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A CANONICAL FORM FOR THE REDUCTION OF LINEAR SCALAR SYSTEMS

by

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Abstract

Consideration is given to the problem of the reduction of order of a scalar system S(A,B,C) described by a transfer function g(s). On the assumption that the reduced order model is to be used for feedback control systems design, a canonical form is derived equivalent to a system decomposition related to the asymptotes, intercepts and finite zeros of the system root-locus. A model reduction procedure, based on the canonical form, is suggested and shown to be capable of providing a good approximation to both the dominant pole and dominant zeros of g(s) and to make possible the matching of the desired number of high and low frequency moments. The canonical form can also be used to provide an estimate of a suitable reduced model order. Two examples are described.

Key to Diagrams

Fig. 1: = decomposition of $g(s)$ into a forward path and a feedback loop.

Fig. 2: = decomposition of $g(s)$ into a sequence of nested feedback loops.

Fig. 3: = simulated outputs of feedback loops in example 1.

.... $\hat{y}_1(t)$

.... $\hat{y}_2(t)$

.... $\hat{y}_3(t)$

.... $\hat{y}_4(t)$

Fig. 4: = unit step responses for example 1.

.... (a) step response of $g(s)$

.... (b) step response of $\hat{g}(s)$

Fig. 5: = unit step responses for example 2.

.... (a) step response of $g(s)$

.... (b) step response of $\hat{g}_1(s)$

.... (c) step response of $\hat{g}_2(s)$

Fig. 6 = Unit step responses for example 2.

.... (a) step response of $g(s)$

.... (b) step response of $\hat{g}_3(s)$

.... (c) step response of $\hat{g}_4(s)$

1. Introduction

The problem of approximation of the input-output dynamics of a single-input-single-output linear, time-invariant dynamical system described by the state space model S(A,B,C)

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t) \in R^n$$
$$y(t) = Cx(t) \tag{1}$$

by a reduced order model $S(A_R,B_RC_R)$ of state dimension $n_R << n$ has been the subject of considerable study. Most analytical techniques for systematic model reduction are initiated by expressing the system state-space model or transfer function in some defined canonical form and subsequently applying truncation and/or approximation methods to produce a reduced order model prossessing a combination of the following properties:

(i) Moment matching about specified points in the frequency domain.[1-10]

(ii) Dominant pole retention.[10-13]

(iii) Matching of time domain dynamics.[14,15]

(iv) Retention of overall open-loop stability characteristics.[6-9,12,13]

The choice of reduced model order is, in general, guided by heuristic rules or trial and error.

In the opinion of the authors, it is important that any reduction technique be based on a systematic analysis of the geometric and algebraic properties of the system state-space model and transfer function guided by intuitive rules relating system structure to the input-output response. If the reduced order model is also to be used for feedback controller design studies, it must retain the important closed-loop characteristics of the original system. Some of the most important feedback characteristics of a system are the asymptotes and intercepts of the system root-locus and the overall structure of the dominant zeros of the system transfer function. In this sense it is anticipated that the most suitable canonical form for S(A,B,C) should be obtained by an investigation of those geometric and algebraic properties governing the behaviour of these features of the root-locus diagram.

In section 2 a suitable canonical form for S(A,B,C) is suggested, based, in its simplest form, on a decomposition of the system transfer function and subsequently converted to a state-space canonical form analogous to the Schwarz canonical form(17) for (A,B) extended to matrix triples (A,B,C). An important feature of the result is that the behaviour of the unbounded roots and finite zeros of the root-locus are described by separate well-defined subsystems within the system structure. In section 3 the application of the decomposition to model reduction is described. The approach provides a technique for obtaining a reduced order model satisfying (i) - (iv) above and has the advantage that simulation of the open-loop system S(A,B,C) in canonical form provides an estimate of a suitable reduced model order. In section 4, the results are illustrated by application to two examples. In particular, consideration is given to an oscillatory, non-minimum phase 10^{th} order example illustrating the technique for selecting a suitable reduced model order and the freedom available for matching high and low frequency behaviour by the application of previously derived reduction methods to subsystems. The degrees of freedom available offer enough flexibility to cover a wide range of applications.

2. Canonical Decomposition of S(A,B,C)

Consider the system model defined by equation (1) with transfer function

$$g(s) = C(sI_n - A)^{-1} B \equiv \frac{\beta_1 \prod_{j=1}^{n_z} (s-z_j)}{\prod_{j=1}^{n} (s-p_j)} \quad , \beta_1 \neq 0 \qquad (2)$$

where p_j, $1 \leq j \leq n$, are the system poles, z_j, $1 \leq j \leq n_z$, are the system zeros, $n_z < n$, and β_1 is the overall system gain. It is easily seen that the inverse transfer function has a unique decomposition

$$g^{-1}(s) = g_1^{-1}(s) - \frac{1}{\beta_1} h_1(s) \tag{3}$$

where $h_1(s)$ is a strictly proper transfer function of order n_z possessing poles $\{z_j\}$, and $g_1(s)$ is a transfer function of order $k_1 = n - n_z$ possessing no zeros i.e.

$$g_1(s) = \frac{\beta_1}{s^{k_1} + \alpha_{11} s^{k_1 - 1} + \ldots\ldots + \alpha_{1 k_1}} \tag{4}$$

Writing equation (3) in the equivalent form,

$$g(s) = \frac{g_1(s)}{1 - g_1(s) \frac{1}{\beta_1} h_1(s)} \tag{5}$$

it is easily seen (Fig. 1) that $g_1(s)$ can be interpreted as a forward path element and $h_1(s)$ as a feedback element describing a inherent state feedback within the system structure[17], hence providing some physical justification for the decomposition. An important observation in the following sections is that the parameters k_1, β_1, α_{11} in $g_1(s)$ are sufficient to define the asymptotes and intercepts of the system root-locus plot, suggesting that any model reduction procedure should retain $g_1(s)$ exactly. In practice this can easily be achieved by applying the model reduction procedure directly to $h_1(s)$ to generate a strictly proper reduced order model $h_{1A}(s)$. A suitable reduced order model for $g(s)$ retaining the structure of $g_1(s)$ is given by (c.f. equation (5))

$$g_A(s) = \frac{g_1(s)}{1 - g_1(s) \frac{1}{\beta_1} h_{1A}(s)} \tag{6}$$

If h_{1A} is proper but not strictly proper then the coefficient α_{1k_1}

becomes $\alpha_{1k_1} - \beta_1^{-1} h_{1A}(\infty)$. The approximation of $h_1(s)$ can be regarded as the approximation of the dominant zero structure of the system implying that some care must be taken if the essential features of the root-locus are to be retained.

