Monograph:
Owens, D.H. (1977) Solution of Differential-Algebraic Systems with Control and State Constraints. Research Report. ACSE Research Report 54 . Department of Automatic Control and Systems Engineering

Solution of Differential-algebraic Systems with Control and State Constraints

by

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January 1977

Research Report No. 54

SRC Vacation School on Systems Modelling and Optimization - 1977

Optimization of Industrial Processes II

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1. Introduction

In many applications of mathematics in science and engineering, the concept of optimization is used as a tool rather than a direct objective. Typical examples of this observation can be found in classical mechanics and, for example, the solution of the algebraic problem $f(\underline{x}) = 0$, $\underline{x} \in \Omega \subset R^n$ as a solution of the optimization problem

$$\min \quad \|F(\underline{x})\|^2 \quad , \qquad x \in \Omega \qquad \ldots(1)$$

This class of problem must be distinguished from the well-known optimal control problem where the objective of the design is to optimize a known performance criterion.

In the analysis of many control systems[1], it is necessary to investigate the existence of and calculate a solution of linear equations of the form,

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \quad , \\ R^m \ni \underline{0} &= E\underline{u}(t) + F\underline{x}(t) \quad , \\ \underline{x}(o) &= \underline{x}_o \quad , \quad \underline{x}(T) \in S \subset R^n \\ \underline{u}(t) &\in \Omega_u(t) \subset R^\ell \quad , \quad \underline{x}(t) \in \Omega_x(t) \subset R^n \\ & 0 \leqslant t \leqslant T \quad \text{(fixed)} \end{aligned} \qquad \ldots(2)$$

ie the problem is the choice of an input trajectory $\underline{u}(t) \in \Omega_u(t)$, $0 \leqslant t \leqslant T$, generating a solution $\underline{x}(t)$ of the algebraic equations and differential equations satisfying the terminal state constraint $\underline{x}(T) \in S$ and

$\underline{x}(t) \in \Omega_x(t)$, $0 \leq t \leq T$. Note that a solution may fail to exist, may be unique or there may be an infinite number of controllers $\underline{u}(\cdot)$ generating states $\underline{x}(\cdot)$ satisfying (2). Note also that the problem is not an optimal control problem, and the engineering problem is not that of optimization in any sense.

The discrete form of (2) is the set of equations,

$$\begin{aligned} \underline{x}(k+1) &= A\,\underline{x}(k) + B\,u(k+1) \\ \underline{0} &= E\,\underline{u}(k+1) + F\,\underline{x}(k) \quad , \qquad 0 \leq k \leq N-1 \\ \underline{x}(o) &= \underline{x}_o \quad , \qquad \underline{x}(N) \in S \qquad \ldots(3) \\ \underline{u}(k) &\in \Omega_u(k) \quad , \qquad \underline{x}(k) \in \Omega_x(k) \\ & \qquad 1 \leq k \leq N \qquad \ldots(4) \end{aligned}$$

In both cases the existence of algebraic equalities usually arises from regulating fast stable time constants in the system and/or from algebraic conservation relations.

The problem is best set in the geometric framework of the mathematical method called functional analysis[2,3]. Our attention will be restricted to the discrete problem but the approach provides significant geometric insight into the solution of both problems and enables the solution of a much large class of problems by application of the same mathematical theorems.

2. Mathematical Background[2,3]

A real vector space X is a set of elements called vectors together with two operations called vector addition and scalar multiplication which associates with each pair $\underline{x}, \underline{y} \in X$ and scalar $\lambda \in R$ the sum $\underline{x}+\underline{y} \in X$ and $\lambda\underline{x} \in X$, both operations satisfying the normal commutative, associative and distributive laws. The zero vector in X is denoted $\underline{0}$ and $(-1)\underline{x}$ $(\underline{x} \in X)$ is denoted $-\underline{x}$.

A real inner product space X is a real vector space with an inner product $<\cdot,\cdot>$ which associates with each pair $\underline{x},\underline{y} \in X$ a real number $<\underline{x},\underline{y}>$ such that

$$<\underline{x},\underline{y}> \;=\; <\underline{y},\underline{x}>$$

$$<\underline{x},\underline{x}> \geqslant 0 \quad \text{and} \quad \text{equality holds iff} \quad \underline{x} = \underline{0}$$

$$<\underline{x}_1+\underline{x}_2,\underline{y}> \;=\; <\underline{x}_1,\underline{y}> + <\underline{x}_2,\underline{y}>$$

$$\lambda <\underline{x},\underline{y}> \;=\; <\lambda\underline{x},\underline{y}> \qquad \forall \quad \lambda \in R \qquad \ldots(5)$$

The metric or distance function on X is termed the norm and, $\underline{x} \in X$,

$$||\underline{x}|| \triangleq \{ <\underline{x},\underline{x}> \}^{\frac{1}{2}} \qquad \ldots(6)$$

A Cauchy sequence in a real inner product space is a sequence of vectors $\{\underline{x}_1,\underline{x}_2,\ldots\}$ such that for every real number $\varepsilon>0$, there exists a positive integer $N\geqslant 1$ such that $||\underline{x}_n-\underline{x}_m|| < \varepsilon$ for all $n,m \geqslant N$.

