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Nazarov, Maxim and Sklyanin, Evgeny [orcid.org/0000-0003-2194-0643](https://orcid.org/0000-0003-2194-0643) (2015) Lax operator for Macdonald symmetric functions. *Letters in Mathematical Physics*. pp. 901-916. ISSN 1573-0530

<https://doi.org/10.1007/s11005-015-0770-1>

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# Lax operator for Macdonald symmetric functions

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**Abstract:** Using the Lax operator formalism, we construct a family of pairwise commuting operators such that the Macdonald symmetric functions of infinitely many variables  $x_1, x_2, \dots$  and of two parameters  $q, t$  are their eigenfunctions. We express our operators in terms of the Hall-Littlewood symmetric functions of the variables  $x_1, x_2, \dots$  and of the parameter  $t$  corresponding to the partitions with one part only. Our expression is based on the notion of Baker-Akhiezer function.

**Key words.** Baker-Akhiezer function, Lax operator, Macdonald polynomials

## Introduction

Over the last two decades the Macdonald polynomials [9] have been the subject of much attention in combinatorics and representation theory. These polynomials are symmetric in  $N$  variables  $x_1, \dots, x_N$  and also depend on two parameters denoted by  $q$  and  $t$ . They are labelled by partitions of  $0, 1, 2, \dots$  with no more than  $N$  parts. Up to normalization, they can be defined as eigenvectors of certain family of commuting linear operators acting on the space  $\Lambda_N$  of all symmetric polynomials in the variables  $x_1, \dots, x_N$  with coefficients from the field  $\mathbb{Q}(q, t)$ . These operators were introduced by Macdonald [9] as the coefficients of a certain operator valued polynomial of degree  $N$  in a variable  $u$  with the constant term 1. In particular, Macdonald observed that all eigenvalues of the operator coefficient at  $u$  are already free from multiplicities. Hence this coefficient alone can be used to define the Macdonald polynomials.

It is quite common in combinatorics to extend various symmetric polynomials to an infinite set of variables. These extensions are called *symmetric functions*. The ring  $\Lambda$  of symmetric functions is defined as the inverse limit of the sequence  $\Lambda_1 \leftarrow \Lambda_2 \leftarrow \dots$  in the category of graded algebras. The defining homomorphism  $\Lambda_{N-1} \leftarrow \Lambda_N$  here is just the substitution  $x_N = 0$ . In particular, the Macdonald polynomials are extended to infinitely many variables  $x_1, x_2, \dots$  by using their *stability property* [9] and by passing to their limits as  $N \rightarrow \infty$ . The limits are the Macdonald symmetric functions. They belong to the ring  $\Lambda$  and are labelled by partitions of  $0, 1, 2, \dots$ . They have been also studied very well. In particular, the limit at  $N \rightarrow \infty$  of the renormalized Macdonald operator coefficient at  $u$  was considered in [9]. Other expressions for the same limit were given in [2, 5].

Since the higher operator coefficients are not required to define the Macdonald polynomials, the limits of these coefficients at  $N \rightarrow \infty$  received due attention only later. From the geometric point of view they were studied in [15], see also the works [6, 14, 19]. Explicit expressions for the limits were given in [1, 4, 16] and independently in [11]. All these expressions involved Hall-Littlewood symmetric functions [9] in the variables  $x_1, x_2, \dots$ . These symmetric functions are labelled by partitions of  $0, 1, 2, \dots$  but depend on the parameter  $t$  only. They also emerge in the calculus of *vertex operators* [7] which underlies the results of [1, 2, 5, 6, 16].

In the present article we construct a different family of commuting operators on  $\Lambda$  such that the Macdonald symmetric functions are their eigenvectors. Unlike in [1, 4, 11] our operators are expressed in terms of the Hall-Littlewood symmetric functions corresponding to the partitions with one part only. Our construction uses the Lax operator formalism, see Subsection 2.1 for details. Our Theorem 1 gives a relation between the new family of commuting operators and the one we constructed in [11]. The proof is based on the notion of Baker-Akhiezer function corresponding to the Lax operator, see Subsection 2.2. In our case this function is given by Theorem 2. The proof of the latter theorem is given in Subsection 2.3.

To find the eigenvalues of our new operators we still need the results of [11]. It would be interesting to prove directly that the eigenvectors of these operators are the Macdonald symmetric functions, see for instance [8]. Also notice that by setting  $q = t^\alpha$  and tending  $t \rightarrow 1$  one obtains the *Jack symmetric functions* [9] as limits of Macdonald symmetric functions. Our new Lax operator can be regarded as a discretization of the operator we found in the limiting case [10, 12]. The latter operator has been in turn a quantized version of the Lax operator for the classical Benjamin-Ono equation [3, 13]. Our new Lax operator is a quantized version of the one for the classical Benjamin-Ono equation with discrete Laplacian [17, 18].

In this article we generally keep to the notation of the book [9] for symmetric functions. When using results from [9] we simply indicate their numbers within the book. For example, the statement (2.15) from Chapter III of the book will be referred to as [III.2.15] assuming it is from [9].

