Monograph:
Billings, S.A. and Fakhouri, S.Y. (1977) Identification of a Class of Nonlinear Systems Using Correlation Analysis. Research Report. ACSE Research Report 56 . Department of Automatic Control and Systems Engineering

IDENTIFICATION OF A CLASS OF NONLINEAR SYSTEMS USING CORRELATION ANALYSIS

by

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Research Report No.56

March 1977

1. INTRODUCTION

Most control systems encountered in practice are non-linear to some extent, and although it may be possible to represent systems which are only 'mildly' non-linear or which operate over a restricted range by a linear model, in general non-linear processes can only be adequately characterised by a non-linear model. Since the analysis of control systems is often dependent upon finding a mathematical description which defines the relationship between the system input and output, system identification is of fundamental importance in control theory. However, whereas system identification methods are well established for linear systems the identification of non-linear systems appears to have been largely neglected. This can of course be attributed to the inherent complexity which results in the analysis of non-linear systems and the absence of a general theory for such systems.

It is well known that for a linear time-invariant system, the output response y(t) to an input x(t) may be computed from a knowledge of the system impulse response h(t) using the Convolution Integral[1]

$$y(t) \;=\; \int_{\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \qquad (1)$$

Methods of measuring h(t) by correlation techniques are well documented, and numerous parameter estimation schemes[2] have been developed to identify the parameters in the pulse transfer function representation of h(t).

A functional representation of non-linear systems which is a generalisation of the linear convolution integral, was first studied by Volterra[3] early in the twentieth century. Volterra showed that the explicit input output relationship for a time-invariant non-linear system which produces a continuous and bounded output y(t) when excited by a continuous and bounded input x(t) can be expressed as

$$y(t) = \int_{-\infty}^{\infty} g_1(\tau_1)x(t-\tau_1)d\tau_1 + \iint_{-\infty}^{\infty} g_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

$$+ \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n \text{ integrals}} g_n(\tau_1,\tau_2\cdots\tau_n)x(t-\tau_1)x(t-\tau_2)\cdots x(t-\tau_n)d\tau_1 d\tau_2\cdots d\tau_n \qquad (2)$$

where the function $g_n(\tau_1,\tau_2\cdots\tau_n)$ is termed the Volterra kernel of order n. The Volterra kernels are symmetric, continuous in their arguments for all $\tau_i>0$, and for a non-aticipative system are zero for any $\tau_i<0$. Systems which contain non-linear memory elements such as hysteresis or backlash are excluded from the description of eqn (2).

In 1942, Wiener[4] applied the functional Volterra series to the study of a non-linear electrical circuit problem, and was able to derive expressions for the output moments and autocorrelation functions for a Gaussian white input process. A major part of Wiener's non-linear theory was the development of the G-functionals which form an orthogonal set of Volterra functions for the representation of a non-linear system when the input is a white Gaussian process.

More recently, the functional representation has been investigated by a number of workers, notably Zadeh[5], Bose[6], Barrett[7], Brilliant[8] and George[9]. Bose introduced the concept of gate functions, which by partitioning the function space of the past of the input into non-overlapping cells overcame many of the difficulties of Wiener's method. An algebra for the analysis of continuous non-linear systems was suggested by Brilliant, and later George applied this approach and developed Multidimensional Laplace Transforms. Mention should also be made of the work of Ku and Wolf[10], Alper[11], Flake[12] and numerous other researchers who have all contributed to the functional series representation of non-linear systems.

A method of measuring the kernels in Wiener's G-functional expansion by cross-correlation using a Gaussian white noise input, was developed by Lee and Schetzen[13] and used by Widnall[14] to determine the second kernel of a simple system. Kadri[15], Gyftopoulos and Hooper[16], Douce and Weedon[17], and Krempl[18] have investigated the use of various discrete level pseudo-random signals in the identification of non-linear systems, and various authors including Ream[19], Barker and Pradisthayon[20] have studied the properties of such signals. A review of the use of correlation techniques for the identification of non-linear systems has been compiled by Simpson and Power[21].

Although the functional series expansion of non-linear systems is now well established very few researchers have attempted to identify practical non-linear systems based on this representation. This can be attributed to the practical difficulty associated with the identification of the system kernels and the excessive computational requirements necessary to characterise non-linear systems using the Volterra series. Consequently, various authors have turned their attention to a restricted class of non-linear systems; notably cascade systems composed of linear subsystems with memory and non-linear no-memory subsystems. Cooper and Falkner[22], Gardiner[23,24], Webb[25], and Economakos[26] have investigated the identification of such systems. However, the identification schemes derived by these authors necessitate a series of tests on a system using different input amplitudes. This will inevitably result in a long measurement time and may be prohibitive in an industrial environment.

In the present study correlation analysis is used to identify the component linear and non-linear subsystems in this class of non-linear systems when the input processes have white noise properties. It is shown that correlation analysis decouples the problem into the identification of the linear subsystems and the characterisation of the non-linear element. Parameterisation of the identified linear system impulse responses and the non-linear system characteristic is discussed and the results are extended to include the case of coloured noise inputs.

2. PROBLEM FORMULATION

The class of non-linear systems considered in the present study consists of a linear system with impulse response $h_1(t)$ in cascade with a non-linear no-memory element and a linear system with impulse response $h_2(t)$, as illustrated in FIG 1.