A real Hilbert space H is a real inner product space such that, for every Cauchy sequence $\{\underline{x}_1,\underline{x}_2,\ldots\}$ in H, there exists $\underline{x} \in H$ such that $\lim_{n\to\infty} ||\underline{x}_n-\underline{x}|| = 0$. The sequence $\{\underline{x}_n\}_{n\geqslant 1}$ is said to converge to $\underline{x}$ or $\lim_{n\to\infty} \underline{x} = \underline{x}$.

The Cartesian Product of two Hilbert spaces H_1,H_2 is denoted $H = H_1 x H_2$ and is a Hilbert space of pairs $(\underline{x}_1,\underline{x}_2)$, $\underline{x}_i \in H_i$, $i = 1,2$, with

$$(\underline{x}_1,\underline{x}_2) + (\underline{y}_1,\underline{y}_2) \triangleq (\underline{x}_1 + \underline{y}_1, \underline{x}_2 + \underline{y}_2)$$

$$\lambda(\underline{x}_1,\underline{x}_2) \triangleq (\lambda\underline{x}_1, \lambda\underline{x}_2) \qquad \ldots(7)$$

and inner product and norm,

$$<(\underline{x}_1,\underline{x}_2), (\underline{y}_1,\underline{y}_2)> \triangleq <\underline{x}_1,\underline{y}_1> + <\underline{x}_2,\underline{y}_2>$$

$$||(\underline{x}_1,\underline{x}_2)|| = \{ ||\underline{x}_1||^2 + ||\underline{x}_2||^2 \}^{\frac{1}{2}} \qquad \ldots(8)$$

A closed set K in a real Hilbert space H is a set of elements of H such that if $\underline{x}_i \in K$, $i\geqslant 1$, and $\lim_{i\to\infty} \underline{x}_i = \underline{x} \in H$ then $\underline{x} \in K$.

A convex set K in a real Hilbert space H is a set of elements of H such that $\lambda\underline{x}+(1-\lambda)\underline{y} \in K$ for all $\underline{x},\underline{y} \in K$ and for all $0<\lambda<1$.
A linear variety $K = \underline{a}+M$ in a real Hilbert space H is generated by a vector $\underline{a} \in H$ and a linear subspace $M \subset H$ and consists of all vectors of the form $\underline{a}+\underline{x}$ ($\underline{x} \in M$).

The fundamental theorem used in this lecture is[2] the theorem characterizing the minimum distance to a convex set.

Theorem[2] 1

Let x be a vector in a Hilbert space H and K a closed, convex subset of H. Then there is a unique vector $\underline{x}_o \in K$ such that

$$\|\underline{x}-\underline{x}_o\| \leqslant \|\underline{x}-\underline{k}\| \qquad \ldots(9)$$

for all $\underline{k} \in K$. Furthermore, a necessary and sufficient condition that $\underline{x}_o$ be the unique minimizing vector is that, for all $\underline{k} \in K$,

$$\langle \underline{x}-\underline{x}_o \, , \, \underline{k}-\underline{x}_o \rangle \leqslant 0 \qquad \ldots(10)$$

Finally, if K is a linear variety the inequality in (10) is replaced by equality.

3. Problem Formulation[4]

Consider the solution of the discrete system (3)-(4). In general it is impossible to obtain a solution by inspection and an iterative solution technique is necessary which, after a small finite number of iterations, generates an approximate solution to (3)-(4) in some sense.

The problem is best posed in the context of a suitable Hilbert space[4] H. We define H_1 to be the Hilbert space of sequences $\underline{x} = \{\underline{x}(1),\underline{x}(2),\ldots,\underline{x}(N)\}$ with addition and scalar multiplication defined by

$$\{\underline{x}(1),\ldots,\underline{x}(N)\} + \{\underline{y}(1),\ldots,\underline{y}(N)\} = \{\underline{x}(1) + \underline{y}(1),\ldots,\underline{x}(N) + \underline{y}(N)\}$$

$$\lambda\{\underline{x}(1),\ldots,\underline{x}(N)\} = \{\lambda\underline{x}(1),\ldots,\lambda\underline{x}(N)\} \qquad \ldots(11)$$

with inner product of $\underline{x} = \{\underline{x}(1),\ldots,\underline{x}(N)\}$ and $\underline{y} = \{\underline{y}(1),\ldots,\underline{y}(N)\}$ defined by

$$\langle\underline{x},\underline{y}\rangle_1 = \sum_{j=1}^{N} \underline{x}(j)^T Q_j\, \underline{y}(j) \qquad \ldots(12)$$

and the corresponding induced norm

$$\|\underline{x}\|_1 = \{\sum_{j=1}^{N} \underline{x}(j)^T Q_j\, \underline{x}(j)\}^{\frac{1}{2}} \qquad \ldots(13)$$

Here Q_j, $1 \leqslant j \leqslant N$, are symmetric positive-definite matrices, which for the problem under consideration are unspecified, although they do provide a technique for conditioning of the algorithm.

