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AUTOMORPHISMS OF η-LIKE COMPUTABLE LINEAR ORDERINGS AND KIERSTEAD'S CONJECTURE

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Abstract. We develop an approach to the longstanding conjecture of H.A. Kierstead concerning the character of strongly nontrivial automorphisms of computable linear orderings. Our main result is that for any η-like computable linear ordering $\mathscr{B}$, such that $\mathscr{B}$ has no interval of order type η, and such that the order type of $\mathscr{B}$ is determined by a $\mathbf{0'}$-limitwise monotonic maximal block function, there exists computable $\mathscr{L} \cong \mathscr{B}$ such that $\mathscr{L}$ has no nontrivial Π^0_1 automorphism.

1. Introduction

The computability theoretic character of relational structure, in a real context, is directly related to the balance between logic and information. Given the need for a basic deconstructive understanding of computation and definability of relations in a given structure, the characterising of basic automorphisms within specific computational contexts underpins progress in the area. Linear orderings provide a relatively simple context within which basic ingredients and corresponding techniques can be clarified and further refined.

In this paper we address the longstanding conjecture of Kierstead concerning the nature of strongly nontrivial automorphisms of computable linear orderings. Our approach is developed via the theory of η-like linear orderings. The class of η-like linear orderings provides a particularly apposite context in which to test properties of countably infinite (and, in the present case, computable) linear orderings for two reasons. The first is that there is a straightforward and fundamentally uniform method of describing any member of this class, namely that its order type is of the form $\sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ for some function $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$. The second reason is the inherent generality of η-like orderings, as underlined by the fact[1] that any countably infinite linear ordering that does not contain an interval of order type ω or ω^* is η-like up to a finite number of elements—in the sense that it has order type $n_1 + \gamma + n_2$, where n_1 and n_2 are finite (perhaps zero) and γ is η-like.

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[1]This is a standard result with straightforward proof. See for example Lemma 2.10 of [McC03].

In his 1987 paper [Kie87] Kierstead investigated the particular case of η-like computable linear orderings of order type $\tau = 2 \cdot \eta$. The paper concludes with 3 conjectures. The main conjecture (Conjecture 1 on page 688), stated in full generality, is as follows.

Conjecture [Kie87]. Every computable copy of a computable linear ordering $\mathscr{B} = \langle B, <_{\mathscr{B}} \rangle$ has a strongly nontrivial—meaning that, for some $x \in B$, the interval between x and $f(x)$ is infinite—automorphism f which is Π^0_1, if and only if the order type τ of $\mathscr{B}$ contains an interval of order type η.

Kierstead proved the truth of his conjecture for the case when $\mathscr{B}$ has the order type $\tau = 2 \cdot \eta$. Further support for the truth of the latter was subsequently provided by the result by Downey and Moses [DM89] that every discrete[2] computable linear ordering $\mathscr{L}$ has a computable copy with no strongly nontrivial Π^0_1 self embedding. In the work presented below we extend these results by proving that Kierstead's conjecture is true for a quite general subclass of η-like computable linear orderings. In so doing, we provide key tools and make potentially valuable progress towards a full resolution.

Our argument starts by noting that every nontrivial automorphism of an η-like linear ordering is strongly nontrivial, and that every computable linear ordering with an interval of order type η has a computable (strongly) nontrivial automorphism. Thus resolution of Kierstead's conjecture for η-like computable linear orderings is equivalent to answering the following question.

Question 1. If $\mathscr{B}$ is an η-like computable linear ordering with no interval of order type η then does there exist computable $\mathscr{L} \cong \mathscr{B}$ such that $\mathscr{L}$ has no nontrivial Π^0_1 automorphism?

To establish the scope of our work our argument proceeds via the recent result [Fro12, Har14a] that, if computable $\mathscr{B}$ is either (a) strongly η-like or (b) η-like but with no strongly η-like interval, then there exists a $\mathbf{0'}$-limitwise monotonic function $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ such that $\mathscr{B}$ has order type $\sum\{ F(q) \mid q \in \mathbb{Q} \}$. Using the term *maximal block function* to denote such F, we then prove in our main result (Theorem 3.11) that, for any η-like computable linear ordering whose order type is determined by a $\mathbf{0'}$-limitwise monotonic maximal block function, Question 1 has an affirmative answer. We also observe (Note 3.4) that our proof is framed in such a way that if, in the statement of Question 1, we replace the class of Π^0_1 functions by any class of of functions whose graphs lie within a $\mathbf{0'}$-uniform class—such as, for example, the class of ω-c.e. functions—we obtain the same (affirmative) result.