## 1. Symmetric functions

*1.1. Power sums.* Fix any field  $\mathbb{F}$ . Denote by  $\Lambda$  the  $\mathbb{F}$ -algebra of symmetric functions in infinitely many variables  $x_1, x_2, \dots$ . Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be any partition of  $0, 1, 2, \dots$ . We will always assume that  $\lambda_1 \geq \lambda_2 \geq \dots$ . The number of non-zero parts is called the *length* of  $\lambda$  and denoted by  $\ell(\lambda)$ . Let  $k_1, k_2, \dots$  be the multiplicities of the parts  $1, 2, \dots$  of  $\lambda$  respectively. Then  $k_1 + k_2 + \dots = \ell(\lambda)$ .

For  $n = 1, 2, \dots$  let  $p_n \in \Lambda$  be the *power sum symmetric function* of degree  $n$ :

$$p_n = x_1^n + x_2^n + \dots$$

More generally, for any partition  $\lambda$  put  $p_\lambda = p_{\lambda_1} \dots p_{\lambda_k}$  where  $k = \ell(\lambda)$ . The elements  $p_\lambda$  form a basis of  $\Lambda$ . In other words, the elements  $p_1, p_2, \dots$  are free generators of the commutative algebra  $\Lambda$  over  $\mathbb{F}$ .

Define a bilinear form  $\langle \cdot, \cdot \rangle$  on  $\Lambda$  by setting for any two partitions  $\lambda$  and  $\mu$

$$\langle p_\lambda, p_\mu \rangle = k_\lambda \delta_{\lambda\mu} \quad \text{where} \quad k_\lambda = 1^{k_1} k_1! 2^{k_2} k_2! \dots \quad (1.1)$$

in the above notation. This form is obviously symmetric and non-degenerate. We will indicate by the superscript  $\perp$  the operator conjugation relative to this form. In particular, by (1.1) for the operator conjugate to the multiplication in  $\Lambda$  by  $p_n$  with  $n \geq 1$

$$p_n^\perp = n \partial / \partial p_n. \quad (1.2)$$

*1.2. Elementary and complete symmetric functions.* For  $n = 1, 2, \dots$  let  $e_n \in \Lambda$  be the *elementary symmetric function* of degree  $n$ . By definition,

$$e_n = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}.$$

We will also use a formal power series in the variable  $z$

$$E(z) = 1 + e_1 z + e_2 z^2 + \dots = \prod_{i \geq 1} (1 + x_i z).$$

By taking logarithms of the left and right hand side of the above display and then exponentiating,

$$E(z) = \exp \left( - \sum_{n \geq 1} \frac{p_n}{n} (-z)^n \right). \quad (1.3)$$

The *complete symmetric functions*  $h_1, h_2, \dots$  can be determined by the relation

$$E(-z)H(z) = 1 \quad (1.4)$$

where

$$H(z) = 1 + h_1 z + h_2 z^2 + \dots$$

The degree of the element  $h_n \in \Lambda$  is  $n$ . Furthermore, by (1.3) we get an equality

$$H(z) = \exp\left(\sum_{n \geq 1} \frac{p_n}{n} z^n\right). \quad (1.5)$$

The elements  $h_1, h_2, \dots$  as well as the elements  $e_1, e_2, \dots$  are free generators of the commutative algebra  $\Lambda$  over the field  $\mathbb{F}$ .

*1.3. Hall-Littlewood functions.* Let  $\mathbb{F}$  be the field  $\mathbb{Q}(t)$  with  $t$  a parameter. Put

$$Q(z) = E(-tz)H(z) = 1 + Q_1 z + Q_2 z^2 + \dots$$

Note that then by (1.3) and (1.4) we have

$$Q(z) = \prod_{i \geq 1} \frac{1 - t x_i z}{1 - x_i z} = \exp\left(\sum_{n \geq 1} \frac{1 - t^n}{n} p_n z^n\right). \quad (1.6)$$

In this article we will employ the *Jing vertex operator*

$$J(z) = Q(z) E^\perp(-z^{-1}). \quad (1.7)$$

This is a formal series in  $z$  with coefficients acting on  $\Lambda$  as linear operators. These operators do not commute, see [7, Proposition 2.12] for commutation relations between them. Using another variable  $w$  instead of  $z$  in the equalities (1.6) we get

$$\begin{aligned} E^\perp(-z^{-1})(Q(w)) &= \exp\left(-\sum_{n \geq 1} z^{-n} \partial / \partial p_n\right)(Q(w)) = \\ &= Q(w) \exp\left(-\sum_{n \geq 1} \frac{1 - t^n}{n} z^{-n} w^n\right) = Q(w) \frac{z - w}{z - t w} \end{aligned} \quad (1.8)$$

due to (1.2) and (1.3). The fraction at the right hand side of the equalities (1.8) should be expanded as a power series in the ratio  $w/z$ . It follows by (1.4) that

$$H^\perp(z^{-1})(Q(w)) = Q(w) \frac{z - t w}{z - w}. \quad (1.9)$$

Following [7, Proposition 3.9] we will use the relation

$$J(z)(Q(z_1) \dots Q(z_k)) = Q(z) Q(z_1) \dots Q(z_k) \prod_{j=1}^k \frac{z - z_j}{z - t z_j}. \quad (1.10)$$

To prove (1.10) note that due to [Ex. I.5.25] the series  $E^\perp(-z^{-1})$  with operator coefficients showing in the definition (1.7) is comultiplicative, so that

$$E^\perp(-z^{-1})(Q(z_1) \dots Q(z_k)) = E^\perp(-z^{-1})(Q(z_1)) \dots E^\perp(-z^{-1})(Q(z_k)).$$

Hence the equality (1.10) is obtained by using (1.8) with  $w = z_1, \dots, z_k$ .