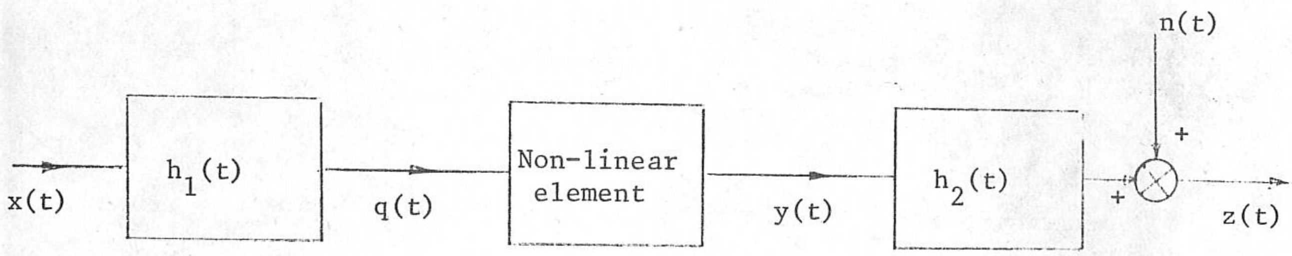

FIG 1. A class of non-linear systems

Many chemical and other industrial processes are of this type.

It is assumed that the non-linear no-memory element can be represented by a transfer characteristic of the form

$$y(t) = \gamma_1 q(t) + \gamma_2 q^2(t) + \ldots + \gamma_k q^k(t) \quad (3)$$

According to the theorem of Weierstrass[27], any function which is continuous within an interval may be approximated to any required degree of accuracy by polynomials in this interval. Thus, even violent non-linearities such as an ideal half-wave rectifier can, theoretically, be very closely approximated by a polynomial.

From FIG 1, the output y(t) of the non-linear element can be expressed as

$$y(t) = \gamma_1 \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 + \gamma_2 \iint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

$$+ \dots + \gamma_k \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_1(\tau_1) \dots h_1(\tau_k)x(t-\tau_1) \dots x(t-\tau_k)d\tau_1 \dots d\tau_k \qquad (4)$$

and the measured system output can be represented by the functional series

$$z(t) = \gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)x(t-\tau_1-\tau_2)d\tau_1 d\tau_2$$

$$+ \gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)x(t-\tau_1-\tau_3)x(t-\tau_2-\tau_3)d\tau_1 d\tau_2 d\tau_3$$

$$+ \vdots$$

$$+ \gamma_k \int_{-\infty}^{\infty} \dots \int h_1(\tau_1)\dots h_1(\tau_k)h_2(\tau_{k+1})x(t-\tau_1-\tau_{k+1})$$

$$\dots x(t-\tau_k-\tau_{k+1})d\tau_1 \dots d\tau_{k+1} + n(t) \qquad (5)$$

Comparison with the Volterra series eqn (2) shows that for this class of non-linear systems the mth order Volterra kernel is given by

$$g_m(\tau_1 \dots \tau_m) = \gamma_m \int_{-\infty}^{\infty} h_2(\sigma) \prod_{p=1}^{m} h_1(\tau_p - \sigma)d\sigma \qquad (6)$$

Thus for the class of non-linear systems depicted in FIG 1, the Volterra kernels have a known structure.

3. IDENTIFICATION OF THE LINEAR SUB-SYSTEMS

If the input signal x(t) is a zero mean white Gaussian process with a spectral density of 1 watt per cycle, then its ith dimensional autocorrelation function[28] is

$$\overline{x(t_1)x(t_2)\ \ldots\ x(t_i)} = 0 \quad \text{for } i \text{ odd}$$

$$= \sum_{i}\ \prod_{n\neq m}\ \delta(t_n - t_m) \quad \text{for } i \text{ even} \qquad (7)$$

where the summation is over all the ways of dividing (i+1) objects into pairs. Thus, for example

$$\overline{x(t)} = 0$$

$$\overline{x(t_1)x(t_2)} = \delta(t_2 - t_1)$$

$$\overline{x(t_1)x(t_2)x(t_3)} = 0$$

$$\overline{x(t_1)x(t_2)x(t_3)x(t_4)} = \delta(t_2-t_1)\delta(t_4-t_3) + \delta(t_3-t_1)\delta(t_4-t_2)+\delta(t_4-t_1)\delta(t_3-t_2)$$

etc.

where — means time average.

It will be assumed throughout that all random signals are ergodic, such that ensemble averages may be replaced by time averages over one sample function.

Consider the system illustrated in FIG 1 when the input signal comprises a Gaussian white signal x(t) with a mean level b. From the functional series expansion eqn (5), the measured output z(t) can be expressed as

$$z(t) = \gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)\{x(t-\tau_1-\tau_2)+b\}d\tau_1 d\tau_2$$

$$+ \gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{x(t-\tau_1-\tau_3)+b\}\{x(t-\tau_2-\tau_3)+b\}d\tau_1 d\tau_2 d\tau_3$$

$$+ \ldots \qquad + n(t)$$

$$= w_1(t)+w_2(t) + \ldots w_k(t)+n(t) \qquad (8)$$

Utilizing the 'a priori' information concerning the structure of the kernels in the functional series expansion eqn (8), it is shown that correlation analysis can be used to decouple the identification of the linear subsystems from the identification of the non-linear characteristic.

Defining $z'(t) = z(t) - \overline{z(t)}$

$$w_i'(t) = w_i(t) - \overline{w_i(t)}$$

the first order cross-correlation function is given by

$$\phi_{xz'}(\sigma) = E\left[z'(t)x(t-\sigma)\right]$$

$$= \overline{w_1'(t)x(t-\sigma)} + \overline{w_2'(t)x(t-\sigma)} + \ldots + \overline{w_k'(t)x(t-\sigma)}$$

$$+ \overline{n(t)x(t-\sigma)} \qquad (9)$$

Evaluating the first term on the rhs of eqn (9)

$$\overline{w_1'(t)x(t-\sigma)} = \overline{\{w_1(t)-\overline{w_1(t)}\}x(t-\sigma)}$$

where $\overline{w_1(t)} = b\gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)d\tau_1 d\tau_2$