The decomposition defined by equation (3) can be systematically extended using the polynomial division algorithm,

$$h_o(s) \triangleq g(s)$$
$$h_{j-1}^{-1}(s) \triangleq g_j^{-1}(s) - \frac{1}{\beta_j} h_j(s) \quad , \quad j \geq 1 \tag{7}$$

defined whenever $h_{j-1}(s) \not\equiv 0$. The decomposition is unique if $h_j(s)$, $j \geq o$, is defined to be strictly proper and $g_j(s)$, $j \geq 1$, possesses no zeros and is of the form,

$$g_j(s) = \frac{\beta_j}{s^{k_j} + \alpha_{j1} s^{k_j - 1} + \ldots. + \alpha_{jk_j}} \tag{8}$$

The system transfer function takes the canonical form,

$$g(s) = \{g_1^{-1}(s) - \frac{1}{\beta_1}\{g_2^{-1}(s) - \frac{1}{\beta 2}\{\ldots. - \frac{1}{\beta_{j-1}}\{g_j^{-1}(s) - \frac{1}{\beta_j} h_j(s)\}^{-1}..\}^{-1}\}^{-1} \tag{9}$$

and can be represented as a sequence of nested feedback loops as shown in Fig. 2. The algorithm terminates at the smallest integer $\ell \geq 1$ such that $h_\ell(s) \equiv 0$. If the system $S(A,B,C)$ is controllable and observable then

$$\bar{n} \triangleq \sum_{j=1}^{\ell} k_j = n \tag{10}$$

If, however, the system is either uncontrollable or unobservable then $\bar{n}$ is equal to the dimension of a minimal realization of $g(s)$ i.e. the polynomial division algorithm factors out the uncontrollable and unobservable modes of the system.

The algebraic canonical form for g(s) defined above has an immediate interpretation in terms of a canonical form for the matrix triple (A,B,C). Using the notation of Fig. 2 with $\beta_{-1} = \beta_o = 1$, $y_o(t) \equiv u(t)$ and $y_{\ell+1}(t) \equiv 0$, the (loop output) $y_j(t)$ is generated by $\beta_{j-2}y_{j-1}(t) + y_{j+1}(t)$ through the dynamics $g_j(s)$ i.e., $1 \leq j \leq \ell$,

$$\dot{x}_j(t) = A_j x_j(t) + \beta_j e_1 (\beta_{j-2}y_{j-1}(t) + y_{j+1}(t))$$

$$y_j(t) = \frac{1}{\beta_{j-1}} e_{k_j}^T x_j(t) \qquad , x_j(t) \in R^{k_j} \tag{11}$$

where e_r is a unit vector of approppriate dimension with zero elements everywhere apart from a unit element in the r^{th} position, and A_j is a companion matrix of dimension $k_j \times k_j$, of the form,

$$A_j = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -\alpha_{jk_j} \\ 1 & 0 & & & & & & & \cdot \\ 0 & 1 & & & & & & & \cdot \\ \cdot & & \cdot & & & & & & \cdot \\ \cdot & & & \cdot & & & & & \cdot \\ \cdot & & & & \cdot & & & & \\ \cdot & & & & & \cdot & & 0 & -\alpha_{j2} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -\alpha_{j1} \end{bmatrix} \tag{12}$$

Combining equation (11) to form a composite state space model with state $\hat{x}(t)^T = \{x_1(t)^T, \ldots, x_\ell(t)^T\}^T$ yields the following result.

<u>Theorem 1</u>

A controllable and observable linear, time-invariant single-input-single-output system S(A,B,C) in R^n is similarly equivalent to the system $S(\hat{A}, \hat{B}, \hat{C})$ of the form

$$\hat{A} = \begin{bmatrix} A_1 & E_1 & 0 & \cdots & \cdots & 0 \\ B_2 & A_2 & E_2 & 0 & & \vdots \\ 0 & B_3 & A_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & E_{\ell-1} \\ 0 & \cdots & \cdots & \cdots & B_\ell & A_\ell \end{bmatrix}$$

$$\hat{C} = e_{k_1}^T \quad , \quad \hat{B} = \beta_1 e_1 \tag{13}$$

where $$B_j = \beta_j e_1 e_{k_{j-1}}^T \quad , \quad E_j = e_1 e_{k_{j+1}}^T \tag{14}$$

The following propositions follow from inspection of $\hat{A},\hat{B},\hat{C}$:

Proposition 1

The system $S(A,B,C)$ is represented by the triple $(\hat{A},\hat{B},\hat{C})$ with respect to a basis for R_n of the form,

$$\{d_1 , Ad_1 , \ldots , A^{k_1-1} d_1, d_2, Ad_2, \ldots , A^{k_\ell-1} d_\ell\} \tag{15}$$

where $d_1 \triangleq \beta_1^{-1} B$, $i > 1$,

$$A^{k_i} d_i - d_{i-1} - \beta_j d_{i+1} \in \text{span} \{d_i, Ad_i, \ldots, A^{k_i-1} d_i\} \tag{16}$$

Proposition 2

If a system $S(\hat{A},\hat{B},\hat{C})$ taking the form of equation (13) is such that $\beta_{j+1} = 0$ for same $1 \leqslant j \leqslant \ell-1$, then it is uncontrollable and input-output equivalent to a system $S(\overline{A},\overline{B},\overline{C})$ of dimensions $\tilde{n} = \sum_{i=1}^{j} k_i$ defined by

$$\overline{A} = \begin{bmatrix} A_1 & E_1 & 0 & \cdots & & & 0 \\ B_2 & A_2 & E_2 & & & & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & E_{j-1} \\ 0 & \cdots & & & & B_j & A_j \end{bmatrix}$$

$$\overline{B} = \beta_1 e_1 \qquad , \qquad \overline{C} = e_{k_1}^T \tag{17}$$

From proposition 2 it follows that, if $\beta_{j+1} = 0$, the poles of $S(\overline{A},\overline{B},\overline{C})$ are a proper subset of the poles of $S(\hat{A},\hat{B},\hat{C})$ and hence, if β_{j+1} is 'small', the poles of $S(\overline{A},\overline{B},\overline{C})$ are a close approximation to a proper subset of the poles of $S(A,B,C)$. The zeros of $S(A,B,C)$ are characterized by the maximal $\{A,B\}$ invariant subspace in the kernel of C and hence are defined by the eigenvalues of

$$\begin{bmatrix} A_2 & E_2 & 0 & \cdots & & & & 0 \\ B_3 & A_3 & E_3 & & & & & \vdots \\ 0 & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & & 0 \\ \vdots & & & & & \ddots & \ddots & E_{\ell-1} \\ 0 & \cdots & & & & 0 & B_\ell & A_\ell \end{bmatrix} \tag{18}$$

The zeros of $S(\overline{A},\overline{B},\overline{C})$ are, in a similar manner defined by the eigenvalues of

$$\begin{bmatrix} A_2 & E_2 & 0 & \cdots & & & & 0 \\ B_3 & A_3 & E_3 & & & & & \vdots \\ 0 & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & 0 \\ \vdots & & & & \ddots & \ddots & & E_{j-1} \\ 0 & \cdots & & & & 0 & B_j & A_j \end{bmatrix} \tag{19}$$

and hence, (a) if $\beta_{j+1} = 0$, the zeros of $S(\overline{A},\overline{B},\overline{C})$ are a proper subset of the zeros of $S(A,B,C)$ and (b) if β_{j+1} is 'small', the zeros of $S(\overline{A},\overline{B},\overline{C})$ are a good approximation to a proper subset of the zeros of $S(A,B,C)$.