Now let  $\lambda$  be any partition with  $\ell(\lambda) = k$ . Recall that  $\lambda_1 \geq \dots \geq \lambda_k$  by our assumption. Introduce a rational function of the variables  $z_1, \dots, z_k$

$$F(z_1, \dots, z_k) = \prod_{1 \leq i < j \leq k} \frac{z_i - z_j}{z_i - t z_j}. \quad (1.11)$$

Let us expand every factor with  $i < j$  in the product (1.11) as a power series in  $z_j/z_i$  respectively. By [III.2.15] the *Hall-Littlewood symmetric function*  $Q_\lambda \in \Lambda$  is the coefficient at

$$z_1^{\lambda_1} \dots z_k^{\lambda_k}$$

in the formal series

$$Q(z_1) \dots Q(z_k) F(z_1, \dots, z_k).$$

If the partition  $\lambda$  consists of only one part  $n$  then  $Q_\lambda$  is  $Q_n$  by above definition.

The elements  $Q_\lambda$  constitute a basis of the vector space  $\Lambda$ . Furthermore, define a bilinear form  $\langle \cdot, \cdot \rangle_t$  on the vector space  $\Lambda$  by setting for any partitions  $\lambda$  and  $\mu$

$$\langle p_\lambda, p_\mu \rangle_t = k_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1 - t^{\lambda_i}} \quad (1.12)$$

in the notation (1.1). It is obviously symmetric and non-degenerate. By [III.4.9]

$$\langle Q_\lambda, Q_\mu \rangle_t = b_\lambda(t) \delta_{\lambda\mu} \quad (1.13)$$

where

$$b_\lambda(t) = \prod_{i \geq 1} \prod_{j=1}^{k_i} (1 - t^j).$$

Along with the symmetric function  $Q_\lambda$  it is convenient to use the symmetric function  $P_\lambda$  which is a scalar multiple of  $Q_\lambda$ . By definition,

$$Q_\lambda = b_\lambda(t) P_\lambda \quad (1.14)$$

so that due to (1.13)

$$\langle P_\lambda, Q_\mu \rangle_t = \delta_{\lambda\mu}.$$

*1.4. Macdonald functions.* Now let  $\mathbb{F}$  be the field  $\mathbb{Q}(q, t)$  with  $q$  and  $t$  parameters independent of each other. Generalizing (1.12) define a bilinear form  $\langle \cdot, \cdot \rangle_{q,t}$  on  $\Lambda$  by setting for any partitions  $\lambda$  and  $\mu$

$$\langle p_\lambda, p_\mu \rangle_{q,t} = k_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \quad (1.15)$$

in the notation of (1.1). This form is again symmetric and non-degenerate. We will indicate by the superscript  $*$  the operator conjugation relative to the latter form. In particular, by (1.2) and (1.15) for any  $n \geq 1$  we have

$$p_n^* = \frac{1 - q^n}{1 - t^n} p_n^\perp.$$

Hence by (1.6) we get

$$Q^*(z) = 1 + Q_1^* z + Q_2^* z^2 + \dots = \exp \left( \sum_{n \geq 1} \frac{1 - q^n}{n} p_n^\perp z^n \right). \quad (1.16)$$

Using (1.14) when  $\lambda$  consists of only one part  $n$  we get  $P_n = Q_n/(1 - t)$ . Now consider the linear operator acting on the vector space  $\Lambda$  as the sum

$$\sum_{n \geq 1} q^{-n} Q_n P_n^* = \sum_{n \geq 1} q^{-n} Q_n Q_n^*/(1 - t). \quad (1.17)$$

For future discussion note that (1.17) equals the coefficient at 1 of the series in  $z$

$$(Q(z) Q^*(q^{-1} z^{-1}) - 1)/(1 - t).$$

The operator (1.17) is clearly self-conjugate relative to the bilinear form (1.15). By [2, Eq. 32] for any partition  $\lambda$  the *Macdonald symmetric function*  $M_\lambda \in \Lambda$  can be defined up to normalization as an eigenvector of (1.17) with the eigenvalue

$$\sum_{i \geq 1} (q^{-\lambda_i} - 1) t^{i-1}. \quad (1.18)$$

For different partitions  $\lambda$  the eigenvalues (1.18) are pairwise distinct in  $\mathbb{Q}(q, t)$ . It follows that the eigenvectors  $M_\lambda$  with different  $\lambda$  are pairwise orthogonal relative to (1.15). In the present article we will not be choosing any normalization of  $M_\lambda$ . We will use only the fact [VI.4.7] that the  $M_\lambda$  form a basis of the vector space  $\Lambda$ .