Hence

$$\overline{w_1'(t)x(t-\sigma)} = \overline{\{\gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)x(t-\tau_1-\tau_2)d\tau_1 d\tau_2\}\, x(t-\sigma)}$$

$$= \gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)\overline{x(t-\tau_1-\tau_2)x(t-\sigma)}\,d\tau_1 d\tau_2$$

$$= \gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)\delta(\tau_1+\tau_2-\sigma)d\tau_1 d\tau_2$$

$$\overline{w_1'(t)x(t-\sigma)} = \gamma_1 \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 \qquad (10)$$

Considering the second term on the rhs of eqn (9)

$$\overline{w_2'(t)x(t-\sigma)} = \overline{\{w_2(t)-\overline{w_2(t)}\}x(t-\sigma)} = \overline{w_2(t)x(t-\sigma)} - \overline{\overline{w_2(t)}x(t-\sigma)}$$

where

$$\overline{w_2(t)} = \gamma_2 \overline{\iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{x(t-\tau_1-\tau_2)+b\}\{x(t-\tau_2-\tau_3)+b\}d\tau_1 d\tau_2 d\tau_3}$$

$$= \gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{\delta(\tau_1-\tau_2)+b^2\}d\tau_1 d\tau_2 d\tau_3$$

$$= \gamma_2 \iint_{-\infty}^{\infty} h_1^{\,2}(\tau_1)h_2(\tau_3)d\tau_1 d\tau_3 + b^2\gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)d\tau_1 d\tau_2 d\tau_3 \tag{11}$$

Consequently

$$\overline{w_2{}'(t)x(t-\sigma)} = \overline{\{\gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{x(t-\tau_1-\tau_2)+b\}\{x(t-\tau_2-\tau_3)+b\}}$$

$$\overline{.d\tau_1 d\tau_2 d\tau_3\}x(t-\sigma)}$$

$$-\overline{\{\gamma_2 \iint_{-\infty}^{\infty} h_1^{\,2}(\tau_1)h_2(\tau_3)d\tau_1 d\tau_3\}x(t-\sigma)}$$

$$-\overline{\{b^2\gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)d\tau_1 d\tau_2 d\tau_3\}x(t-\sigma)}$$

$$= b\gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{\delta(\tau_2+\tau_3-\sigma)+\delta(\tau_1+\tau_3-\sigma)\}d\tau_1 d\tau_2 d\tau_3$$

$$= b\gamma_2 \iint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\sigma-\tau_2)d\tau_1 d\tau_2$$

$$+ b\gamma_2 \iint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\sigma-\tau_1)d\tau_1 d\tau_2$$

$$\overline{w_2{}'(t)x(t-\sigma)} = 2b\gamma_2 \int_{-\infty}^{\infty} h_1(\tau_1)d\tau_1\{\int_{-\infty}^{\infty} h_1(\tau_2)h_2(\sigma-\tau_2)d\tau_2\} \tag{12}$$

Similarly, for the third term on the rhs of eqn (9)

$$\overline{w_3{}'(t)x(t-\sigma)} = \{3\gamma_3 \int_{-\infty}^{\infty} h_1^{\,2}(\tau_3)d\tau_3 + 2b^2\gamma_3 \iint_{-\infty}^{\infty} h_1(\tau_2)h_1(\tau_3)d\tau_2 d\tau_3\}$$

$$.\{\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1\} \tag{13}$$

Higher order terms can be evaluated in a similar manner.

Collecting terms

$$\phi_{xz'}(\sigma) = \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 \{\gamma_1 + 2b\gamma_2 \int_{-\infty}^{\infty} h_1(\tau_2)d\tau_2$$

$$+ 3\gamma_3 \int_{-\infty}^{\infty} h_1^{\,2}(\tau_3)d\tau_3 + 2b^2\gamma_3 \iint_{-\infty}^{\infty} h_1(\tau_2)h_1(\tau_3)d\tau_2 d\tau_3$$

$$+ \ldots\ldots\ldots\ldots\} \quad + \overline{n(t)x(t-\sigma)} \qquad (14)$$

Assuming that the input signal x(t) and the noise process n(t) are statistically independent, $\overline{n(t)x(t-\sigma)} = 0 \;\forall\; \sigma$. Providing the linear subsystems are stable, bounded inputs-bounded outputs, the terms enclosed in square brackets eqn (14) are constants and the first order cross-correlation function can be expressed as

$$\phi_{xz'}(\sigma) = \beta \,.\, \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 \qquad (15)$$

where β is a constant.

Therefore, by applying a white Gaussian process with mean level b to the system illustrated in FIG 1 and computing the first order cross-correlation function $\phi_{xz'}(\sigma)$, an estimate of the convolution of the linear subsystem impulse responses is obtained. A Gaussian signal with a mean level b, is used as the input signal to ensure that all terms in the functional expansion eqn (8) contribute to the first order cross-correlation function eqn (14). If the input signal had zero mean level, cross-correlation over all the even terms in eqn (8) (i.e. $\phi_{xw_2'}(\sigma)$, $\phi_{xw_4'}(\sigma)$, etc.) would be zero because of the symmetry of the Gaussian density function.

If the system were linear (i.e. $\gamma_n = 0$, $n = 2,3...$) the identification would be complete. However, in order to characterize the non-linear system illustrated in FIG 1 an estimate of the individual linear subsystem impulse responses is required. This can be achieved by computing the second order cross-correlation function which provides a second equation relating $h_1(t)$ and $h_2(t)$.