The canonical form defined by equations (11)-(13) will be used primarily as a representation for simulation purposes. The above observations suggest that, if β_{j+1} is small enough, the 'reduced system' $S(\overline{A},\overline{B},\overline{C})$ can provide a good approximation to the dominant pole and zero structure of $S(A,B,C)$. This idea is pursued in the next section.

Finally, it is noted for simulation purposes that if $S(A,B,C)$ is subject to unity proportional negative output feedback $u(t) = k\{r(t) - y(t)\} = k\{r(t) - \hat{C}\,\hat{x}\,(t)\}$, the structure of the state-space canonical form remains the same for the canonical closed-loop system $S(\hat{A} - k\,\hat{B}\,\hat{C}\;,\; k\,\hat{B},\;\hat{C})$ except that α_{1k_k} is replaced by $\alpha_{1k_1} + k\,\beta_1$ and hence the moment-matching properties of the model are preserved under constant feedback, a particular case of a more general result recently published by Shamash[18].

3. Model Reduction

In the feedback decomposition of $g(s)$ illustrated in Fig. 2, the lower loops have a decreasing significance in the evaluation of the high frequency input-output behaviour of the system, suggestions that a useful reduced model might be obtained by using a lower order approximation of one of the lower loops $h_j(s)$, say. This general procedure corresponds to approximation of the blocks $A_{j+1}, \ldots, A_\ell, E_j, \ldots E_{\ell-1}, B_{j+1}, \ldots, B_\ell$ in equation (13). In the simplest case, if β_{j+1} is small, the discussion of the previous section indicates that a good representation of the dominant pole-zero structure can be obtained by setting $\beta_{j+1} = 0$ and constructing the reduced model $S(\overline{A},\overline{B},\overline{C})$ of equation (17).

The general approach to model reduction suggested can be summarized as follows:

(a) Compute the algebraic canonical form for g(s) by the algorithm defined by equation (7) and store $g_1(s),\ldots, g_\ell(s)$ $h_1(s),\ldots, h_{\ell-1}(s)$.

(b) Retain $g_1(s)$ exactly as β_1, α_{11}, k_1 characterize the asymptotes and intercepts of the system root-locus under dynamic unity negative output feedback control.

(c) By analysis of $h_i(s)$, $1 \leq i \leq \ell-1$, choose an integer $j \geq 1$, and hence a suitable subsystem, $h_j(s)$, for reduction.

(d) Construction of a p^{th} order reduced model of $h_{j,A}(s)$, enables the construction of a reduced model for g(s) of order $\sum_{i=1} k_i + p$, namely,

$$g_A(s) = \{g_1^{-1}(s)\ \frac{1}{\beta_1}\{g_2^{-1}(s) - \ldots \frac{1}{\beta_{j-1}}\{g_j^{-1}(s) - \frac{1}{\beta_j} h_{j,A}(s)\}^{-1}\ldots\}^{-1}\}^{-1} \tag{20}$$

The reduced model $g_A(s)$ retains the asymptotes and intercepts of the root-locus of g(s) and can represent a good approximation to the dominant pole-zero structure of g(s). It also has other important properties.

3.1 Moment Matching Properties

Consider the feedback system of Fig. 1 with $g_1(s)$ of rank k_1 and $h_1(s)$ strictly proper and let $h_{1A}(s)$ be a strictly proper reduced model of $h_1(s)$ generating a reduced model $g_A(s)$ for g(s) defined by equation (6). It is easily verified that

$$g(s) = g_A(s) = g(s)\ \beta_1^{-1}\ \{h_1(s) - h_{1A}(s)\}\ g_A(s) \tag{21}$$

and that g(s) and $g_A(s)$ have the same rank. Suppose that $h_1(s)$ has the series expansion about the point at infinity

$$h_1(s) = \sum_{i=1}^{\infty} s^{-i} M_i \tag{22}$$

where M_i, $i \geq 1$, are the system Markov parameters and a series expansion about the point s = o

$$h_1(s) = \sum_{i=o}^{\infty} s^i N_i \tag{23}$$

If $h_{1A}(s)$ matches the first m_o Markov parameters of $h_1(s)$ and the time moments N_i, $o \leq i \leq n_o - 1$ of $h_1(s)$, then it follows directly from equation (21) that $g_A(s)$ matches the first $2k_1 + m_o$ Markov parameters of $g(s)$ and the first n_o time moments of $g(s)$.

The general case of approximation of $h_j(s)$ by the reduced model $h_{j,A}(s)$ is easily examined by application of the above techniques to the lowest loop in Fig. 2 and applying induction. The general result is as follows:

Theorem 2

If $h_{j,A}(s)$ is a strictly proper reduced model of $h_j(s)$ matching the first m_o Markov parameters and n_o time moments of $h_j(s)$, then the reduced model $g_A(s)$ defined by equation (20) matches the first $m_o + 2 \sum_{i=1}^{j} k_i$ Markov parameters and n_o time moments of $g(s)$.

This result is of great generality and does not presuppose the type of model reduction procedure applied to $h_j(s)$. The authors feel that a steady-state approximation to $h_j(s)$, for a suitable choice of j, will often give a satisfactory reduced model. In general, however, the canonical decomposition of $g(s)$ will facilitate a preliminary matching of high frequency behavior to be followed by the analysis and reduction of a lower-order system, $h_j(s)$.

3.2 Estimation of Reduced Model Order

A problem frequently encountered in the application of model reduction procedures is that of choosing a suitable order for the model i.e. one for which the reduction technique in use will generate a model that preserves the stability characteristics of the original system. In general, trial and error methods are employed. The reduction techniques suggested above, while not yielding any exact characterisations of the model's stability properties, can be used to give very effective guidelines as to which orders will produce satisfactory models.