In a similar way H_2 is defined to be the Hilbert space of sequences $\underline{u} = \{\underline{u}(1),\ldots,\underline{u}(N)\}$ with addition and scalar multiplication defined in an analogous way to (11) and inner product and norm defined as in (12), (13) with Q_j, $1 \leqslant j \leqslant N$, replaced by symmetric positive definite matrices R_j, $1 \leqslant j \leqslant N$.

The Hilbert space of interest here is the cartesian product space $H = H_1 x H_2$ which is a Hilbert space with inner product and norm defined as in section (2). It is convenient to characterize (3) and (4) in terms of subsets of H, by noting that any solution of (3)-(4) can be regarded as an element of H. Let

$H \supset D \overset{\Delta}{=}$ set of all solutions of (3) regarded as points of H ...(14)

$H \supset A \overset{\Delta}{=}$ set of all solutions of (4) regarded as points of H ...(15)

and, to avoid trivialities, assume that both D and A are non-empty. The engineering problem can now be posed in the abstract form of the search for a point in the interaction of D and A. This is a complex problem unless extra structure is included in the problem. It is assumed therefore that $\Omega_u(k)$, $\Omega_x(k)$, $1 \leq k \leq N$ and S are closed and convex, from which both D and A are closed, convex subsets of H. The geometric representation of the problem is illustrated in Fig.1 and is the intuitive key to a solution of the problem.

Consider the geometric construction illustrated in Fig.2 where $\underline{k}_o \in H$ is an initial guess at the solution of (3),(4) not necessarily in D or A. Let $\underline{k}_1$ be that unique point in D nearest to $\underline{k}_o$, $\underline{k}_2$ be that point in A nearest to $\underline{k}_1$, $\underline{k}_3$ that point in D nearest to $\underline{k}_2$ etc then, intuitively, the sequence $\{\underline{k}_1, \underline{k}_2, \underline{k}_3, \ldots\}$ tends in the limit to a point $\underline{k}_\infty \in D \cap A$. Formally[4],

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<u>Theorem 2</u>

Let $\underline{k}_o \in H$ and $D \cap A$ be nonempty where D and A are closed convex subsets of the real Hilbert space H. Defining $\underline{k}_1$ to be that unique vector in D such that

$$\|\underline{k}_1 - \underline{k}_o\| = \inf_{\underline{k} \in D} \|\underline{k} - \underline{k}_o\| \qquad \ldots(16)$$

and $\underline{k}_i$ by

$$\|\underline{k}_{2i} - \underline{k}_{2i-1}\| = \inf_{\underline{k} \in A} \|\underline{k} - \underline{k}_{2i-1}\| \quad , \quad \underline{k}_{2i} \in A$$

$$\|\underline{k}_{2i+1} - \underline{k}_{2i}\| = \inf_{\underline{k} \in D} \|\underline{k} - \underline{k}_{2i}\| \quad , \quad \underline{k}_{2i+1} \in D \qquad \ldots(17)$$

then the sequence $\{\underline{k}_1, \underline{k}_2, \ldots\}$ is defined uniquely and, for any point $\underline{x} \in D \cap A$

$$\|\underline{x} - \underline{k}_1\|^2 \geq \sum_{j=1}^{\infty} \|\underline{k}_{j+1} - \underline{k}_j\|^2 \qquad \ldots(18)$$

Moreover, for any $\varepsilon > 0$, there exists an integer N such that, for $j \geq N$,

$$\inf_{\underline{k}\in D} \|\underline{k}_j - \underline{k}\| + \inf_{\underline{k}\in A} \|\underline{k}_j - \underline{k}\| < \varepsilon \qquad \ldots(19)$$

Finally, if H is finite-dimensional, there exists a point $\underline{k}_\infty \in D \cap A$ such that

$$\lim_{j\to\infty} \underline{k}_j = \underline{k}_\infty \qquad \ldots(20)$$

Proof (Outline)

By theorem 1, the sequence is defined uniquely and $\langle \underline{x}-\underline{k}_{j+1}, \underline{k}_{j+1}-\underline{k}_j \rangle \geq 0$, $j\geq 1$ ie

$$\|\underline{x} - \underline{k}_j\|^2 = \|\underline{x} - \underline{k}_{j+1} + \underline{k}_{j+1} - \underline{k}_j\|^2$$

$$= \|\underline{x} - \underline{k}_{j+1}\|^2 + \|\underline{k}_{j+1} - \underline{k}_j\|^2 + 2 \langle \underline{x} - \underline{k}_{j+1}, \underline{k}_{j+1} - \underline{k}_j \rangle$$

$$\geq \|\underline{x} - \underline{k}_{j+1}\|^2 + \|\underline{k}_{j+1} - \underline{k}_j\|^2 \qquad \ldots(21)$$

Equation (18) follows directly from induction and hence that $\lim_{j\to\infty} \|\underline{k}_{j+1} - \underline{k}_j\| = 0$ so that (19) follows by noting that $\underline{k}_{j+1} \in D$, $\underline{k}_j \in A$ (or vice versa). Equation (20) can be proved[4] by noting from the above that $\{\underline{k}_1, \underline{k}_2, \ldots\}$ is relatively compact and hence has a cluster point $\underline{k}_\infty \in D \cap A$ (by (19)) which is unique by convexity.