Consider the system illustrated in FIG 1 when the input signal comprises a Gaussian white signal x(t) with a mean level b. From the functional series expansion eqn (8), the second order cross-correlation function is defined as

$$\begin{aligned}\phi_{xxz'}(\sigma_1,\sigma_2) &= E\left[z'(t)x(t-\sigma_1)x(t-\sigma_2)\right] \\ &= \overline{w_1'(t)x(t-\sigma_1)x(t-\sigma_2)} + \overline{w_2'(t)x(t-\sigma_1)x(t-\sigma_2)} \ldots \\ &\ldots + \overline{w_k'(t)x(t-\sigma_1)x(t-\sigma_2)} + \overline{n(t)x(t-\sigma_1)x(t-\sigma_2)}\end{aligned} \quad (16)$$

Evaluating the first term on the rhs of eqn (16)

$$\overline{w_1'(t)x(t-\sigma_1)x(t-\sigma_2)} = \overline{\{w_1(t)-\overline{w_1(t)}\}x(t-\sigma_1)x(t-\sigma_2)}$$

where $\overline{w_1(t)} = b\gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)d\tau_1 d\tau_2$

Hence

$$\begin{aligned}\overline{w_1'(t)x(t-\sigma_1)x(t-\sigma_2)} &= \overline{\{\gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)\{x(t-\tau_1-\tau_2)+b\}d\tau_1 d\tau_2\}x(t-\sigma_1)x(t-\sigma_2)} \\ &\quad - \overline{\{b\gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)d\tau_1 d\tau_2\}x(t-\sigma_1)x(t-\sigma_2)} \\ &= \gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)\{\overline{x(t-\tau_1-\tau_2)x(t-\sigma_1)x(t-\sigma_2)} \\ &\qquad + \overline{bx(t-\sigma_1)x(t-\sigma_2)}\}d\tau_1 d\tau_2 \\ &\quad - b\gamma_1 \iint_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)\overline{x(t-\sigma_1)x(t-\sigma_2)}\, d\tau_1 d\tau_2\end{aligned} \quad (17)$$

$$\overline{w_1'(t)x(t-\sigma_1)x(t-\sigma_2)} = 0 \;\forall\; \sigma_1,\sigma_2 \tag{18}$$

Considering the second term on the rhs of eqn (16)

$$\overline{w_2'(t)x(t-\sigma_1)x(t-\sigma_2)} = \overline{\{w_2(t)-\overline{w_2(t)}\}x(t-\sigma_1)x(t-\sigma_2)}$$

and substituting for $\overline{w_2(t)}$ from eqn (11) gives

$$\begin{aligned}
\overline{w_2'(t)x(t-\sigma_1)x(t-\sigma_2)} = {} & \overline{\{\gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{x(t-\tau_1-\tau_3)+b\}.} \\
& \overline{\{x(t-\tau_2-\tau_3)+b\}d\tau_1 d\tau_2 d\tau_3\}x(t-\sigma_1)x(t-\sigma_2)} \\
& -\{\gamma_2 \overline{\iint_{-\infty}^{\infty} h_1^{\,2}(\tau_1)h_2(\tau_3)d\tau_1 d\tau_3} \\
& + b^2\gamma_2 \overline{\iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)d\tau_1 d\tau_2 d\tau_3\}x(t-\sigma_1).} \\
& \overline{x(t-\sigma_2)} \\
= {} & \gamma_2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)\{\delta(\tau_1-\tau_2)\delta(\sigma_1-\sigma_2)+\delta(\tau_1+\tau_3-\sigma_1)\delta(\tau_2+\tau_3-\sigma_2) \\
& + \delta(\tau_1+\tau_3-\sigma_2)\delta(\tau_2+\tau_3-\sigma_2)+b^2\delta(\sigma_1-\sigma_2)\}d\tau_1 d\tau_2 d\tau_3 \\
& - \delta(\sigma_1-\sigma_2)\gamma_2 \{\iint_{-\infty}^{\infty} h_1^{\,2}(\tau_1)h_2(\tau_3)d\tau_1 d\tau_3 \\
& + b^2 \iiint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)h_2(\tau_3)d\tau_1 d\tau_2 d\tau_3\}
\end{aligned}$$

$$= 2\gamma_2 \int_{-\infty}^{\infty} h_1(\sigma_1-\tau_3)h_1(\sigma_2-\tau_3)h_2(\tau_3)d\tau_3 \tag{19}$$

Setting $\sigma_1 = \sigma_2 = \sigma$ gives

$$\overline{w_2'(t)x(t-\sigma)x(t-\sigma)} = 2\gamma_2 \int_{-\infty}^{\infty} h_1^{\,2}(\sigma-\tau_3)h_2(\tau_3)d\tau_3 \tag{20}$$

Similarly for the third term on the rhs of eqn (16)

$$\overline{w_3'(t)x(t-\sigma)x(t-\sigma)} = \int_{-\infty}^{\infty} h_1^2(\sigma-\tau_4)h_2(\tau_4)d\tau_4\{6b\gamma_3 \int_{-\infty}^{\infty} h_1(\tau_3)d\tau_3\} \quad (21)$$

Higher order terms can be evaluated in a similar manner.

Collecting terms

$$\phi_{xxz'}(\sigma,\sigma) = \int_{-\infty}^{\infty} h_1^2(\sigma-\tau_1)h_2(\tau_1)d\tau_1 \{0+2\gamma_2+6b\gamma_3 \int_{-\infty}^{\infty} h_1(\tau_3)d\tau_3 + \ldots\ldots\} + \overline{n(t)x(t-\sigma)x(t-\sigma)} \quad (22)$$

Assuming that the input signal x(t) and the noise process are statistically independent, $\overline{n(t)x(t-\sigma)} = 0 \forall \sigma$ it can readily be shown that $\overline{n(t)x(t-\sigma)x(t-\sigma)} = 0 \forall \sigma$. Providing the linear subsystems are stable, bounded inputs bounded outputs, the terms enclosed in square brackets eqn (22) are constants and the second order cross-correlation function can be expressed as

$$\phi_{xxz'}(\sigma,\sigma) = \alpha \int_{-\infty}^{\infty} h_1^2(\sigma-\tau_1)h_2(\tau_1)d\tau_1 \quad (23)$$

where α is a constant.