*1.5. Higher Hamiltonians.* In [11] we introduced the following generalization of the operator (1.17). For any  $k = 0, 1, 2, \dots$  consider the linear operator on  $\Lambda$

$$A^{(k)} = \sum_{\ell(\lambda)=k} q^{-\lambda_1 - \lambda_2 - \dots} Q_\lambda P_\lambda^*.$$

Then  $A^{(0)} = 1$  while  $A^{(1)}$  is the operator (1.17). For any  $k$  the operator  $A^{(k)}$  is obviously self-conjugate relative to (1.15). Consider a series in a variable  $u$

$$A(u) = \sum_{k \geq 0} A^{(k)}/(u; t^{-1})_k \quad (1.19)$$

where as usual

$$(u; t^{-1})_k = \prod_{j=0}^{k-1} (1 - u t^{-j}).$$

In [11] we proved

$$A(u) M_\lambda = M_\lambda \prod_{i \geq 1} \frac{q^{-\lambda_i} - u t^{1-i}}{1 - u t^{1-i}}. \quad (1.20)$$

It follows from (1.20) that the operators  $A^{(1)}, A^{(2)}, \dots$  on  $\Lambda$  pairwise commute. The eigenvalue (1.18) of the operator  $A^{(1)}$  can also be obtained from this equality.

The equality (1.20) can be derived from the results of [1, Sec. 3] which in turn are modifications of those of [16, Sec. 9]. Our proof [11, Sec. 3] was independent of all those results. Let us now establish a relation between the works [1] and [11].

For  $k \geq 1$  denote by  $S^{(k)}$  the constant term of the formal series in  $z_1, \dots, z_k$

$$Q(z_1) \dots Q(z_k) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_k^{-1}) F(z_1, \dots, z_k). \quad (1.21)$$

Here the product (1.11) is regarded as a series in  $z_1, \dots, z_k$  using the expansion rule explained just after displaying it. This  $S^{(k)}$  is a certain linear operator on the vector space  $\Lambda$ . It is convenient to set  $S^{(0)} = 1$ . It turns out that  $M_\lambda$  for each  $\lambda$  is an eigenvector of the operators  $S^{(1)}, S^{(2)}, \dots$  like that of  $A^{(1)}, A^{(2)}, \dots$ . This fact goes back to [16]. It also follows from (1.20) by the next proposition.

**Proposition.** *We have the relation*

$$(u^{-1}; t)_\infty A(u) = \sum_{k \geq 0} (ut)^{-k} S^{(k)} / (t^{-1}; t^{-1})_k$$

where as usual

$$(u^{-1}; t)_\infty = \prod_{j=0}^{\infty} (1 - u^{-1}t^j).$$

*Proof.* For every partition  $\lambda$  let us denote by  $P_\lambda(z_1, \dots, z_k)$  the specialization of the symmetric function  $P_\lambda$  to  $x_1 = z_1, \dots, x_k = z_k$  and  $x_{k+1} = x_{k+2} = \dots = 0$ . This is a homogeneous symmetric polynomial in the variables  $z_1, \dots, z_k$  of degree  $\lambda_1 + \lambda_2 + \dots$ . By using the first equality in (1.6) and then the expansion [III.4.4]

$$Q(z_1) \dots Q(z_k) = \prod_{i \geq 1} \prod_{j=1}^k \frac{1 - tx_i z_j}{1 - x_i z_j} = \sum_{\ell(\lambda) \leq k} Q_\lambda P_\lambda(z_1, \dots, z_k).$$

It follows from the latter equality that

$$Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_k^{-1}) = \sum_{\ell(\mu) \leq k} q^{-\mu_1 - \mu_2 - \dots} P_\mu^* Q_\mu(z_1^{-1}, \dots, z_k^{-1}).$$

Hence

$$S^{(k)} = \sum_{\ell(\lambda), \ell(\mu) \leq k} q^{-\mu_1 - \mu_2 - \dots} Q_\lambda P_\mu^* a_{\lambda\mu}(t)$$

where  $a_{\lambda\mu}(t)$  denotes the constant term of the formal series in  $z_1, \dots, z_k$

$$P_\lambda(z_1, \dots, z_k) Q_\mu(z_1^{-1}, \dots, z_k^{-1}) F(z_1, \dots, z_k).$$

It is known that

$$a_{\lambda\mu}(t) = \delta_{\lambda\mu} \frac{(t; t)_k}{(t; t)_{k-\ell(\lambda)}}$$

see [1, App. B] for an elementary proof of this fact. Thus we obtain the relation

$$S^{(k)} = \sum_{i=0}^k A^{(i)} \frac{(t; t)_k}{(t; t)_{k-i}}. \quad (1.22)$$



By substituting the latter expression for  $S^{(k)}$  in our Proposition and by using the definition (1.19) with the running index  $k$  replaced by  $i$  it remains to prove

$$\sum_{i \geq 0} A^{(i)} \frac{(u^{-1}; t)_{\infty}}{(u; t^{-1})_i} = \sum_{k \geq 0} \sum_{i=0}^k A^{(i)} \frac{(t; t)_k}{(ut)^k (t^{-1}; t^{-1})_k (t; t)_{k-i}}.$$