Finally we note that many of the original ideas and techniques used below were first developed in the second author's PhD Thesis [Lee11].

2. Preliminaries.

We assume $\{W_e\}_{e \in \mathbb{N}}$ to be a standard listing of c.e. sets with associated c.e. approximation $\{W_{e,s}\}_{e,s \in \mathbb{N}}$. $\emptyset'$ denotes the standard halting set for Turing machines in this context, i.e. the set $\{ e \mid e \in W_e \}$, and $\mathbf{0'}$ denotes the Turing degree of $\emptyset'$. We suppose $Q_{\mathbb{N}} : \mathbb{N} \to \mathbb{Q}$ to be a computable bijection and we use $q_0, q_1, q_2, \ldots$ to denote the resulting listing of $\mathbb{Q}$, i.e. such that $q_n = Q_{\mathbb{N}}(n)$ for all $n \geq 0$. We also

[2] A linear ordering is *discrete* if every element has both an immediate predecessor and an immediate successor except for the possible first and last elements.

assume $\langle x, y\rangle$ to be a standard computable pairing function over $\mathbb{N}$ extended to use over $\mathbb{Q}$ via the above listing. We use $X^{[e]}$ to denote the set $\{\, \langle e, x\rangle \mid \langle e, x\rangle \in X \,\}$. χ_Y denotes the characteristic function of Y, and $Y(n) = \chi_Y(n)$ is the shorthand that we will use in place of χ_Y below. For any set X and string α, $|X|$ denotes the cardinality of X whereas $|\alpha|$ denotes the length of α.

For any function f with domain and range in $\mathbb{N}$ or $\mathbb{Q}$ we use $\operatorname{Graph} f$ to denote the set $\{\, \langle x, y\rangle \mid f(x)\downarrow = y \,\}$, i.e. the graph of f coded into $\mathbb{N}$ via the pairing function $\langle\cdot,\cdot\rangle$. (Note that in this context we identify a pair (x, y) with its code $\langle x, y\rangle$ so that, for example, the shorthand $\operatorname{Graph} f \subseteq \mathbb{N}\times\mathbb{N}$ makes sense.) For any function f we use $\operatorname{Dom} f$ and $\operatorname{Ran} f$ to denote, respectively, the domain and range of f. Following standard usage, we define f to be Γ, for some arithmetical predicate of sets Γ, if $\operatorname{Graph} f \in \Gamma$. We extend this notation in the obvious way to classes of functions. Thus for example the class of functions $\mathcal{F}$ is Π^0_1 if $\operatorname{Graph} f \in \Pi^0_1$ for all $f \in \mathcal{F}$.

For a binary (ternary) function f we use the shorthand f_e ($f_{e,s}$) for $\lambda n f(e, n)$ ($\lambda n f(e, n, s)$). Given degree $\boldsymbol{a}$, and using the standard identification (mentioned above) of a set with its characteristic function, we define a class of sets $\mathcal{C}$ to be $\boldsymbol{a}$-uniform [Joc72] if there is a binary $\boldsymbol{a}$-computable function C such that $\mathcal{C} = \{C_e\}_{e\in\mathbb{N}}$.

We use the standard notation $f(n)\downarrow$ to denote that the function f is defined at argument n and we use similar ($\downarrow$) notation to denote the convergence of computations and limits of functions. For the latter we also use shorthand of the form "$\liminf_{s\to\infty} f_s(x) = \infty$" to denote that $\liminf_{s\to\infty} f_s(x)$ tends to infinity.

In the context of linear orderings we use ω and ω^* to denote the order types of the nonnegative and negative integers respectively. We use η to denote the order type of $\mathbb{Q}$ whereas n denotes the finite order type with n elements. For linear orders $\mathscr{L}_\beta = \langle L_\beta, <_{\mathscr{L}_\beta}\rangle$ and $\mathscr{L}_\gamma = \langle L_\gamma, <_{\mathscr{L}_\gamma}\rangle$ of order type β and γ respectively, $\beta\cdot\gamma$ denotes the order type of $\mathscr{L}_\beta \times \mathscr{L}_\gamma$ under lexicographical ordering (from the right). For example $2\cdot\eta$ denotes the order type of a linear ordering formed by taking a copy of the rational numbers and replacing every element by an ordered pair.