Q.E.D.

Interpreting this result in terms of the solution of the discrete problem (3)-(4), we can write, $i\geq 0$,

$$\underline{k}_i = (\{\underline{x}_i(1), \underline{x}_i(2), \ldots, x_i(N)\}, \{\underline{u}_i(1), \underline{u}_i(2), \ldots, \underline{u}_i(N)\})$$

so that $\underline{k}_i$ (i odd) is generated by the solution of the optimization problem

$$\min \sum_{j=1}^{N} \left[(\underline{x}_i(j) - \underline{x}_{i-1}(j))^T Q_j (\underline{x}_i(j) - \underline{x}_{i-1}(j)) \right.$$

$$\left. + (\underline{u}_i(j) - \underline{u}_{i-1}(j))^T R_j (\underline{u}_i(j) - \underline{u}_{i-1}(j)) \right] \qquad \ldots(22)$$

subject to

$$\underline{x}_i(k+1) = A\,\underline{x}_i(k) + B\,\underline{u}_i(k+1)$$

$$\underline{0} = E\,\underline{u}_i(k+1) + F\,\underline{x}_i(k) \quad , \quad 0 \leqslant k \leqslant N-1$$

$$\underline{x}_i(o) = \underline{x}_o \quad , \quad \underline{x}_i(N) \in S \qquad \ldots(23)$$

or, if i is even, $\underline{k}_i$ is the solution of the optimization problem

$$\min\ (22) \quad \text{subject to} \quad \underline{u}_i(k) \in \Omega_u(k),\ \underline{x}_i(k) \in \Omega_x(k)$$
$$1 \leqslant k \leqslant N \qquad \ldots(24)$$

Problem (22), (23) can be solved in many cases for example by application of discrete dynamic programming techniques and problem (24) is soluble by standard algebraic minimization routines. The algorithm has guaranteed convergence but, in practice, can suffer from slow convergence problems due to the geometry of D and A in H as illustrated in Fig.3. In this case an accelerated algorithm can be obtained if D is a linear variety ie $S = R^n$. The general idea is illustrated in Fig.4 and consists of the use of tangent hyperplanes to A to construct an improved iterate. Formally[4],

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Theorem 3

If D is a closed linear variety in H, $\underline{k}_o \in H$ and $\underline{r}_1$ is the unique solution of the optimization problem

$$\|\underline{r}_1 - \underline{k}_o\| = \inf_{\underline{r}\in D} \|\underline{r} - \underline{k}_o\| \quad , \quad \underline{r}_1 \in D \qquad \ldots(25)$$

and, $i \geqslant 1$

$$\|\underline{k}_i - \underline{r}_i\| = \inf_{\underline{k}\in A} \|\underline{k} - \underline{r}_i\| \quad , \quad \underline{k}_i \in A$$

$$\|\underline{s}_i - \underline{k}_i\| = \inf_{\underline{s}\in D} \|\underline{s} - \underline{k}_i\| \quad , \quad \underline{s}_i \in D \qquad \ldots(26)$$

$$\underline{r}_{i+1} = \underline{r}_i + \lambda_i (\underline{s}_i - \underline{r}_i)$$

$$1 \leq \lambda_i \leq \frac{\|\underline{k}_i - \underline{r}_i\|^2}{\|\underline{s}_i - \underline{r}_i\|^2} \qquad \ldots(27)$$

then the sequence $\{\underline{r}_1, \underline{k}_1, \underline{s}_1, \underline{r}_2, \underline{k}_2, \ldots\}$ is well-defined for each $\underline{k}_o \in H$. Moreover, for any $\varepsilon>0$, there exists an integer N such that, for $j \geq N$,

$$\inf_{\underline{k} \in A} \|\underline{k} - \underline{r}_j\| + \inf_{\underline{r} \in D} \|\underline{r} - \underline{k}_j\| < \varepsilon \qquad \ldots(28)$$

Finally, if H is finite-dimensional, there exists a point $\underline{r}_\infty \in D \cap A$ such that

$$\lim_{j \to \infty} \underline{r}_j = \underline{r}_\infty \qquad \ldots(29)$$

Note that

(a) The flexibility in choice of λ_i, $i \geq 1$, implicit in (27) is very useful in practice as a large extrapolation factor λ_i tends to magnify numerical errors.

(b) The algorithm has a similar interpretation to that of theorem 1, except that the constraint $\underline{x}_i(N) \in S$ is now trivial.

(c) Both theorem 2 and 3 have similar interpretations for the continuous system (2) but, in this case the Hilbert space of interest is infinite-dimensional so the existence of a limit point is not proved. All is not lost however as (28) and (19) indicate that it is always possible to generate an approximate solution (in the sense of the norm) with an arbitrary degree of accuracy.