Therefore by injecting a Gaussian white signal with mean level b into the non-linear system illustrated in FIG 1, and computing the first and second order cross-correlation functions as defined by eqns (9) and (16), the individual linear subsystem impulse responses $\mu_1 h_1(t)$ and $\mu_2 h_2(t)$ can be identified using eqns (15) and (23) where μ_1 and μ_2 are constants. An algorithm for estimating the parameters in the pulse transfer function representation of $h_1(t)$ and $h_2(t)$ is presented in Section 4.

If the input signal is white Gaussian with a non-zero mean level all the terms in eqn (16) except the first, contribute to $\phi_{xxz'}(\sigma,\sigma)$, and hence a result similar to eqn (23) can be obtained for all non-linearities which are continuous.

Inspection of eqn (19) shows that the mean level of the measured system output z(t) gives rise to an impulse when $\sigma_1 = \sigma_2 = \sigma$. Theoretically this would make the second order cross-correlation function infinite for $\sigma = \sigma_1 = \sigma_2$. However, by normalising the system output prior to computing the second order cross-correlation function, as in eqn (16), this problem can be avoided.

Notice that $\overline{w_1'(t)x(t-\sigma_1)x(t-\sigma_2)} = 0 \;\forall\; \sigma_1,\sigma_2$. This provides a convenient test for linearity, since if the system is linear (i.e. $\gamma_n = 0$, n = 2,3,4...) $\overline{z'(t)x(t-\sigma_1)x(t-\sigma_2)} = 0 \;\forall\; \sigma_1,\sigma_2$, and the identification problem is solved by computing just the first order cross-correlation function eqn (15).

4. PARAMETERISATION OF THE LINEAR SUBSYSTEMS

In order to identify the non-linear element in FIG 1 it is necessary to isolate $h_1(t)$ and $h_2(t)$ from the results of eqns (15) and (23). Since the identification will normally be performed with the aid of a digital computer, the first and second order cross-correlation functions, eqns (15) and (23), will be in sampled data form

$$\phi_{xz'}(i) = \beta \sum_{j=o}^{N} h_1(j)h_2(i-j) \qquad (24)$$

$$\phi_{xxz'}(i,i) = \alpha \sum_{j=o}^{N} h_1^{\,2}(j)h_2(i-j) \qquad (25)$$

By fitting a pulse transfer function model to the sampled cross-correlation functions each linear subsystem model can be isolated using a least squares algorithm[30].

Define the following pulse transfer functions

$$Z\{\beta h_1(t)*h_2(t)\} = \frac{B(z^{-1})}{A(z^{-1})} \tag{26}$$

$$Z\{\alpha h_1^2(t)*h_2(t)\} = \frac{F(z^{-1})}{E(z^{-1})} \tag{27}$$

$$Z\{\mu_1 h_1(t)\} = \frac{N_1(z^{-1})}{D_1(z^{-1})} \tag{28}$$

$$Z\{\mu_2 h_2(t)\} = \frac{N_2(z^{-1})}{D_2(z^{-1})} \tag{29}$$

$$Z\{\mu_3 h_1^2(t)\} = \frac{N_3(z^{-1})}{D_3(z^{-1})} \tag{30}$$

where * denotes convolution, z^{-1} is the backward shift operator and α,β and μ_i, $i = 1,2,3$ are constants.

The sampled first and second order cross-correlation functions $\phi_{xz'}(i)$, $\phi_{xxz'}(i,i)$ can be visualised as the response $r(i)$ to an impulse $u(o) = 1$, $u(j) = 0 \;\forall\; j \neq 0$. Thus for example for the first order cross-correlation function, from eqn (26)

$$r(i) = \frac{B(z^{-1})}{A(z^{-1})} u(i) \tag{31}$$

or expanding

$$r(i) = b_o u(i) + b_1 u(i-1) + b_2 u(i-2) \;\ldots\; + b_m u(i-m) - a_1 r(i-1)$$
$$- a_2 r(i-2) \;\ldots\; - a_m r(i-m) \tag{32}$$

where $u(o) = 1$, $u(i) = 0 \;\forall\; i \neq 0$. Substituting the values of $u(i)$ into eqn (32) gives the matrix equation

$$\begin{bmatrix} r(0) \\ r(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ r(N) \end{bmatrix} = \left[\begin{array}{cccc|ccc} 0 & 0 & \cdots\cdots & 0 & u(0) & 0 & 0 \\ -r(0) & 0 & \cdots\cdots & 0 & 0 & u(0) & 0 \\ -r(1) & -r(0) & \cdots\cdots & 0 & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ -r(N-1) & -r(N-2) & \cdots & -r(N-m) & 0 & \cdots\cdots & 0 \end{array}\right] \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \\ \hline b_o \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{m+1} \end{bmatrix}$$

$$R = Q\theta \tag{33}$$

Solving for θ using least squares[29] gives a consistent estimate of the unknown parameters

$$\hat{\theta} = (Q^TQ)^{-1}Q^TR \tag{34}$$

A similar approach can be used to estimate the parameters (e_i, f_i) in the pulse transfer function representation of the second order cross-correlation function.