As shown in section 2, if any one of the parameters, β_{j+1}, in the systems canonical realisation, (13) is very small, then the states corresponding to blocks $A_{j+1}, A_{j+2}, \ldots A_{\ell}$ will be 'almost uncontrollable'. This means that the feedback loops $j+1, j+2, \ldots, \ell$ in the feedback decomposition of Fig. 2, have only small effect on the input-output behaviour of the system. Consequently, an examination of the relative magnitudes of the β_j, $1 \leq j \leq \ell$, in the manner of Arumugam and Ramamoorty[16] may indicate a choice of j to give a reduced model and hence a suitable model order. A better indication is given, however, by considering a simulated step response (say) of the systems in the canonical form, (13). The input to the j^{th} loop is just $\beta_{j-2}, y_{j-1}(t) + y_{j+1}(t)$. If $|y_{j+1}(t)|$ is significantly smaller than $|\beta_{j-2} \cdot y_{j-1}(t)|$ at all times, t, of interest, then a good approximation to the system dynamic behaviour should be anticipated if the signal $y_{j+1}(t)$ is neglected. Equivalently, a good reduced model can be obtained by approximating $h_j(s)$ by some low-order transfer function. Furthermore, close inspection of $y_{j+1}(t)$ and $\beta_{j-2} \cdot y_{j-1}(t)$ will indicate how the reduced model performance will match that of the original system over different intervals of time. If $|y_{j+1}(t)| << | \beta_{j-2} y_{j-1}(t)|$ for all times of interest, then the model obtained by simply neglecting $h_j(s)$ (termed the 'truncated' model) or by replacing $h_j(s)$ by its steady-state value, $h_{j+1}(o)$ (the 'steady-state corrected' model) should be highly accurate. Both cases suggest that a suitable reduced model order is simply $\sum_{i=1}^{j} k_i$. The steady-state corrected model will however only match $2 \sum_{i=1}^{j} k_i - 1$ Markov parameters and is essentially a generalisation to matrix triples (A,B,C) of the model derived by Arumugam and Ramamoorty[16] for matrix pairs (A,B).

It is convenient to define the following loop output variables,

$$\hat{y}_j(t) = \frac{y_j(t)}{\beta_{j-3} \cdot \beta_{j-5} \cdots \beta_1 \, \beta_{-1}} \quad \text{if } j \text{ is even}, \; j > 2$$

$$y_1(t) = y(t)$$

$$\hat{y}_j(t) = \frac{y_j(t)}{\beta_{j-3} \cdot \beta_{j-5} \cdots \beta_2 \, \beta_o} \quad \text{if } j \text{ is odd}, \; j > 3$$

Then the identity $\left| \frac{\beta_{j-2}(t) \, . \, y_{j-1}(t)}{y_{j+1}(t)} \right| = \left| \frac{\hat{y}_{j-1}(t)}{y_{j+1}(t)} \right|$, $2 \leqslant j \leqslant \ell-1$, indicates that the comparison of $\left| \beta_{j-2} \; y_{j-1}(t) \right|$ and $\left| \hat{y}_{j+1}(t) \right|$ can be replaced by a comparison of $\left| \hat{y}_{j-1}(t) \right|$ and $\left| \hat{y}_{j+1}(t) \right|$.

4. Examples

The two examples presented below illustrate how the above methods can be applied to obtain reduced models. The first example is of a very simple system with approximate pole-zero cancellation. The second is of a more complex system whose oscillatory nature makes it more difficult to model successfully.

(1) Consider a system described by the transfer function

$$g(s) = \frac{6.0 \; (s-2)(s+3)(s+3.9)(s+6)}{(s+2+2j)(s+2-2j)(s+4)(s+5)(s+8)}$$

Note that the zero at $s = -3.9$ and the pole at $s = -4.0$ 'approximately' cancel. With the notation of preceding sections, the parameters in the canonical decomposition of $g(s)$ are:

$\beta_1 = 6.0$; $g_1(s) = 6.0 \; (s + 10.1)^{-1}$

$\beta_2 = 30.61$; $g_2(s) = - \; 30.6 \; (s-2.96)^{-1}$

$\beta_3 = -6.92$; $g_3(s) = -6.92 \; (s + 5.52)^{-1}$

$\beta_4 = 2.05$; $g_4(s) = 2.05 \; (s + 4.41)^{-1}$

$\beta_5 = -.094$; $g_5(s) = -0.94 \; (s + 3.93)^{-1}$

As expected, β_5 is significantly smaller than the other β_i's, reflecting the approximate pole-zero cancellation. The simulated outputs $\hat{y}_1(t)$, $1 \leqslant i \leqslant 4$, of the feedback loops following a unit step input to the system are illustrated in Fig. 3 ($\hat{y}_5(t)$ is not shown, as $|\hat{y}_5(t)| < .005$ for all t). The magnitude of $\hat{y}_2(t)$ is more than 10 times as great as that of $\hat{y}_4(t)$, suggesting that a suitable reduced order model of state dimension 3 should be obtainable. An adequate model, $\hat{g}(s)$, is given by approximating $h_g(s)$ by its steady-state value, where

$$\hat{g}(s) = \frac{6.0\ (s - 1.98)\ (s + 4.07)}{(s + 2.2 + 2.14j)(s + 2.2 - 2.14j)(s + 7.8)}$$

The step responses of g(s) and $\hat{g}(s)$ are compared in Fig. 4. It is easily checked that (i) the dominant pole-zero structures of g(s) and $\hat{g}(s)$ are almost identical

(ii) the asymptotes and intercepts of the root-loci of g(s) and $\hat{g}(s)$ are identical and

(iii) $\hat{g}(s)$ matches one time moment and 5 Markov parameters of g(s). Overall the reduction is highly successful.

(2) Consider now a system described by the transfer function,

$$g(s) = \frac{(s-1)(s+4)(3+7.5)(s+12)(s+15)(s+25)}{(s+1+2j)(s+1-2j)(s+2)(s+7)(s+10)(s+10)(s+12.5)(s+18)(s+20)(s+30)}$$

While there is again a degree of pole-zero cancellation, the system is very oscillatory due to the pair of complex poles near the imaginary axis, and again has a right-half-plane zero, at s = 1. The parameters of the decomposition are:

$\beta_1 = 1.0$, $g_1(s) = 1.0\ (s^4+49s^3+720.5s^2+4055s+5451)^{-1}$

$\beta_2 = -109010$; $g_2(s) = -109010\ (s+20.79)^{-1}$

$\beta_3 = 98.49$; $g_3(s) = 98.49\ (s+3.32)^{-1}$

$\beta_4 = -13.86$; $g_4(s) = -13.86\ (s+18.97)^{-1}$

$\beta_5 = -52.81$; $g_5(s) = -52.81\ (s+.107)^{-1}$

$\beta_6 = .019$; $g_6(s) = .019\ (s-266.8)^{-1}$

$\beta_7 = -76445$; $g_7(s) = -76445\ (s+286.2)^{-1}$

The simulated values of $\hat{y}_i(t)$ following a unit step input to the system are shown in Table 1.