4. Numerical Examples

Example One: consider the problem of calculating a solution of the scalar equations

$$\dot{x}(t) = u(t) \ , \quad x(o) = 0 \ , \quad x(1) = 1 \ , \quad \int_o^1 tu(t)\,dt = 1 \qquad \ldots(30)$$

This is an infinite dimensional problem and the appropriate choice of H is $H = L_2\ (0,1)$ with norm

$$\|k\|^2 = \tfrac{1}{2}\int_o^1 (k(t))^2 dt \ , \qquad k \in H \qquad \ldots(31)$$

Let

$$D \triangleq \{u \in H : \int_o^1 u(t)\,dt = 1\} \qquad \text{(closed linear variety)}$$

$$A \triangleq \{u \in H : \int_o^1 tu(t)\,dt = 1\} \qquad \text{(closed hyperplane)} \qquad \ldots(32)$$

Applying theorem 3 with $k_o(t) \equiv 0$, $0 \leq t \leq 1$, then $r_1(t)$ is the solution of the problem,

$$\min \int_o^1 (u(t))^2 dt \ \ \text{subject to} \ \ \dot{x}(t) = u(t),\ x(o) = 0,\ x(1) = 1 \qquad \ldots(33)$$

Application of optimal control theory yields the solution

$$r_1(t) \equiv 1 \ , \qquad 0 \leq t \leq 1 \qquad \ldots(34)$$

and in a similar manner

$$k_1(t) = 1 + \frac{3}{2}t$$

$$s_1(t) = \frac{1}{4} + \frac{3}{2}t \qquad \ldots(35)$$

Choosing $\lambda_1 = \|k_1 - r_1\|^2 / \|s_1 - r_1\|^2 = 4$, then

$$r_2(t) = 1 + 4\{\frac{1}{4} + \frac{3}{2}t - 1\} = -2 + 6t \qquad \ldots(36)$$

Moreover $r_2 \in D \cap A$ and the algorithm converges in two iterations.

<u>Example Two:</u> The power distribution P(x,t) in a one-dimensional thermal nuclear reactor of length L can be approximately described by the relation

$$P(x,t) = \sum_{j=1}^{4} u_j(t) \sqrt{\frac{2}{L}} \sin \frac{j\pi x}{L}$$

$$0 \leqslant x \leqslant L \;, \qquad 0 \leqslant t \leqslant T \qquad \ldots(37)$$

where the flux mode amplitudes $u_j(t)$, $1 \leqslant j \leqslant 4$, are regarded as constrained system inputs. There are three other inputs $u_5(t)$, $u_6(t)$, $u_7(t)$ representing bulk control action and two trimming control mechanism. A detailed description of the dynamics of the spatial power distribution is outside the scope of this lecture[1,5]. However on the iodine-xenon time scale (10–30 hours) it should be noted that large, thermal, power reactors can exhibit spatial instability in the flux mode $u_2(t)$ despite the presence of bulk control action $u_5(t)$ if no trimming control action is used. A linearized discrete model of the process can be derived[4] of the form given by (3) with $S = R^n$, $n = 8$, $\ell = 7$ and $m = 5$ where $\underline{x}(t)$ is a state vector of iodine and xenon concentrations through the reactor core and $\underline{u}(t) = \{u_1(t),\ldots,u_7(t)\}^T$. With the initial condition

$$\underline{x}(o) = \{10^{14}, -10^{14}, -10^{14}, 10^{14}, 0, 0, 0, 0\}^T \qquad \ldots(38)$$

and T = 40 hours, N = 20, the solution of (3) with bulk control action and no trimming control is unique and $u_2(t)$ is as illustrated in Fig.5. It is noted that the system is unstable, with peak magnitude of $u_2(t)$ equal to 5.39×10^{14}.

The problem considered in this example is the choice of trimming control action in the time interval of interest to stabilize the system and, in particular to induce transient behaviour from the initial condition (38) satisfying the constraint,

$$|u_2(k)| \leqslant 0.4 \times 10^{14} \qquad \ldots(39)$$

which would obviously be a great improvement on the open-loop behaviour. Note again that the problem is not an optimal control problem!

In the notation of section 3, the constraint sets Ω_u, Ω_x are

$$\Omega_x(k) = R^8 \quad , \quad 1 \leqslant k \leqslant N$$

$$\Omega_u(k) = \{\underline{u} \in R^7 : |u_2| \leqslant 0.4 \times 10^{14}\} \quad , \quad 1 \leqslant k \leqslant N \qquad \ldots(40)$$

Applying the algorithm defined by theorem 1, with

$$Q_j = I_8 \quad , \quad 1 \leqslant j \leqslant N$$

$$R_j = \text{diag}\,\{1.0,1.0,1.0,1.0,10^5,10^8,10^8\} \quad , \quad 1 \leq j \leq N \qquad \ldots(41)$$

the following results were obtained, with a zero initial guess,

Iteration	$\|\underline{s}_i - \underline{k}_i\|$
1	50.4
10	42.8
50	16.9

illustrating the extremely slow convergence of the unaccelerated algorithm. In fact, after 50 iterations, the algorithm had not converged to an acceptable degree of accuracy.