Although the parameters in the pulse transfer function representations of the linear subsystems eqns (28) and (29) cannot be determined uniquely from measurements of the input-output sequences, they can be identified to within constant scale factors μ_1 and μ_2. From eqns (26) to (30)

$$A(z^{-1})D_3(z^{-1}) = E(z^{-1})D_1(z^{-1}) \tag{35}$$

Define $T(z^{-1}) = A(z^{-1})D_3(z^{-1}) = (1+a_1z^{-1}\ldots+a_mz^{-m})(1+d_{3,1}z^{-1}$

$$+\ldots d_{3,2\ell}z^{-2\ell}) = t_o+t_1z^{-1} \ldots +t_{2\ell+m}z^{-2\ell-m} \tag{36}$$

and

$$u(z^{-1}) = E(z^{-1})D_1(z^{-1}) = (1+e_1z^{-1}+\ldots+e_kz^{-k})(1+d_{1,1}z^{-1}$$

$$+ \ldots + d_{1,\ell}z^{-\ell}) = u_o+u_1z^{-1}\ldots+u_{\ell+k}z^{-\ell-k}. \tag{37}$$

where $\ell+m = k$, $t_o = 1$, $u_o = 1$, $t_i = q_i(a,d_3)$ and $u_i = w_i(e,d_1)$ for $i = 1,2...2\ell+m$. Replacing a,e by their estimates $\hat{a},\hat{e}$, and introducing error terms ρ_i gives

$$t_o = u_o$$
$$q_i(\hat{a},d_3) = w_i(\hat{e},d_1)+\rho_i(\delta_e,\delta_a) \qquad i = 1,2...2\ell+m \qquad (38)$$

Each ρ_i is a random variable which is a function of the estimation errors, $\delta_a = a-\hat{a}$, $\delta_e = e-\hat{e}$. Equation (38) provides a set of linear equations to which the least squares solution can be applied to estimate the unknown parameters $d_{3,i}$ and $d_{1,i}$.

By equating coefficients in the identity

$$B(z^{-1})N_3(z^{-1}) = F(z^{-1})N_1(z^{-1}) \qquad (39)$$

and following the above procedure least squares estimates for the unknown parameters $n_{3,1},n_{3,2}\cdots n_{3,2\ell}$; $n_{1,1},n_{1,2}\cdots n_{1,\ell}$ can be obtained. Similarly, estimates of the parameters $n_{2,1},n_{2,2}\cdots n_{2,q};d_{2,1},d_{2,2}\cdots d_{2,q}$ can be computed using the identities

$$A(z^{-1}) = D_1(z^{-1})D_2(z^{-1}) \qquad (40)$$
$$B(z^{-1}) = N_1(z^{-1})N_2(z^{-1}) \qquad (41)$$

5. IDENTIFICATION OF THE NON-LINEAR ELEMENT

Consider the schematic diagram of the identification procedure illustrated in FIG 2.

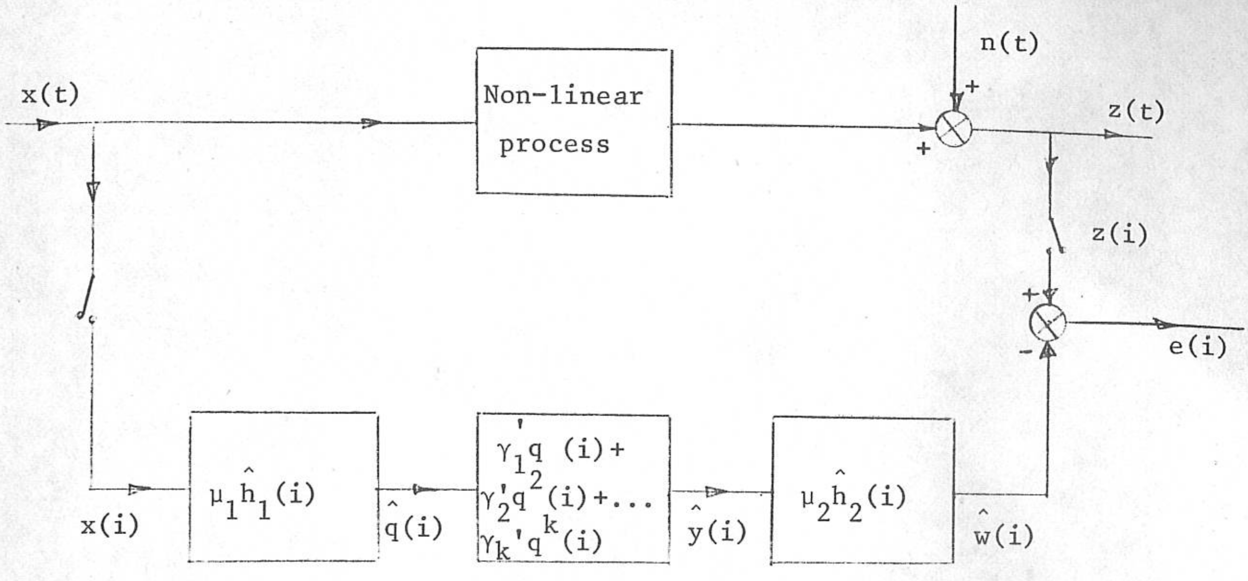

FIG 2. A schematic diagram of the identification procedure

The error e(i) between the sampled process output z(i) and the output of the model $\hat{w}(i)$ can be defined as

$$e(i) = z(i) - \hat{w}(i) \tag{42}$$

where
$$\hat{w}(i) = \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{y}(i-j)$$

$$= \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\{\gamma_1'\hat{q}(i-j) + \gamma_2'\hat{q}^2(i-j) + \ldots$$

$$\ldots + \gamma_k'\hat{q}^k(i-j)\} \tag{43}$$

$$\hat{q}(i) = \hat{n}_{1,1}x(\ell-1) + \hat{n}_{1,2}x(\ell-2)\ldots + \hat{n}_{1,\ell}x(i-\ell) - \hat{d}_{1,1}\hat{q}(i-1)$$
$$\ldots - \hat{d}_{1,\ell}\hat{q}(i-\ell) \tag{44}$$

and $\gamma_t = \mu_1^t\mu_2\gamma_t'$, $t = 1,2\ldots k$. Substituting eqn (43) into eqn (42) and considering N measurements of the sampled process input and output gives the matrix equation