By equating here the coefficients at  $A^{(i)}$  we have to prove that for every  $i \geq 0$

$$\frac{(u^{-1}; t)_{\infty}}{(u; t^{-1})_i} = \sum_{k \geq i} \frac{(t; t)_k}{(ut)^k (t^{-1}; t^{-1})_k (t; t)_{k-i}}.$$

But the last relation follows by setting  $v = u^{-1}t^i$  and  $j = k - i$  in the equality

$$(v; t)_{\infty} = \sum_{j \geq 0} \frac{(-v)^j t^{j(j-1)/2}}{(t; t)_j}. \quad \square$$

Note that the relation (1.22) established above is equivalent to [1, Eq. 3.3]. By using a variation of the Möbius inversion [4, Lemma 5.1] we get from (1.22)

$$A^{(k)} = \sum_{i=0}^k S^{(i)} \frac{(-1)^{k-i} t^{(k-i)(k-i-1)/2}}{(t; t)_i (t; t)_{k-i}}.$$

## 2. Lax operator and Baker-Akhiezer function

*2.1. Lax operator.* In this section we will construct yet another family of pairwise commuting operators on  $\Lambda$  with the Macdonald symmetric functions  $M_{\lambda}$  being their eigenvectors. Let

$$\Lambda^{\infty} = z^{-1}\Lambda[z^{-1}]$$

be the ring of polynomials in  $z^{-1}$  with the coefficients from  $\Lambda$  but without the constant term. Introduce the linear operator  $\mathcal{U}$  on the vector space  $\Lambda^{\infty}$  by setting

$$\mathcal{U} : f(z) \mapsto [Q(z)f(z)]_{-}$$

where the symbol  $[ ]_{-}$  means taking the only negative degree terms of the series.

Let us extend the bilinear form (1.15) from  $\Lambda$  to  $\Lambda^{\infty}$  so that the subspaces

$$z^{-1}\Lambda, z^{-2}\Lambda, \dots \subset \Lambda^{\infty} \quad (2.1)$$

are orthogonal to each other, while each one carries the bilinear form determined by identifying that subspace with  $\Lambda$ . For the operator  $\mathcal{U}^*$  on  $\Lambda^{\infty}$  conjugate to  $\mathcal{U}$  we then have

$$\mathcal{U}^* : f(z) \mapsto Q^*(z^{-1})f(z).$$

Moreover, using the decomposition of  $\Lambda^{\infty}$  into the direct sum of subspaces (2.1) the operators  $\mathcal{U}$  and  $\mathcal{U}^*$  are represented by infinite matrices with operator entries

$$\begin{pmatrix} 1 & Q_1 & Q_2 & \cdots \\ 0 & 1 & Q_1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & \cdots \\ Q_1^* & 1 & 0 & \cdots \\ Q_2^* & Q_1^* & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Further, let  $\mathcal{D}_t$  be the linear operator on  $\Lambda^\infty$  defined by setting

$$\mathcal{D}_t : f(z) \mapsto f(z t^{-1}).$$

Accordingly, let

$$\mathcal{D}_q : f(z) \mapsto f(z q^{-1}).$$

The operators  $\mathcal{D}_t$  and  $\mathcal{D}_q$  are clearly self-conjugate relative to the bilinear form on  $\Lambda^\infty$  defined above. They are represented by diagonal matrices with the entries  $t, t^2, \dots$  and  $q, q^2, \dots$  respectively. Our *Lax operator* on  $\Lambda^\infty$  is the composition

$$\mathcal{L} = \mathcal{U}^* \mathcal{D}_t \mathcal{U}.$$

Furthermore introduce the operator  $\mathcal{Q} : \Lambda^\infty \rightarrow \Lambda$  by setting

$$\mathcal{Q} : f(z) \mapsto [Q(z)f(z)]_0$$

where  $[ ]_0$  means taking the constant term of a series in  $z$ . The conjugate operator  $\mathcal{Q}^* : \Lambda \rightarrow \Lambda^\infty$  is the application of  $Q^*(z^{-1}) - 1$  to elements of  $\Lambda$ . The operators  $\mathcal{Q}$  and  $\mathcal{Q}^*$  are represented by an infinite row and a column with operator entries

$$(Q_1 \ Q_2 \ \dots) \quad \text{and} \quad \begin{pmatrix} Q_1^* \\ Q_2^* \\ \vdots \end{pmatrix}.$$

Now put

$$I(u) = \mathcal{Q} (u \mathcal{D}_q - \mathcal{L})^{-1} \mathcal{Q}^*. \quad (2.2)$$

It expands as a formal power series in  $u^{-1}$  without constant term. The coefficients of that series are self-conjugate operators on  $\Lambda$  by definition. These operators will form our new commuting family. The commutativity follows from the theorem below which relates  $I(u)$  to  $A(u)$ . The proof of the theorem shall be given later.

**Theorem 1.** *We have an equality of formal power series in  $u^{-1}$  with operator coefficients acting on  $\Lambda$*

$$\frac{u I(u)}{u - 1} = 1 - \frac{A(u)}{A(u t^{-1})}.$$

By using this theorem, the eigenvalues of the operator coefficients of the series  $I(u)$  on the Macdonald symmetric functions  $M_\lambda \in \Lambda$  can be obtained from (1.20).