Time	$\hat{y}_1(t)$	$\hat{y}_2(t)$	$\hat{y}_3(t)$	$\hat{y}_4(t)$	$\hat{y}_5(t)$	$\hat{y}_6(t)$	$\hat{y}_7(t)$
0.2	.121 E-4	- .391 E-1	.136 E-5	.446 E-3	- .470 E-7	.346 E-7	.145 E-5
0.4	.512 E-4	- .293	.213 E-4	.101 E-1	- .266 E-5	.673 E-5	.305 E-3
0.6	.812 E-4	- .715	.763 E-4	.379 E-1	- .181 E-4	.867 E-4	.404 E-2
0.8	.850 E-4	- 1.18	.160 E-3	.745 E-1	- .560 E-3	.398 E-3	.187 E-1
1.0	.629 E-4	- 1.59	.256 E-3	.106	- .117 E-3	.108 E-2	.512 E-1
1.2	.228 E-4	- 1.89	.350 E-3	.123	- .195 E-3	.215 E-2	.102
1.4	- .256 E-4	- 2.08	.429 E-3	.124	- .278 E-3	.350 E-2	.167
1.6	- .735 E-4	- 2.16	.487 E-3	.111	- .355 E-3	.497 E-2	.237
1.8	- .114 E-3	- 2.15	.523 E-3	.893 E-1	- .420 E-3	.636 E-2	.305
2.0	- .145 E-3	- 2.08	.538 E-3	.637 E-1	- .468 E-3	.755 E-2	.362
2.2	- .164 E-3	- 1.99	.539 E-3	.388 E-1	- .498 E-3	.845 E-2	.406
2.4	- .173 E-3	- 1.89	.529 E-3	.180 E-1	- .513 E-3	.904 E-2	.435
2.6	- .173 E-3	- 1.81	.515 E-3	.284 E-2	- .515 E-3	.935 E-2	.450
2.8	- .168 E-3	- 1.75	.499 E-3	- .628 E-2	- .509 E-3	.943 E-2	.454
3.0	- .161 E-3	- 1.71	.486 E-3	- .101 E-1	- .499 E-3	.936 E-2	.450

Table 1

As the ratio $\left|\hat{y}_4(t)\right| / \left|\hat{y}_2(t)\right|$ is small for all t, this suggests that a 6th order model will be suitable if derived from the first three loops ($g_1(s)$, $g_2(s)$, $g_3(s)$ or equivalently $g_1(s)$, $g_2(s)$, $h_2(s)$). The 6th order model $\hat{g}_1(s)$ obtained by taking the steady-state approximation to $h_3(s)$ is given by

$$\hat{g}_1(s) = \frac{(s - 1.15)(s + 25.3)}{(s+.508+1.81j)(s+.508-1.81j)(s+6.16)(s+14.7)(s+21.3)(s+30)}$$

The step response for this model is shown in Fig. 5, and is seen to be highly oscillatory. These oscillations can be damped by making a better match of the steady-state behaviour of g(s) i.e. by matching further moments of g(s) about s = o. A 6^{th} order can again be formed by retaining only $g_1(s)$ and $g_2(s)$, and taking a first-order Chen approximation to $h_2(s)$, giving

$$\hat{g}_2(s) = \frac{(s-.959)(s+24.4)}{(s+.89+1.61j)(s+.89-1.61j)(s+5.18)(s+16)(s+19.6)(s+29.9)}$$

The step-response for this model is shown in Fig. 5, and is seen to be more accurate than that of $\hat{g}_1(s)$. As the response of g(s) is dominated by the pair of complex poles at $s = -1 \pm 2_j$, it is suggested that an improved 6^{th} order model will be obtained by retention of these poles i.e. by factoring them out of the transfer function g(s), and reducing the resultant 8^{th} order system, $\bar{g}(s)$. The model, $\hat{g}_3(s)$, obtained by retaining the two complex-poles and using a 4^{th} order steady-state corrected model of $\bar{g}(s)$ is given by

$$\hat{g}_3(s) = \frac{(s-1.12)(s+18.4)}{(s+1+2j)(s+1-2j)(s+3.61)(s+15.6+5.32j)(s+15.6-5.32j)(s+29.6)}$$

Again, better step-response matching can be achieved by matching further moments about s = o of $\bar{g}(s)$. Using a 4^{th} order model of $\bar{g}(s)$ formed by retaining the first two loops of its feedback decomposition and taking a 1^{st} order Chen approximation of the remaining system gives the 6^{th} order model of g(s),

$$\hat{g}_4(s) = \frac{(s+8.94)(s-.978)}{(s+1+2j)(s+1-2j)(s+2.38)(s+11.8+6.25j)(s+11.8-6.25j)(s+29)}$$

Step responses for $\hat{g}_3(s)$, $\hat{g}_4(s)$ are shown in Fig. 6.

The second example presented above illustrates how the techniques suggested in this paper can provide a systematic approach to model reduction, in particular facilitating trade off between high and low frequency behaviour matching. Although only step-response matching was examined, similar analyses could be used to match responses to other kinds of input (impulses, etc.) and different model reduction criteria could be used to approximate the dynamics of the lower loops.

5. Conclusions

The paper has given consideration to the problem of the reduction of order of a single-input-single-output system $S(A,B,C)$ described by a scalar transfer function $g(s)$. On the assumption that the reduced order model is to be used for feedback control systems design and that the reduction procedure should use a canonical form for $S(A,B,C)$ reflecting the structure of the asymptotes, intercepts and finite zeros of the system root-locus, the following results have been obtained:

a) A simple canonical form for $g(s)$ has been derived in the form of a sequential feedback decomposition of the system (see Fig. 2) The important features of the decomposition are best seen in its simplest form (Fig. 1) where the system is structured into a forward path element, $g_1(s)$, describing the asymptotes and intercept of the system root-locus and a feedback element, $h_1(s)$, characterizing the finite zeros of the system. The reduction of the system $g(s)$ for the purposes of feedback controller design can hence be regarded as the approximation of the zero structure of $g(s)$ by deriving a reduced model of $h_1(s)$.

b) The algebraic canonical form for $g(s)$ has an immediate interpretation as a canonical form for the matrix triple (A,B,C). The canonical form, in its simplext form, is directly related to recent results[17] on the decomposition of the system state space as the direct sum of the maximal A,B invariant subspace in the kernel of C(characterising the system zeros) and a subspace of the form $B \oplus AB \oplus \ldots \oplus A^{k_1-1}B$ (characterising the asymptotes of the system root locus). Previous interpretations[17] of such decompositions in terms of inherent state feedback within the system structure provide a physical justification for the use of the proposed canonical form.

c) A simple model reduction procedure based on neglecting feedback elements $g_i(s)$, $i > j$, is shown to be capable of producing a reduced order model approximating to the dominant pole-zero structure of $g(s)$.

d) A more general approach based on the approximation of some $h_j(s)$ (Fig. 2) also enables the matching of high and low frequency moments of the system.

e) The procedure has the advantage of providing an estimate of a suitable reduced model order by simulation of the open-loop step responses of the system in canonical form.

Overall the procedure is highly flexible, making possible the retention of dominant root-locus behaviour, the retention of dominant poles exactly (by factoring them out of $g(s)$), the retention of dominant zeros (by factoring them out of $h_1(s)$) and the matching of high and low frequency behaviour. In this sense the procedures can be regarded as a general framework for reduction for closed and open-loop purposes, where any known reduction technique can be applied to a subsystem, $h_j(s)$, say. The examples discussed illustrate this point.