$$\begin{bmatrix} Z(1) \\ Z(2) \\ \vdots \\ Z(N) \end{bmatrix} = \begin{bmatrix} \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}(1-j), & \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}^2(1-j) & \ldots & \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}^k(1-j) \\ \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}(2-j), & \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}^2(2-j) & \ldots & \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}^k(2-j) \\ \vdots & & & \vdots \\ \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}(N-j), & \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}^2(N-j) & \ldots & \mu_2 \sum_{j=0}^{\ell} \hat{h}_2(j)\hat{q}^k(N-j) \end{bmatrix} \begin{bmatrix} \gamma_1' \\ \gamma_2' \\ \vdots \\ \gamma_k' \end{bmatrix} + \begin{bmatrix} e(1) \\ e(2) \\ \vdots \\ e(N) \end{bmatrix}$$

$$Z = \phi\theta + E \qquad (45)$$

Since all the elements of the Z and ϕ matrices are either measured or estimated, a least squares estimate of the coefficients $\gamma_1', \gamma_2' \ldots \gamma_k'$ in the polynomial series representation of the non-linear element can be readily computed

$$\hat{\theta} = (\phi^T\phi)^{-1}\phi^T Z \qquad (46)$$

and the identification is complete.

Since the identification of the non-linear element is decoupled from the identification of the linear subsystems it may be possible to fit a series of straight line segments, rather than a polynomial to the non-linear characteristic. This would be particularly advantageous when the non-linearity consists of a deadband or saturation.

6. THE HAMMERSTEIN AND WIENER MODELS

The class of non-linear systems illustrated in FIG 1 is often referred to as the "general model". Special cases of this model, known as the Wiener and Hammerstein models are illustrated in FIG 4(a) and 4(b) respectively.

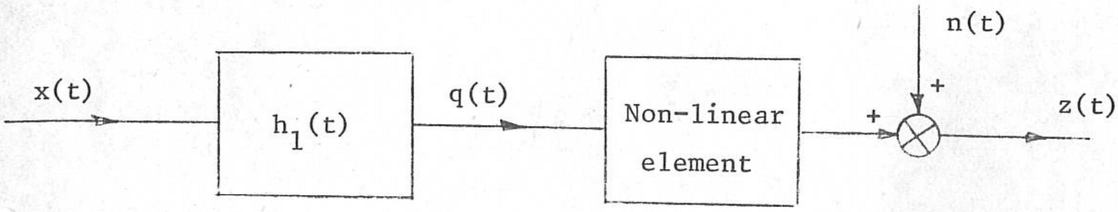

FIG 4(a) THE WIENER MODEL

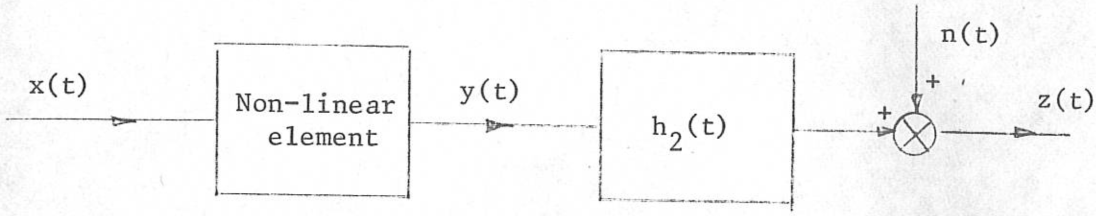

FIG 4(b) THE HAMMERSTEIN MODEL

The Wiener model, FIG 4(a), consists of a linear system followed by a non-linear no-memory element. The Volterra series and the models of Cameron and Martin[31], Wiener, Bose and others belong to this class. By setting $h_2(t) = \delta(t)$ in FIG 1, the general model reduces to the Wiener model, and the identification procedure outlined in previous sections is simplified considerably.

If the input signal has the properties of a Gaussian white process with mean level b, setting $h_2(t) = \delta(t)$ in eqn (15), the first order cross-correlation function becomes directly proportional to the impulse response of the linear element

$$\phi_{xz'}(\sigma) = \beta h_1(\sigma) \qquad (47)$$

Similarly, the complexity of the least squares algorithm for estimating the coefficients in the polynomial series representation of the non-linear element eqn (45), is reduced for the Wiener model.

The Hammerstein model, FIG 4(b), which consists of a non-linear no-memory element followed by a linear system has been investigated by several authors[32,33,34,35]. The linear and non-linear elements in the Hammerstein model can be readily identified by setting $h_1(t) = \delta(t)$ in the results derived for the general model.

7. EXTENSION TO COLOURED NOISE INPUTS

The identification procedure outlined in previous sections can be applied with only slight modification even if the input is non-white, providing that it has a non-zero mean level.

Consider the system illustrated in FIG 5, where the non-white input $x(t)+b$ is regarded as the output of a shaping filter $k_2(t)$ which is driven by a white Guassian process $\zeta(t)+b/g_k$ where g_k is the shaping filter gain.