Application of the accelerated algorithm defined for a zero initial guess yielded the results,

Iteration	λ_i	$\|\underline{s}_i - \underline{k}_i\|$
1	34.6	50.4
2	59.6	10.08
3	1.0	$0.35\text{x}10^{-4}$
4	1.0	$0.107\text{x}10^{-7}$

indicating rapid convergence. In fact, the large values of extrapolation factor used in the initial stages of the algorithm are the main cause of this success.

The converged solution $u_2(t)$, $u_6(t)$, $u_7(t)$ are shown in Fig.6, together with the flux constraint.

5. Summary

The lecture is devoted to a discussion and demonstration of the fact that optimization is not necessarily the objective of a design exercize but can play an important role in the solution of complex engineering design problems. The lecture described a large class of problems soluble by sequential application of optimization methods. The level of mathematics required to formulate the algorithm is relatively sophisticated but, with hindsight, the intuitive appreciation of the methods and proof of convergence are difficult to obtain without the use of such sophisticated methods.

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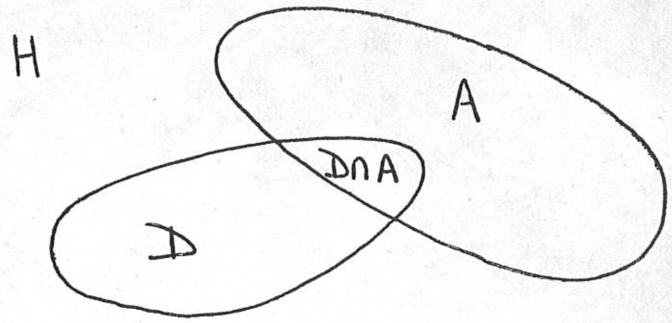


Fig. 1.

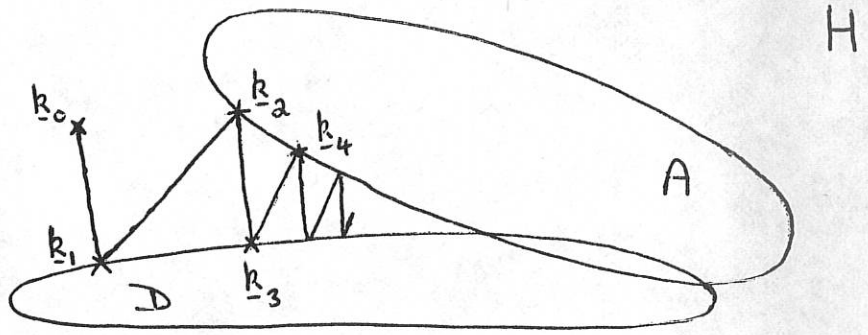


Fig. 2.

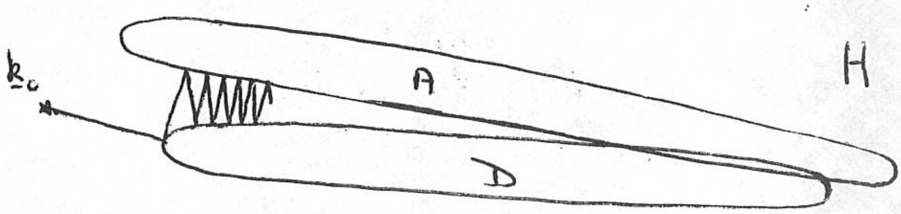


Fig 3

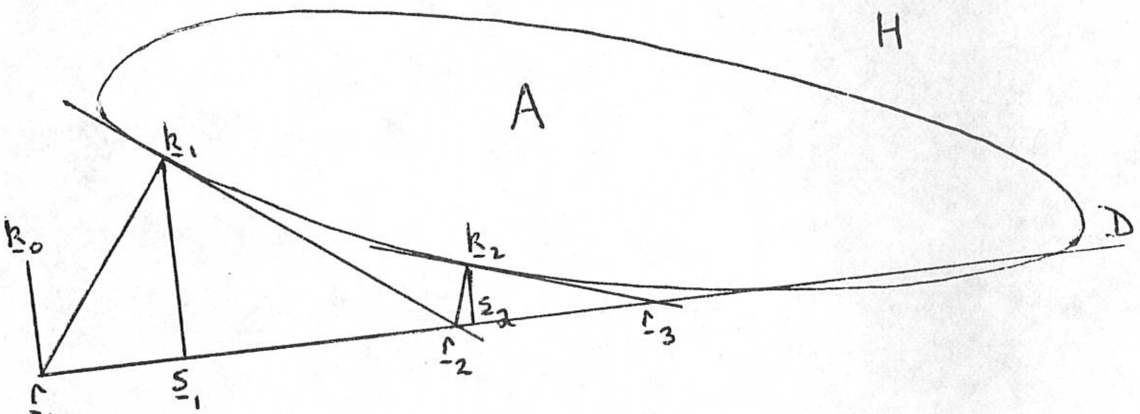


Fig. 4.

