988).

[11] M. J. Aschwanden, L. Fletcher, C. J. Schrijver, and D. Alexander, Astrophys. J. **520**, 880 (1999).
[12] V. M. Nakariakov, L. Ofman, E. E. DeLuca, B. Roberts, and J. M. Davila, Science **285**, 862 (1999).
[13] M. S. Ruderman and B. Roberts, Astrophys. J. **577**, 475 (2002).
[14] M. Goossens, J. Andries, and M. J. Aschwanden, Astron. Astrophys. **394**, L39 (2002).
[15] K. Appert, R. Gruber, and J. Vaclavik, Phys. Fluids **17**, 1471 (1974).
[16] M. Goossens, M. S. Ruderman, and J. V. Hollweg, Solar Phys. **157**, 75 (1995).
[17] R. Erdélyi, Solar Phys. **171**, 49 (1997).
[18] D. O. Gough and M. E. McIntyre, Nature **394**, 755 (1998).
[19] M. Goossens, J. V. Hollweg, and T. Sakurai, Solar Phys. **138**, 233 (1992).
[20] T. Sakurai, M. Goossens, and J. V. Hollweg, Solar Phys. **133**, 227 (1991).