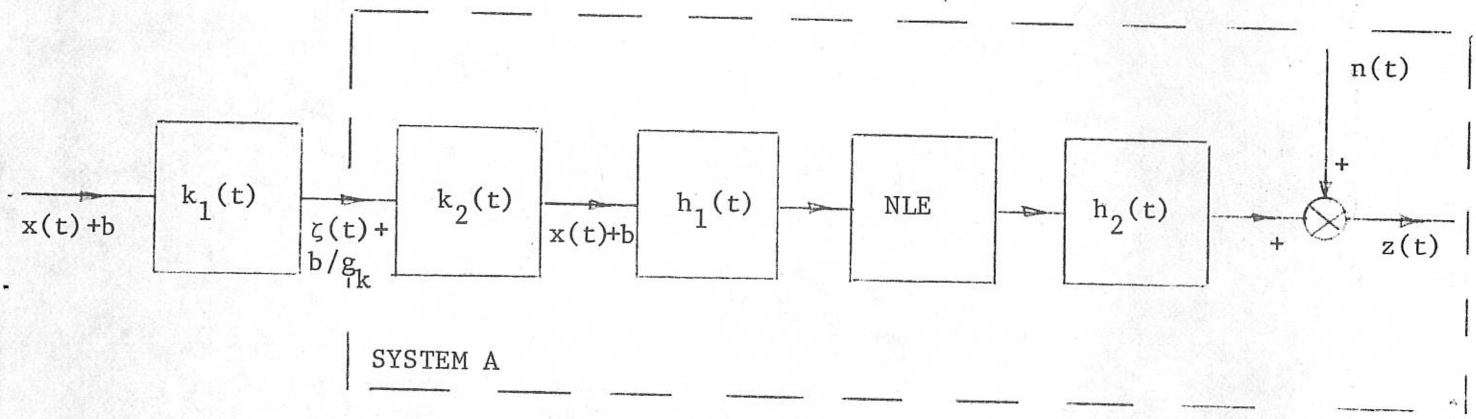

FIG 5. AUGMENTED NON-LINEAR SYSTEM

Providing the power spectral density of $x(t)$ is factorable[1], it can be expressed as

$$S_{xx}(\omega) = S^+_{xx}(\omega)S^-_{xx}(\omega) \qquad (48)$$

where $S^+_{xx}(\omega)$ is the complex conjugate of $S^-_{xx}(\omega)$, and all the poles and zeros of $S^+_{xx}(\omega)$ are in the left half of the complex s-plane. Thus $S^+_{xx}(\omega)$ and $1/S^+_{xx}(\omega)$ are each realizable as the transfer function of a linear system where

$$K_1(\omega) = 1/S^+_{xx}(\omega) \tag{49}$$

$$K_2(\omega) = S^+_{xx}(\omega) \tag{50}$$

Since the input to the system enclosed in dashed lines in FIG 5 is a white Gaussian process with mean level b/g_k the first and second order correlation functions between $\zeta(t)$ and the normalised output $z'(t)$ can be expressed as

$$\phi_{\zeta z'}(\sigma) = \beta\{k_2(t)*h_1(t)\}*h_2(t) \tag{51}$$

$$\phi_{\zeta\zeta z'}(\sigma,\sigma) = \alpha\{k_2(t)*h_1(t)\}^2*h_2(t) \tag{52}$$

where

$$\phi_{xz'}(\sigma) = \int_{-\infty}^{\infty} k_2(\tau)\phi_{\zeta z'}(\tau+\sigma)d\tau \tag{53}$$

$$\phi_{xxz'}(\sigma,\sigma) = \iint_{-\infty}^{\infty} k_2(\tau_1)k_2(\tau_2)\phi_{\zeta\zeta z'}(\tau_1+\sigma,\tau_2+\sigma)d\tau_1 d\tau_2 \tag{54}$$

Considering sampled values and taking the Z-transform of eqns (48) and (49)

$$\phi_{\zeta z'}(k) = \frac{1}{FF(z^{-1})} \cdot \frac{B(z^{-1})}{A(z^{-1})} u(k) \tag{55}$$

$$\phi_{\zeta\zeta z'}(k) = \frac{1}{FF'(z^{-1})} \cdot \frac{F(z^{-1})}{E(z^{-1})} u(k) \tag{56}$$

where $u(0) = 1$, and $u(k) = 0 \; \forall \; k \neq 0$. Estimates of the parameters in the autoregressive model representation of the shaping filter $Z\{k_2(t)\} = 1/FF(z^{-1})$ can be obtained from measurements of $x(k)$ using a least squares algorithm.

Thus by fitting an autoregression $FF(z^{-1})$ to $x(k)$ an estimate of the white noise sequence $\zeta(k)$ can be obtained

$$\hat{\zeta}(k) = FF(z^{-1})x(k) \tag{57}$$

and $\phi_{\zeta z'}(k)$, $\phi_{\zeta\zeta z'}(k)$ can be computed. Since $FF(z^{-1})$ is known, the first order correlation function eqn (52) can be filtered to give

$$FF(z^{-1})\phi_{\zeta z'}(k) = \frac{B(z^{-1})}{A(z^{-1})}\,u(k) \tag{58}$$

and $FF'(z^{-1})$ can be absorbed into $D_3(z^{-1})$

$$\phi_{\zeta\zeta z'}(k,k) = \frac{N_2(z^{-1})}{D_2(z^{-1})}\cdot\frac{N_3(z^{-1})}{D_3'(z^{-1})} \tag{59}$$

Estimates of the parameters in the pulse transfer functions $B(z^{-1})/A(z^{-1})$ and $F(z^{-1})/E'(z^{-1})$ can now be obtained using the procedure outlined in section 4.

8. CONCLUSIONS

A procedure for the identification of a class of non-linear systems has been presented. If the non-linear system has the structure of the general model where the linear and no-memory non-linear elements are separable the form of the kernels in the functional series expansion are known a priori and the identification of the linear and non-linear elements can be decoupled. For a Gaussian white input signal with a non-zero mean level, estimates of the impulse response functions of the linear elements can be obtained by computing the first and second order correlation functions. Once the linear elements have been identified, estimates of the coefficients in the polynomial series representation of the non-linear element can be computed. Identification of the linear subsystems is therefore independent of the non-linear characteristic and consequently the computational burden often associated with the identification of this class of non-linear systems is reduced.

The only necessary data for the characterization of this class of non-linear systems is a record of the white Gaussian input and the system output. Tests involving multiple amplitude input signals and hence excessive experimentation times are avoided.

Because the linear, Hammerstein and Wiener models are all sub-classes of the general model the identification procedure is applicable to systems with these structures. A comparison with the algorithms derived by previous authors for this class of systems demonstrates the simplicity of the present method and emphasises the advantages of decoupling the identification of the linear and non-linear elements.

The results of preliminary investigations using simulated systems confirm the validity of the algorithm and the identification of a non-linear plant is in progress. It is hoped that these results and an investigation into the extension of the method to include other common system structures will be published at a later date.

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