Finally, it is noted that the division algorithm (equation (7)) can, in principle, be directly applied to a system described by an mxm transfer function matrix $G(s)$, yielding a similar system decomposition (Fig. 2). The procedure described in this paper could then be extended to the reduction of multi-input-multi-output systems with essentially the same results. It is possible, however, that the algorithm of equation (7) could break down if $h_{j-1}(s)$ is non-zero but identically singular. It seems therefore that the generalization of the technique to be multi-input-multi-output case will be best approached using geometric methods analogous to those mentioned in (b) above. Work is in progress on this problem.

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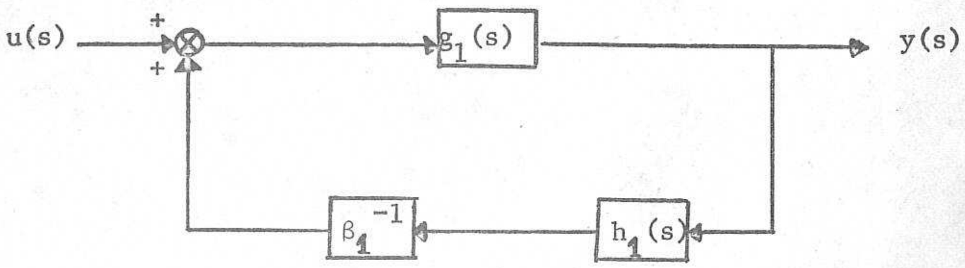
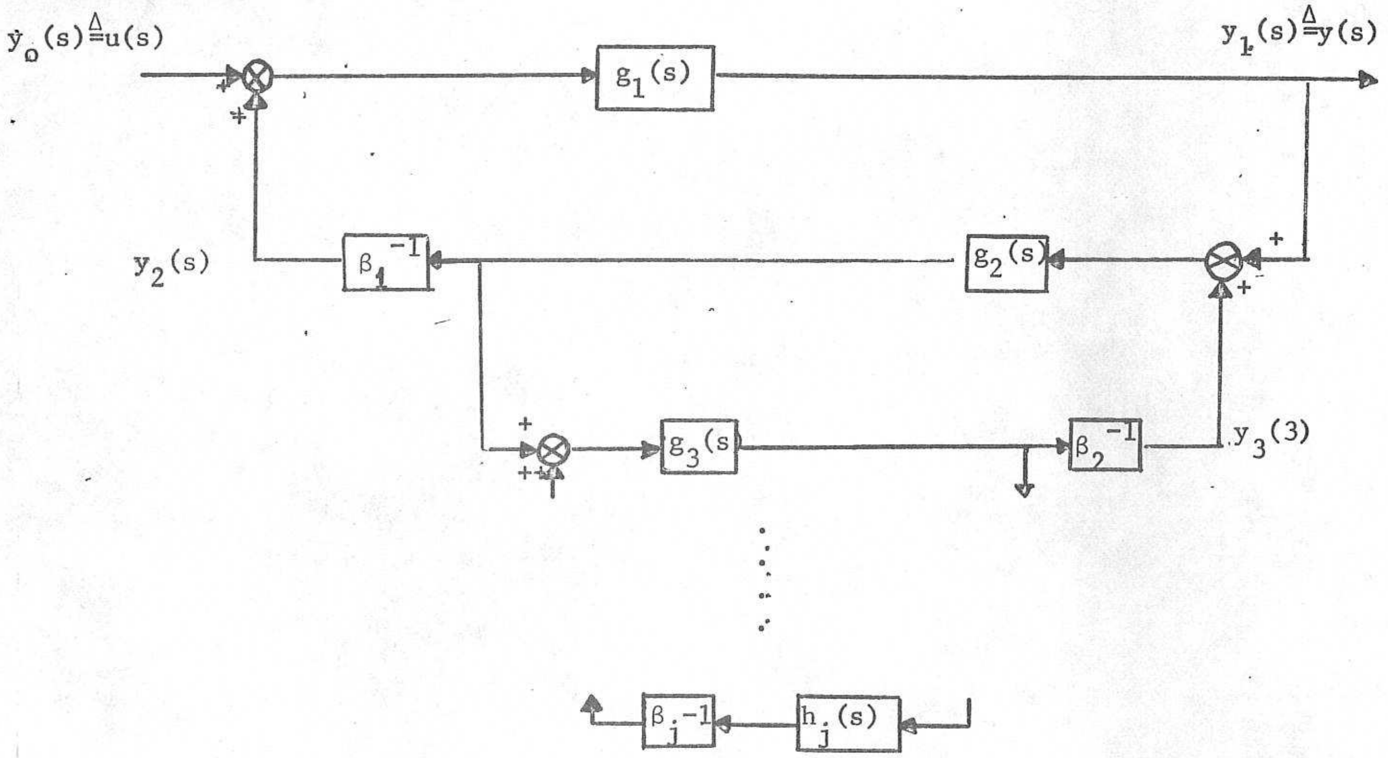
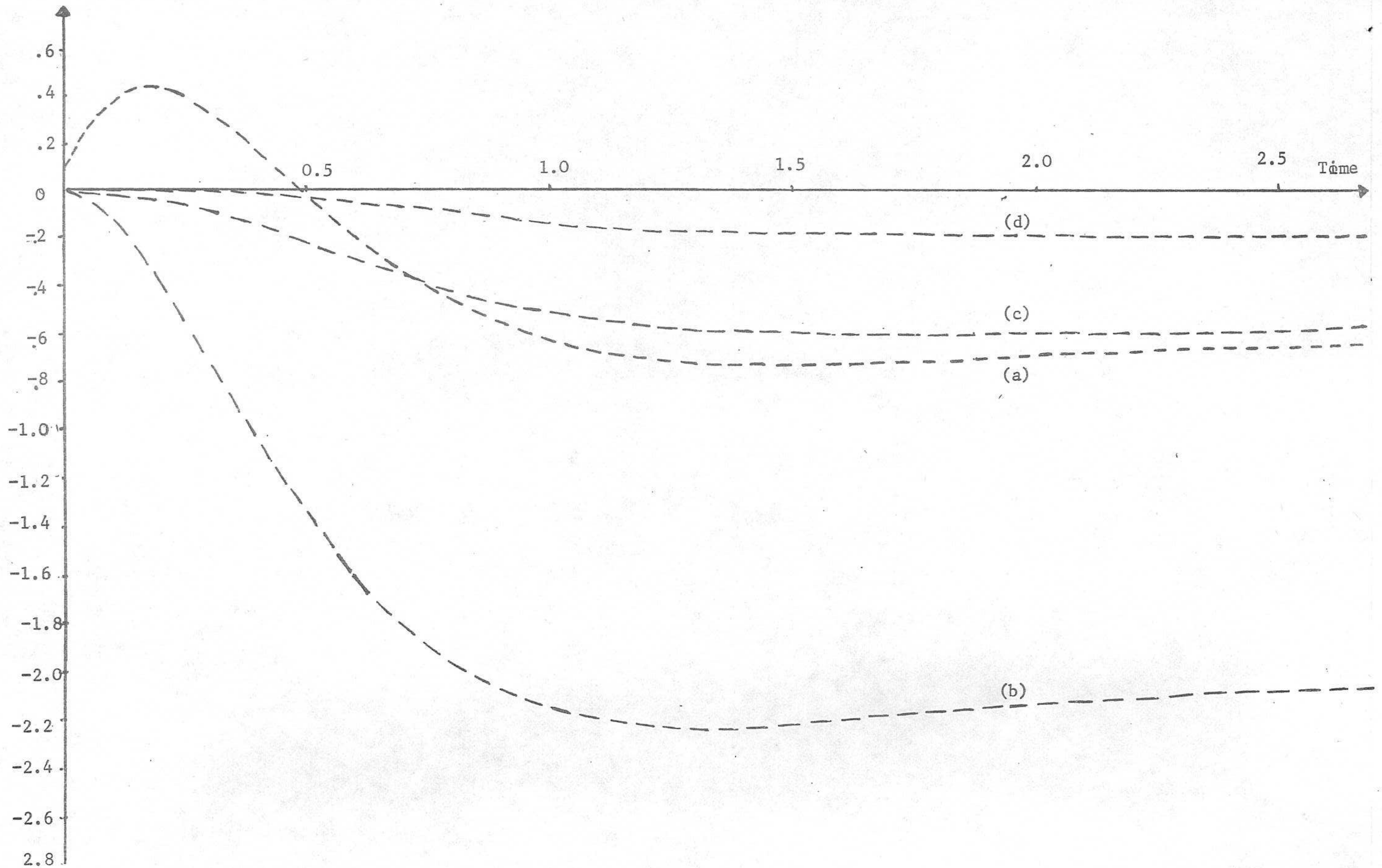