Let $\mathscr{L} = \langle L, <_{\mathscr{L}}\rangle$ be a linear ordering. We call $S \subseteq L$ an *interval* if, for all $a, b \in S$, and any c that lies $<_{\mathscr{L}}$ between a and b, c is also in S. Notice that S does not necessarily have endpoints, also that this terminology refers implicitly to the subordering $\langle S, <_{\mathscr{L}}\rangle$ of $\mathscr{L}$. For any $a, b \in L$, we say that a, b are *finitely far apart*—written $B_{\mathscr{L}}(a, b)$—if the interval S of elements lying between a and b is finite. (By definition $S = \emptyset$ if $a = b$.) We call $B_{\mathscr{L}}$ the *block relation* of $\mathscr{L}$. Note that $B_{\mathscr{L}}$ is an equivalence relation. If $\mathscr{L}$ is countably infinite we define $\mathscr{L}$ and its order type τ to be *η-like* if (i) $\mathscr{L}$ has no $<_{\mathscr{L}}$ least or greatest element and (ii) $\{\, c \mid B_{\mathscr{L}}(a, c) \,\}$ is finite for all $a \in L$ or, equivalently, if $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$ for some function $F : \mathbb{Q} \to \mathbb{N}\setminus\{0\}$. We call any finite interval in $\mathscr{L}$ a *block* and we call the equivalence classes under $B_{\mathscr{L}}$ *maximal* blocks. We say that F is a *maximal block function* of $\mathscr{L}$ and its order type τ (or that τ is *determined* by such F). We say that $\mathscr{L}$ and its order type τ are *strongly η-like* if in addition F has finite range (i.e. the maximal block size is bounded).

For any maximal block I of size $p \geq 1$ (written $|I| = p$) we use terminology of the form $I = \{k_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_p\}$ to denote I and we call k_1 (k_p) the *leftmost* (*rightmost*) element of I. For any distinct elements $a, b \in L$ we say that a and b

are *adjacent*—written $N_{\mathscr{L}}(a,b)$—if the interval of elements lying between a and b is empty. Note that $\neg N_{\mathscr{L}}$ is computably enumerable in $<_{\mathscr{L}}$.

If $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ is countably infinite we derive a listing $l_0, l_1, l_2, \ldots$ of L computable in $<_{\mathscr{L}}$. This allows us to assume that $L = \mathbb{N}$. We say that $\mathscr{L}$ is *computable* if $<_{\mathscr{L}}$ is computable.

We assume the reader to be conversant with the Arithmetical Hiearchy and Turing reducibility ($\leq_{\mathrm{T}}$). We refer the reader to [Coo04, Soa87, Odi89] for further background and notation in computability theory and to [Dow98] for a review of computability theoretic results in the context of linear orderings.

3. Rigidity and η-like Computable Linear Orderings.

We now turn to our main theme, namely the properties of η-like computable linear orderings.

Lemma 3.1 (Folklore). *If $\mathscr{B}$ is an η-like computable linear ordering with an interval S of order type $n \cdot \eta$ such that $n > 1$ ($n = 1$), then $\mathscr{B}$ has a nontrivial Δ^0_2 (computable) automorphism.*

Proof Sketch. Choosing computable S, if $n = 1$, then we can use a standard construction to define a computable automorphism f that fixes all elements not in the interval S and that is defined over S, using the density of the latter, such that $f(x) \neq x$ for all $x \in S$. If $n > 1$ on the other hand then, given $a \in S$, we can construct the maximal block to which a belongs using $n - 1$ finite sets of (adjacency) queries of complexity Π^0_1. Thus we can define f in a similar way to the case $n = 1$ but using a process computable in $\emptyset'$ to deal with the definition of f over the interval S. ☐

Notation. For any class of functions $\mathcal{F}$, we say that a linear ordering $\mathscr{L}$ is *$\mathcal{F}$-rigid* if $\mathcal{F}$ contains no nontrivial automorphism of $\mathscr{L}$.

In Lemma 3.1, if $n > 1$ and we fix $I = \{k_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_n\}$ to be a maximal block in S we see that, for all $1 \leq i \leq n$ the set $T_i = \{\, f^m(k_i) \mid m \geq 0 \,\}$ is an infinite Σ^0_2 subset[3] of a choice set—a choice set of a linear ordering is a set containing precisely one element from each maximal block—containing only the "ith to the leftmost" elements of maximal blocks in S. In fact, more generally, if $\mathscr{B}$ contains a strongly η-like interval $\widehat{S}$ and n is greatest such that $\widehat{S}$ contains infinitely many maximal blocks of size n, then $\widehat{S}$ contains such ("ith to the leftmost") infinite Σ^0_2 sets for each $1 \leq i \leq n$. Thus the powerful choice set technique used by Lerman and Rosenstein [LR82] and Downey and Moses [DM89] in the context of self embeddings of discrete computable linear orderings is not applicable in the context of automorphisms of η-like computable linear orderings that contain a strongly η-like interval.