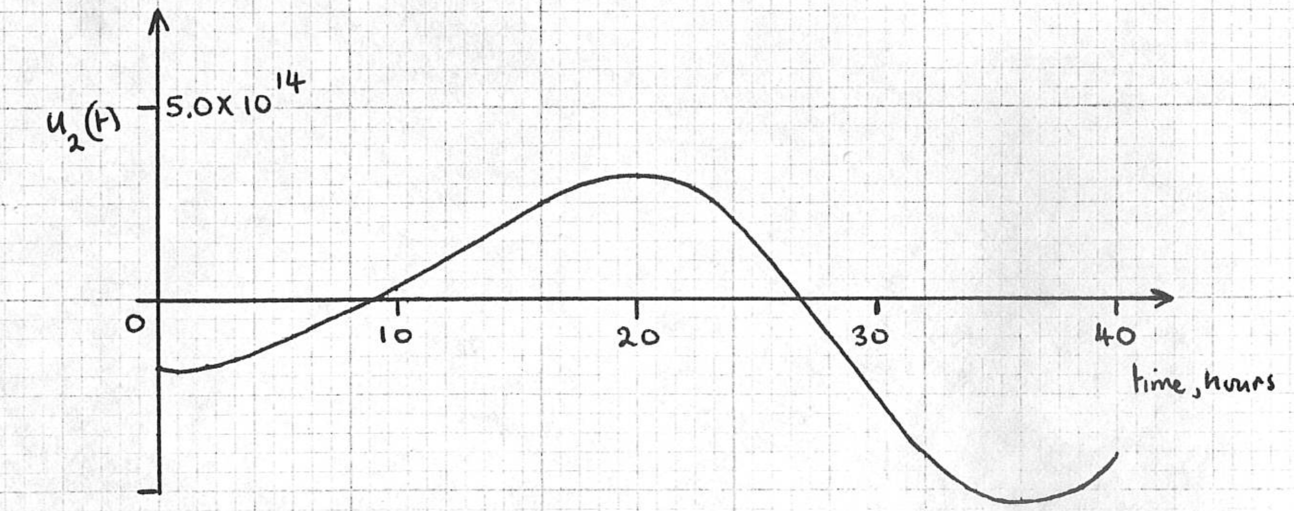


Fig. 5. Open-loop Solution, No Trimming Rods.