*2.2. Baker-Akhiezer function.* Now introduce the formal power series in  $u^{-1}$

$$\Psi(u) = u (u \mathcal{D}_q - \mathcal{L})^{-1} \mathcal{Q}^* A(u t^{-1}). \quad (2.3)$$

The coefficients of this series are operators  $\Lambda \rightarrow \Lambda^\infty$ . By the definitions (2.2), (2.3)

$$u I(u) A(u t^{-1}) = \mathcal{Q} \Psi(u). \quad (2.4)$$

We will call the series  $\Psi(u)$  the *Baker-Akhiezer function* for the Lax operator (2.1). Our proof of Theorem 1 will be based on an expression for  $\Psi(u)$  given next.

**Theorem 2.** *We have an equality of formal power series in  $u^{-1}$  with operator coefficients mapping  $\Lambda$  to  $\Lambda^\infty$*

$$\Psi(u) = E^\perp(-z^{-1}) A(ut^{-1}) H^\perp(z^{-1}q^{-1}) - A(ut^{-1}).$$

We will prove Theorem 2 in the next subsection. Let us now derive Theorem 1 from it. Multiplying both sides of the relation in Theorem 1 by  $(u-1)A(ut^{-1})$  on the right and then using (2.4) we get an equivalent relation to prove:

$$A(ut^{-1}) + \mathcal{Q}\Psi(u) = uA(ut^{-1}) - (u-1)A(u). \quad (2.5)$$

But by using Theorem 2 along with definitions (1.7),(2.1) we get the equalities

$$\begin{aligned} A(ut^{-1}) + \mathcal{Q}\Psi(u) &= [Q(z)E^\perp(-z^{-1})A(ut^{-1})H^\perp(z^{-1}q^{-1})]_0 \\ &= [J(z)A(ut^{-1})H^\perp(z^{-1}q^{-1})]_0. \end{aligned}$$

By our Proposition, the right hand of these equalities can be rewritten as the sum

$$\sum_{k \geq 0} u^{-k} [J(z)S^{(k)}H^\perp(z^{-1}q^{-1})]_0 / (t^{-1}; t^{-1})_k \quad (2.6)$$

divided by  $(u^{-1}t; t)_\infty$ . Note that by using (1.3),(1.5) and then (1.16) we have

$$E^\perp(-z^{-1})H^\perp(z^{-1}q^{-1}) = \exp\left(\sum_{n \geq 1} \frac{1-q^n}{n} p_n^\perp z^{-n} q^{-n}\right) = Q^*(z^{-1}q^{-1}).$$

Therefore by recalling the definition of the operator  $S^{(k)}$  on  $\Lambda$  and then by using the comultiplicativity [Ex. I.5.25] of  $E^\perp(-z^{-1})$ , the factor

$$[J(z)S^{(k)}H^\perp(z^{-1}q^{-1})]_0$$

in the summand of (2.6) is equal to the constant term of the series in  $z_1, \dots, z_k$

$$[J(z)(Q(z_1) \dots Q(z_k))Q^*(z^{-1}q^{-1})Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_k^{-1})]_0 F(z_1, \dots, z_k).$$

By (1.10) and (1.11) this constant term is exactly  $S^{(k+1)}$ . So the sum (2.6) equals

$$\sum_{k \geq 0} u^{-k} S^{(k+1)} / (t^{-1}; t^{-1})_k = \sum_{k \geq 0} u^{1-k} (1-t^{-k}) S^{(k)} / (t^{-1}; t^{-1})_k$$

where we first replaced  $k+1$  by  $k$  and then formally included the zero summand corresponding to  $k=0$ . Using our Proposition, the last displayed sum equals

$$(u^{-1}t; t)_\infty uA(ut^{-1}) - (u^{-1}; t)_\infty uA(u).$$

Dividing this difference by  $(u^{-1}t; t)_\infty$  we get the right hand side of the relation (2.5). This we have derived Theorem 1 from Theorem 2. Let us prove the latter.

2.3. *Proof of Theorem 2.* Due to the definition (2.3) we have to prove the relation

$$u(u\mathcal{D}_q - \mathcal{L})^{-1}\mathcal{Q}^*A(ut^{-1}) = E^\perp(-z^{-1})A(ut^{-1})H^\perp(z^{-1}q^{-1}) - A(ut^{-1}).$$

Let us multiply both sides of this relation by  $u\mathcal{D}_q - \mathcal{L}$  on the left. In this way we get an equivalent relation to prove:

$$u\mathcal{Q}^*A(ut^{-1}) = (u\mathcal{D}_q - \mathcal{L})(E^\perp(-z^{-1})A(ut^{-1})H^\perp(z^{-1}q^{-1}) - A(ut^{-1})).$$

By the definitions of the operators  $\mathcal{Q}^*$  and  $\mathcal{L}$  the latter relation can be written as

$$\begin{aligned} u(Q^*(z^{-1}) - 1)A(ut^{-1}) = \\ (u\mathcal{D}_q - \mathcal{U}^*\mathcal{D}_t\mathcal{U})(E^\perp(-z^{-1})A(ut^{-1})H^\perp(z^{-1}q^{-1}) - A(ut^{-1})). \end{aligned}$$