With the above observations in mind we note that the construction in the proof of Theorem 3.11 is orientated towards exploiting quite general properties possessed by nontrivial automorphisms of η-like linear orderings. It also relies heavily on the properties of the specific type of approximation that we now show to be associated with a class of functions subsuming the Π^0_1 functions.

Definition 3.2. Given computable $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we say that $\{f_{e,s}\}_{e,s\in\mathbb{N}}$ is an *upwards uniform Δ^0_2 approximation* if for all $e, x \in \mathbb{N}$ it satisfies one of the two following conditions.

[3] To see that T_i is Σ^0_2 it suffices to note that T_i is computably enumerable in $\emptyset'$.

(1) $\lim_{s\to\infty} f_{e,s}(x)\downarrow$, or
(2) $\liminf_{s\to\infty} f_{e,s}(x) = \infty$.

Accordingly f defines a class of partial functions $\{f_e\}_{e\in\mathbb{N}}$ such that for every index e, $\mathrm{Dom}(f_e) = \{\, n \mid \lim_{s\to\infty} f_{e,s}(n)\downarrow \}$ and such that, for every $n \in \mathrm{Dom}(f_e)$, $f_e(n) = \lim_{s\to\infty} f_{e,s}(n)$. We say that $\mathcal{F} = \{f_e\}_{e\in\mathbb{N}}$ is *upwards uniform* Δ^0_2.

Lemma 3.3. *There exists an upward uniform* Δ^0_2 *class* $\mathcal{F} = \{f_e\}_{e\in\mathbb{N}}$ *containing the class of (partial)* Π^0_1 *functions.*

Proof. Let $\{U_e\}_{e\in\mathbb{N}}$ be a listing of the class of Π^0_1 sets with associated Π^0_1 approximation $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ defined by setting $U_{e,s} = \mathbb{N} \setminus W_{e,s}$ for all $e, s \in \mathbb{N}$. Define the approximation $\{f_{e,s}\}_{e,s\in\mathbb{N}}$ as follows. For each $e, s \geq 0$, and all $x \in \mathbb{N}$,

$$f_{e,s}(x) \;=\; \begin{cases} \min\{\, z \mid \langle x, z\rangle \in U_{e,s} \} & \text{if } s > 0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

(We assume, for all $e, s \geq 0$, that $|W_{e,s}| \leq s$ so that $f_{e,s}(x) \leq s$.) Then clearly $\{f_{e,s}\}_{e,s\in\mathbb{N}}$ is an upwards uniform Δ^0_2 approximation for the class $\mathcal{F} = \{f_e\}_{e\in\mathbb{N}}$—as prescribed by Definition 3.2—such that $\mathcal{F}$ subsumes the class of Π^0_1 functions. □

Note 3.4. The approximation $\{f_{e,s}\}_{e,s\in\mathbb{N}}$ clearly also satisfies $f_{e,s}(x) \leq f_{e,s+1}(x)$ for all $e, s \geq 0$ and $x \in \mathbb{N}$. However in Theorem 3.11 we restrict ourselves to working with $\mathcal{F} = \{f_e\}_{e\in\mathbb{N}}$ as defined in Definition 3.2. This is because the properties of $\{f_{e,s}\}_{e,s\in\mathbb{N}}$ that are essential to the construction used in the proof of Theorem 3.11 are precisely those stated in Definition 3.2. Moreover, this approach means that Theorem 3.11 is generalisable in a straightforward manner. Indeed, note that any $\mathbf{0}'$-uniform class of sets $\mathcal{U}$ has a listing $\{U_e\}_{e\in\mathbb{N}}$ with computable approximation $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ such that, for all $e \geq 0$ and $x \in \mathbb{N}$, $\lim_{s\to\infty} U_{e,s}(x)\downarrow$. Also—with (3.1) in mind—we can assume that $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ is such that $\{\, \langle x, z\rangle \mid z \geq s \} \subseteq U_{e,s}$ for all $e, s \geq 0$ and $x \in \mathbb{N}$. Accordingly—letting $\mathcal{F}_{\mathcal{U}}$ denote the class of functions whose graphs lie in $\mathcal{U}$—we see that, for $\mathbf{0}'$-uniform $\mathcal{U}$, we can replace the class of Π^0_1 functions in the statement of Lemma 3.3 by $\mathcal{F}_{\mathcal{U}}$, to obtain the upwards uniform Δ^0_2 class $\mathcal{F} = \{f_e\}_{e\in\mathbb{N}}$ such that $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{F}$. This means that in Theorem 3.11 we can replace the class of Π^0_1 functions by the class $\mathcal{F}_{\mathcal{U}}$ for any such $\mathcal{U}$—i.e., given $\mathscr{B}$ as in the statement of Theorem 3.11 the latter will then tell us that there exists computable $\mathscr{L} \cong \mathscr{B}$ such that $\mathscr{L}$ is $\mathcal{F}_{\mathcal{U}}$-rigid. On the other hand, under the classification of the Δ^0_2 sets given by the Ershov Difference Hierarchy [Ers68a, Ers68b, Ers70], Ershov showed (in [Ers68b]) that, for any notation a in Kleene's system of ordinal notations $\mathcal{O}$, the class $\mathcal{U}$ of a-c.e. sets is $\mathbf{0}'$-uniform. (In fact, for any Σ^0_2 set $\mathcal{A} \subseteq \mathcal{O}$ the class $\mathcal{U}$, comprising precisely those sets that are a-c.e. for some $a \in \mathcal{A}$, is $\mathbf{0}'$-uniform [Lee11, BH14].) Hence, as a simple example, we can replace "Π^0_1-rigid" by "ω-c.e.-rigid" in Theorem 3.11.