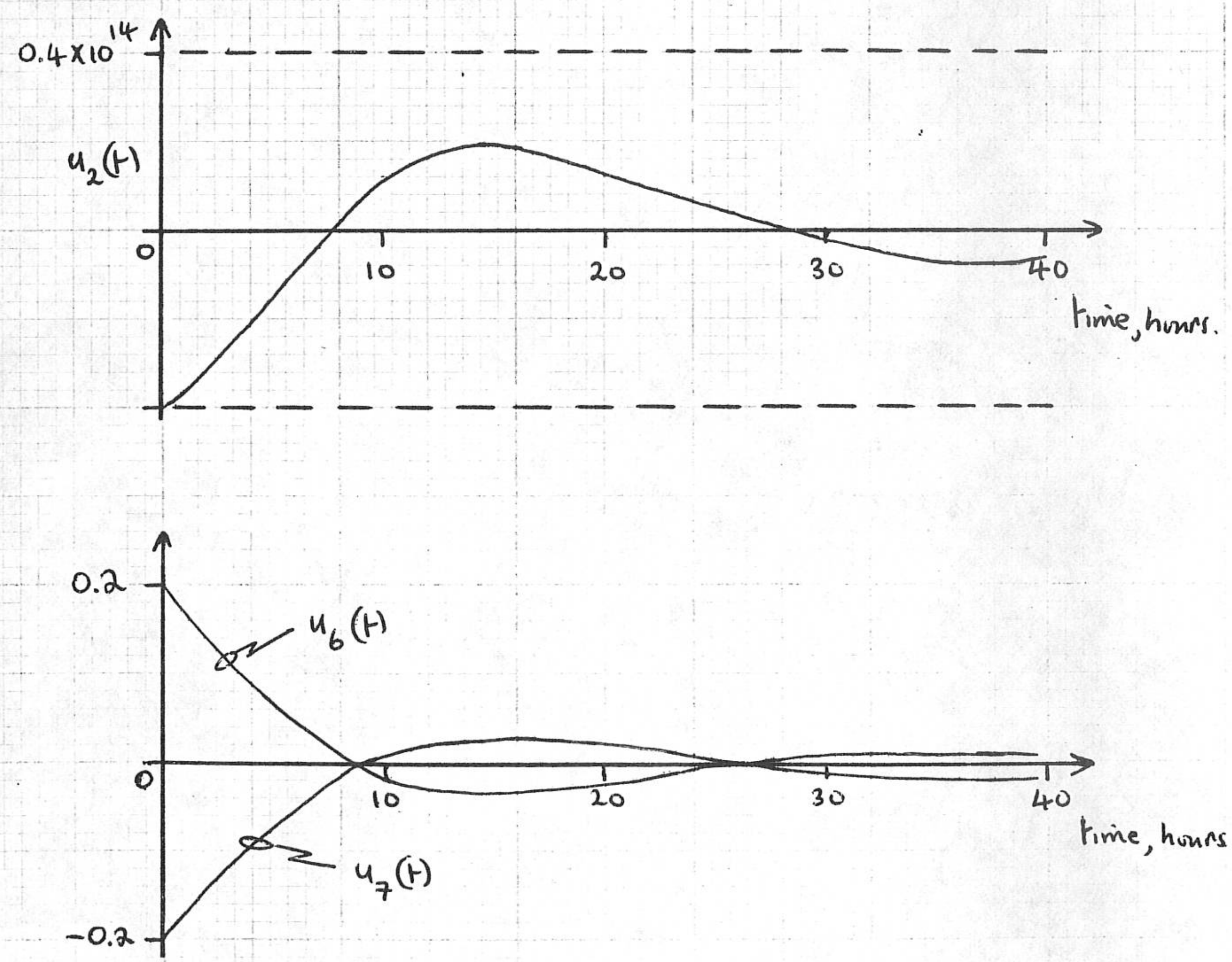
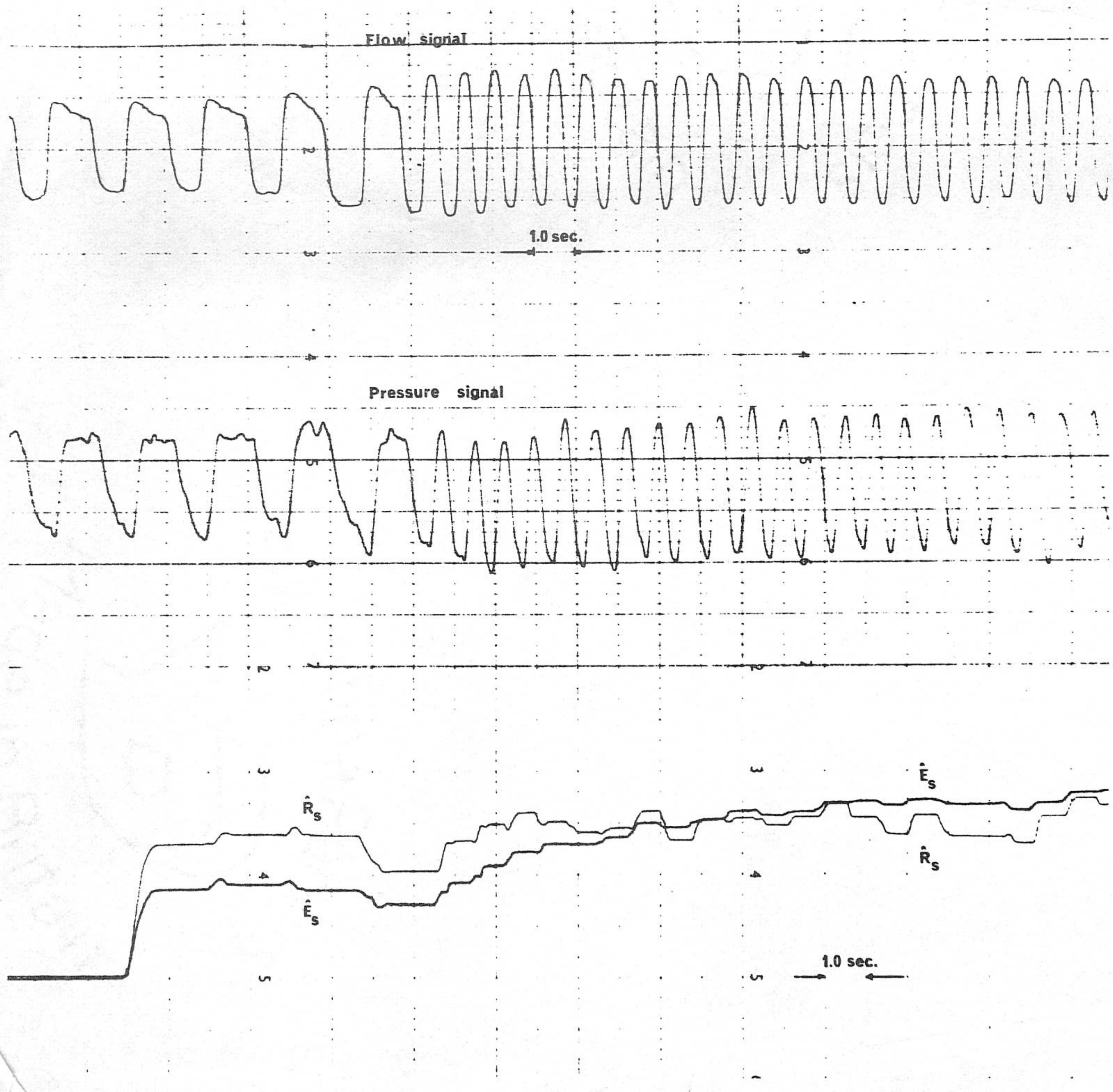


Fig. 6 Converged Solution with Power Constraint.



Flow signal
1.0 sec.
Pressure signal
$\hat{R}_s$
$\hat{E}_s$
$\hat{E}_s$
$\hat{R}_s$
1.0 sec.

Fig (11)