Using the definitions of the operators  $\mathcal{D}_q, \mathcal{D}_t$  and  $\mathcal{U}, \mathcal{U}^*$  it can be rewritten as

$$\begin{aligned} u(Q^*(z^{-1}) - 1)A(ut^{-1}) = u(E^\perp(-z^{-1}q)A(ut^{-1})H^\perp(z^{-1}) - A(ut^{-1})) \\ - Q^*(z^{-1})[Q(zt^{-1})E^\perp(-z^{-1}t)A(ut^{-1})H^\perp(z^{-1}tq^{-1}) - A(ut^{-1})]_-. \end{aligned}$$

Here both sides are series in  $u$  with operator coefficients that map  $\Lambda$  to  $\Lambda^\infty$ . We can also regard both sides as series in  $u$  and  $z$  with operator coefficients mapping  $\Lambda$  to  $\Lambda$ . This allows us to perform obvious cancellations hence getting to prove

$$\begin{aligned} uQ^*(z^{-1})A(ut^{-1}) = uE^\perp(-z^{-1}q)A(ut^{-1})H^\perp(z^{-1}) \\ - Q^*(z^{-1})[Q(zt^{-1})E^\perp(-z^{-1}t)A(ut^{-1})H^\perp(z^{-1}tq^{-1})]_-. \end{aligned}$$

Using the definition (1.7) the last displayed relation can be also written as

$$\begin{aligned} uQ^*(z^{-1})A(ut^{-1}) = uE^\perp(-z^{-1}q)A(ut^{-1})H^\perp(z^{-1}) \\ - Q^*(z^{-1})[J(zt^{-1})A(ut^{-1})H^\perp(z^{-1}tq^{-1})]_-. \end{aligned} \quad (2.7)$$

Here  $A(ut^{-1})$  is a formal power series in  $u^{-1}$  with leading term 1 by (1.19). We also know that

$$E^\perp(-z^{-1}q)H^\perp(z^{-1}) = Q^*(z^{-1}), \quad (2.8)$$

see the previous subsection. Hence the coefficients at  $u$  of both sides of the relation (2.7) coincide. Let us now multiply both sides of the relation (2.7) by  $(u^{-1}; t)_\infty$  and take the coefficients at  $u^{k-1}$  for any  $k \geq 1$ . Due to our Proposition we obtain

$$\begin{aligned} Q^*(z^{-1})S^{(k)}/(t^{-1}; t^{-1})_k = E^\perp(-z^{-1}q)S^{(k)}H^\perp(z^{-1})/(t^{-1}; t^{-1})_k \\ - Q^*(z^{-1})[J(zt^{-1})S^{(k-1)}H^\perp(z^{-1}tq^{-1})]_-/ (t^{-1}; t^{-1})_{k-1}. \end{aligned}$$

Next multiply both sides by  $(t^{-1}; t^{-1})_k$  and divide by  $Q^*(z^{-1})$  on the left. We get

$$\begin{aligned} S^{(k)} = Q^*(z^{-1})^{-1}E^\perp(-z^{-1}q)S^{(k)}H^\perp(z^{-1}) \\ - (1 - t^{-k})[J(zt^{-1})S^{(k-1)}H^\perp(z^{-1}tq^{-1})]_-. \end{aligned} \quad (2.9)$$

The operator  $S^{(k)}$  at the left hand side of the relation (2.9) is the constant term of the series (1.21) in  $z_1, \dots, z_k$ . Using this definition along with (1.9), (2.8)

and the comultiplicativity of  $H^\perp(z^{-1})$ , the first summand of the right hand side of (2.9) is equal to the sum of those terms of the formal series in  $z, z_1, \dots, z_k$

$$Q(z_1) \dots Q(z_k) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_k^{-1}) F(z_1, \dots, z_k) \prod_{j=1}^k \frac{z - z_j}{z - tz_j}.$$

that are free of  $z_1, \dots, z_k$ . To present in a similar way the expression displayed in the second line of (2.9) we will use the following simple lemma. Let  $G(z)$  be a formal series in  $z$  with coefficients in any algebra over  $\mathbb{Q}(t)$ . Let  $[G(z)]_0$  be the constant term of this series. Let  $[G(z)]_-$  be the sum of the negative degree terms.

**Lemma.** *The sum*

$$[G(z)]_0 + (1 - t^{-1}) [G(z t^{-1})]_-$$

*is equal to the sum of those terms of the series in  $z, w$*

$$G(w) \frac{z - w}{z - tw}$$

*that are free of  $w$ . Here the fraction should be expanded as a power series in  $w/z$ .*

Verifying this lemma is straightforward. Let us now set

$$G(z) = J(z) S^{(k-1)} H^\perp(z^{-1}q^{-1}).$$

By the definition of  $S^{(k-1)}$  this  $G(z)$  is the sum of those terms of the series