Figure 1



$y_o(s) \triangleq u(s)$
$y_1(s) \triangleq y(s)$
$g_1(s)$
$y_2(s)$
β_1^{-1}
$g_2(s)$
$g_3(s)$
β_2^{-1}
$y_3(3)$
β_j^{-1}
$h_j(s)$

Figure 2



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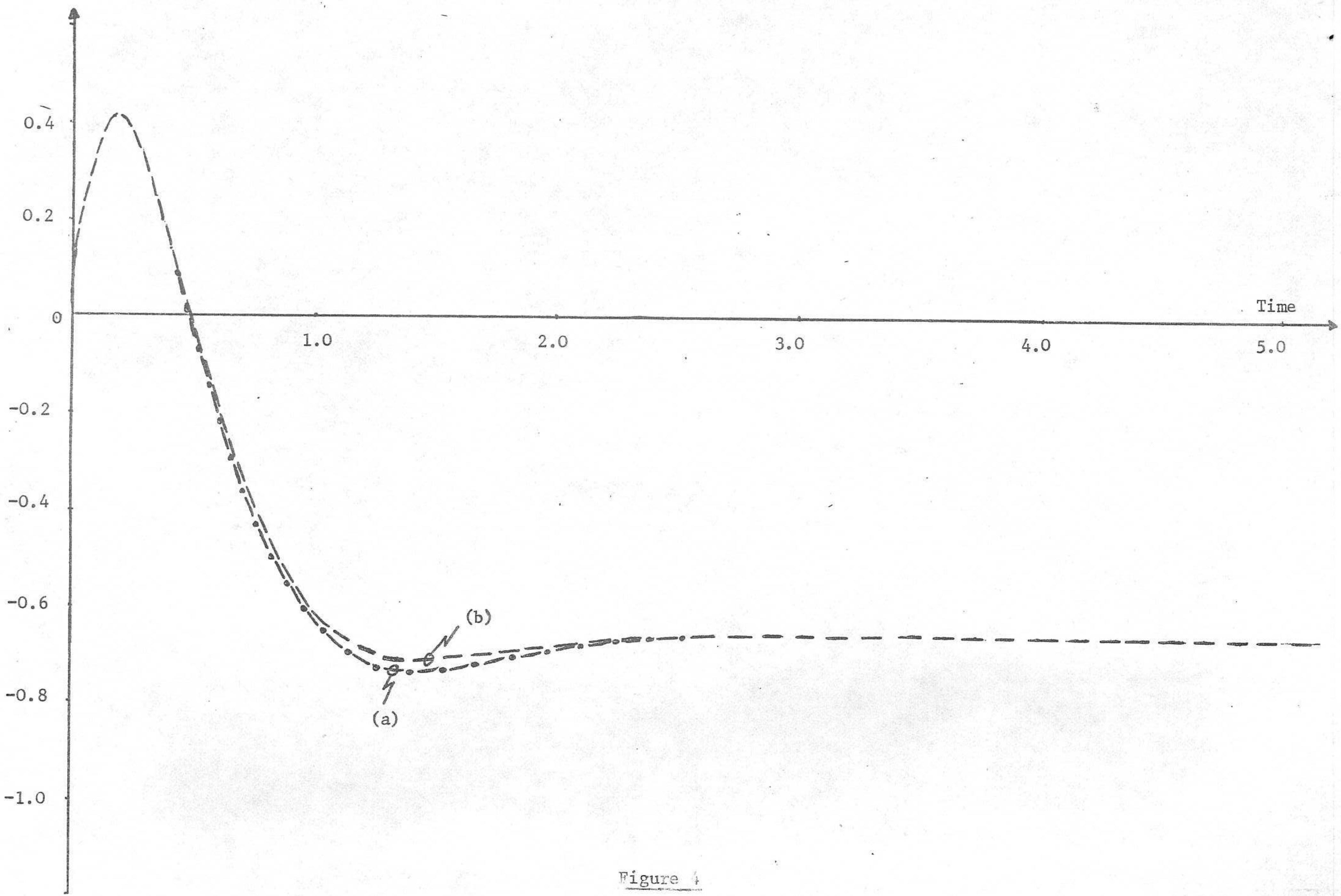