$J(z) Q(z_1) \dots Q(z_{k-1}) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_{k-1}^{-1}) H^\perp(z^{-1}q^{-1}) F(z_1, \dots, z_{k-1})$   
in  $z, z_1, \dots, z_{k-1}$  that are free of  $z_1, \dots, z_{k-1}$ . The last displayed series equals

$$Q(z) Q(z_1) \dots Q(z_{k-1}) Q^*(q^{-1}z^{-1}) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_{k-1}^{-1}) \times \\ F(z, z_1, \dots, z_{k-1})$$

by (1.10) and (1.11). Note that we have used a similar argument in the previous subsection. Denote

$$c_k(t) = (1 - t^k)/(1 - t).$$

Applying the lemma, the expression displayed in the second line of (2.9) is equal to the constant term of the series in  $z, z_1, \dots, z_{k-1}$

$$Q(z) Q(z_1) \dots Q(z_{k-1}) Q^*(q^{-1}z^{-1}) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_{k-1}^{-1}) \times \\ c_k(t^{-1}) F(z, z_1, \dots, z_{k-1}) \quad (2.10)$$

minus those terms of the series in  $z, w, z_1, \dots, z_{k-1}$

$$Q(w) Q(z_1) \dots Q(z_{k-1}) Q^*(q^{-1}w^{-1}) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_{k-1}^{-1}) \times \\ c_k(t^{-1}) F(w, z_1, \dots, z_{k-1}) \frac{z - w}{z - tw} \quad (2.11)$$

that are free of  $w, z_1, \dots, z_{k-1}$ . As we are taking the constant term, the variables  $z, z_1, \dots, z_{k-1}$  in (2.10) can be replaced by  $z_1, \dots, z_k$  respectively. The variables  $w, z_1, \dots, z_{k-1}$  in (2.11) can be replaced by  $z_1, \dots, z_k$  as well.

Recall that the coefficients  $Q_1, Q_2, \dots$  of the series  $Q(z)$  are free generators of the algebra  $\Lambda$ , while the operator product

$$Q(z_1) \dots Q(z_k) Q^*(q^{-1}z_1^{-1}) \dots Q^*(q^{-1}z_k^{-1})$$

is symmetric in  $z_1, \dots, z_k$ . By the above presentation of the terms of (2.9), that relation is equivalent to the equality between the symmetrization of  $F(z_1, \dots, z_k)$  and that of

$$F(z_1, \dots, z_k) \prod_{j=1}^k \frac{z - z_j}{z - t z_j} + c_k(t^{-1}) F(z_1, \dots, z_k) - c_k(t^{-1}) F(z_1, \dots, z_k) \frac{z - z_1}{z - t z_1}. \quad (2.12)$$

Here symmetrizing means taking the sum over all  $k!$  permutations of  $z_1, \dots, z_k$ .

Let us prove the latter equality. At  $z = \infty$  the sum (2.12) equals  $F(z_1, \dots, z_k)$  even before symmetrization. We may assume that  $z_1, \dots, z_k$  are pairwise distinct. Then it suffices to check that that the symmetrization of (2.12) has no poles at  $z = t z_1, \dots, t z_k$ . By the symmetry in  $z_1, \dots, z_k$  taking only  $z = t z_1$  will suffice.

The product over  $j = 1, \dots, k$  showing in the first line of (2.12) is symmetric in  $z_1, \dots, z_k$ . At  $z = t z_1$  it has a simple pole. By [III.1.4] the symmetrization of  $F(z_1, \dots, z_k)$  is

$$c_1(t) \dots c_k(t) \prod_{\substack{1 \leq i, j \leq k \\ i \neq j}} \frac{z_i - z_j}{z_i - t z_j}.$$

Hence the residue at  $z = t z_1$  of the symmetrization of the whole expression in the first line of (2.12) is

$$c_1(t) \dots c_k(t) (t - 1) z_1 \prod_{\substack{1 \leq i, j \leq k \\ i \neq j}} \frac{z_i - z_j}{z_i - t z_j} \cdot \prod_{j=2}^k \frac{t z_1 - z_j}{t(z_1 - z_j)}. \quad (2.13)$$

When symmetrizing the negative term of the difference displayed in the second line of (2.12), we get a pole at  $z = t z_1$  only from the permutations of  $z_1, \dots, z_k$  preserving  $z_1$ . Applying [III.1.4] once again but to the  $k - 1$  variables  $z_2, \dots, z_k$  instead of  $z_1, \dots, z_k$  the residue at  $z = t z_1$  of the symmetrization of the difference displayed in the second line of (2.12) is

$$-c_k(t^{-1}) c_1(t) \dots c_{k-1}(t) (t - 1) z_1 \prod_{\substack{2 \leq i, j \leq k \\ i \neq j}} \frac{z_i - z_j}{z_i - t z_j} \cdot \prod_{j=2}^k \frac{z_1 - z_j}{z_1 - t z_j}. \quad (2.14)$$

To verify that the sum of two products (2.13) and (2.14) is zero, we can cancel in both of them the product of the common factors

$$c_1(t) \dots c_{k-1}(t) (t - 1) z_1 \prod_{\substack{2 \leq i, j \leq k \\ i \neq j}} \frac{z_i - z_j}{z_i - t z_j}$$

and then use the relation  $c_k(t) t^{1-k} = c_k(t^{-1})$ . This verification completes our proof of Theorem 2.

## Acknowledgements

The first named author has been supported by EPSRC grant EP/I014071, and also acknowledges support of IHÉS where he stayed in October–November 2014.

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