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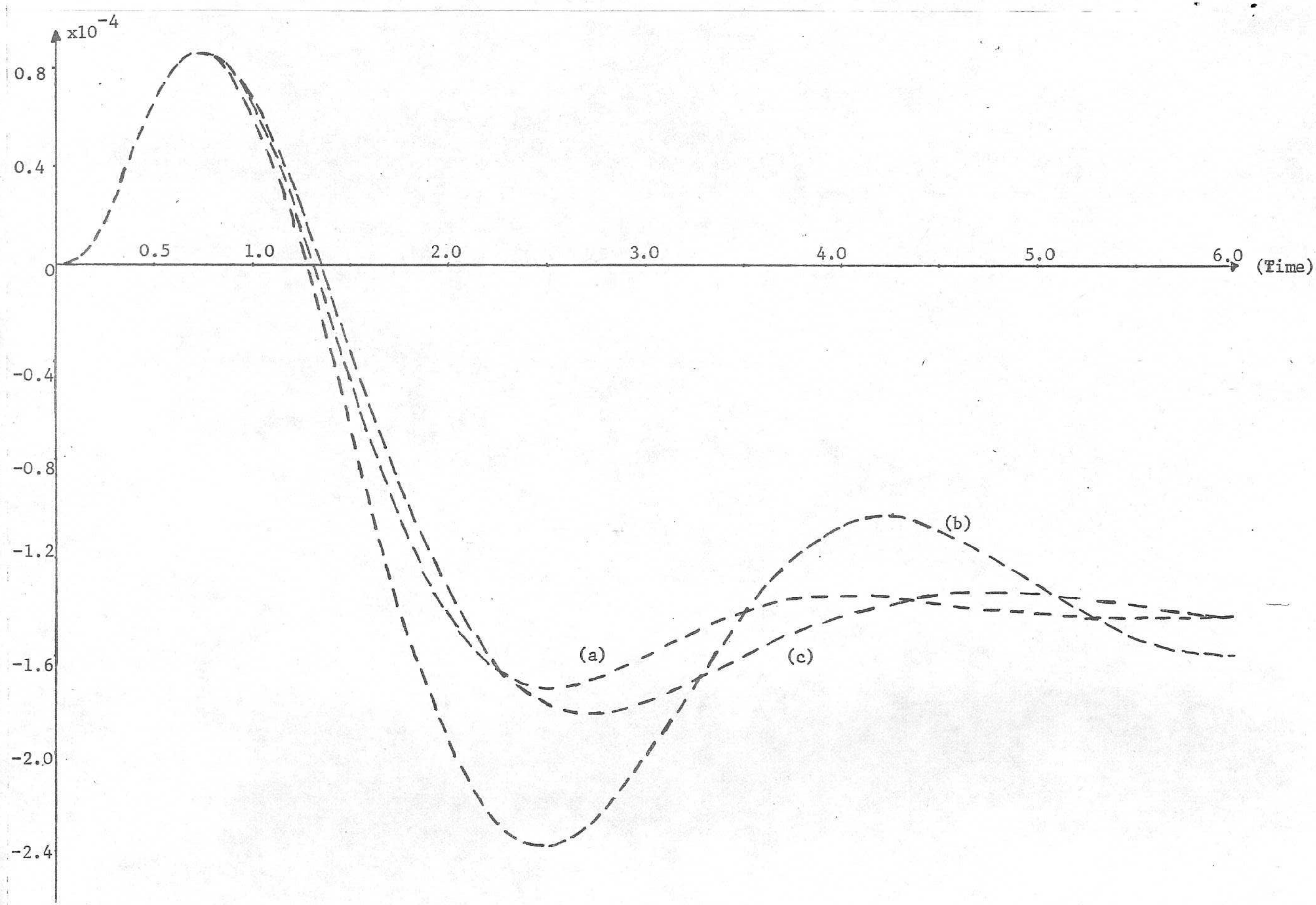

Figure 5

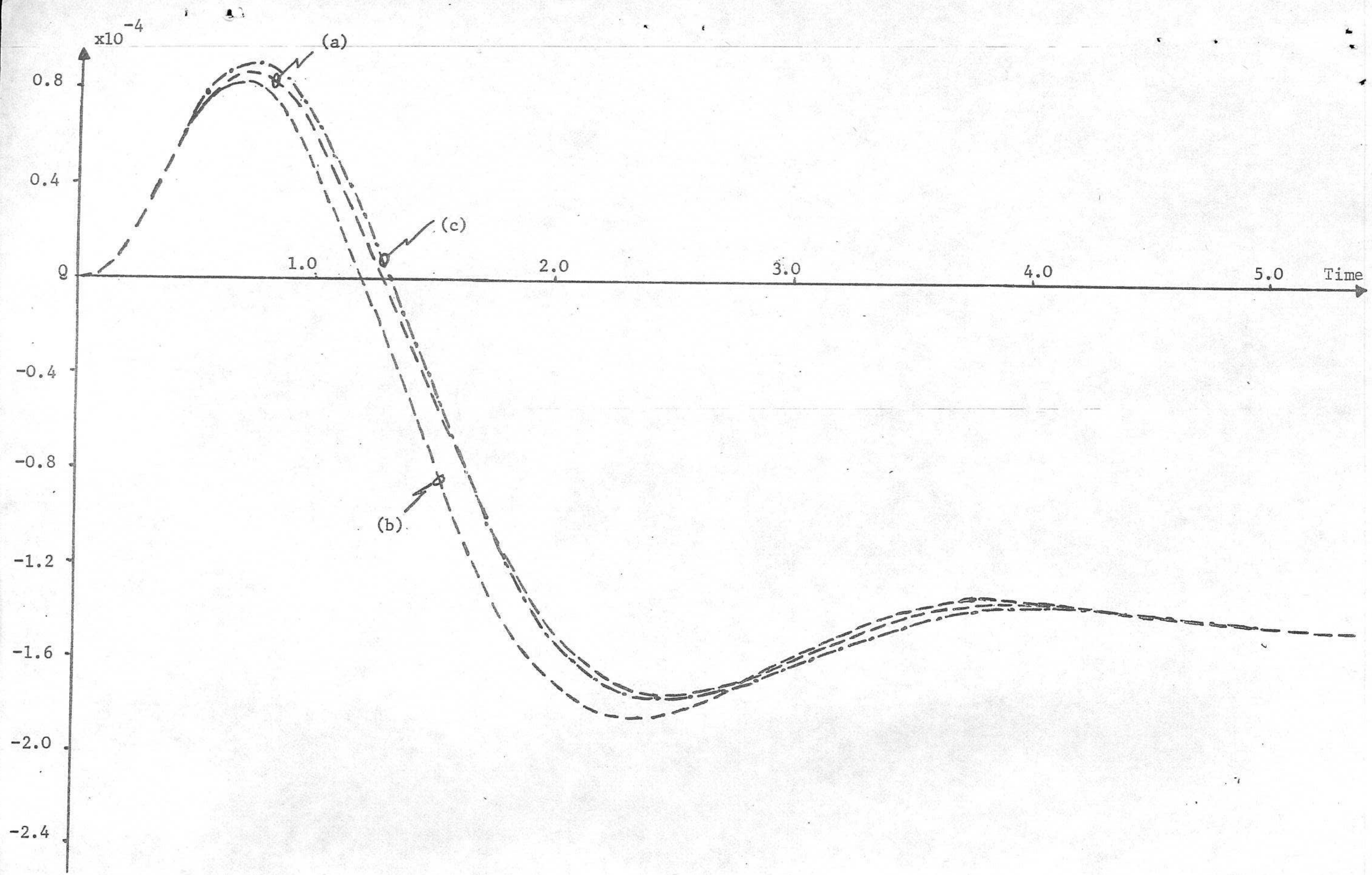

Figure 6