The proof of Theorem 3.11 also depends on the manner in which we approximate the maximal block function F determining the order type of the linear ordering $\mathscr{B}$ in the statement of the latter. We now state the results which specify the relevant properties of F and the approximation that we use.

Definition 3.5. Given degree $\boldsymbol{a}$, we say that $F : \mathbb{N} \to \mathbb{N}$ is $\boldsymbol{a}$*-limitwise monotonic* if there exists $\boldsymbol{a}$-computable $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ satisfying, for all $n, s \geq 0$, the following conditions.

(a) $f(n, s) \leq f(n, s + 1)$.

(b) $\lim_{s\to\infty} f(n,s)$ exists.
(c) $F(n) = \lim_{s\to\infty} f(n,s)$.

If $\boldsymbol{a} = \mathbf{0}$ we simply say that F is *limitwise monotonic.*

Lemma 3.6 ([Har08, Kac08])**.** *For any function $F : \mathbb{N} \to \mathbb{N}$ the following are equivalent.*

(1) *F is $\mathbf{0}'$-limitwise monotonic.*
(2) *There is a computable function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for all $n \geq 0$, $F(n) = \liminf_{s\to\infty} g(n,s)$.*

Note 3.7. By use of the computable bijection $Q_{\mathbb{N}}^{-1}$ defined on page 2 we can apply Definition 3.5 and Lemma 3.6 when F and f have (respectively) domains $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{N}$. This is how we proceed below.

We easily see that the class $\{\, f \mid f : \mathbb{N} \to \mathbb{N} \;\&\; f \in \Pi^0_1 \,\}$ is subsumed by the class of limitwise monotonic functions. Moreover we can show that this subsumption is proper by constructing a limitwise monotonic function g such that $\operatorname{Ran} g \neq \operatorname{Ran} f$ for any Π^0_1 function with domain $\mathbb{N}$ [Har14a]. Then by relativisation—or otherwise via a direct construction—we obtain the following result.

Theorem 3.8 ([Har14b], [Har14a])**.** *There exists an η-like computable linear ordering $\mathscr{A}$ such that, for any Π^0_2 function $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and linear ordering $\mathscr{B}$ of order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$, $\mathscr{A} \ncong \mathscr{B}$.*

Note that in the proof of Theorem 3.8 we construct $\mathscr{A}$ (either directly [Har14b] or indirectly [Har14a]) such that $\mathscr{A}$ has order type $\sum\{\, G(q) \mid q \in \mathbb{Q} \,\}$ where $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ is $\mathbf{0}'$-limitwise monotonic. Turning our attention to the class of such functions, we firstly note that, if an order type τ is determined by $\mathbf{0}'$-limitwise monotonic $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ in the above sense, then τ has a computable presentation.

Proposition 3.9 ([FZ09])**.** *For any $\mathbf{0}'$-limitwise monotonic $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ there exists a computable linear ordering $\mathscr{A}$ with order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$.*

Theorem 3.8 shows that we cannot use the properties of Π^0_2 functions (in their role as maximal block functions) to help us prove general results about η-like computable linear orderings. However its proof suggests that we might be able use the properties of $\mathbf{0}'$-limitwise monotonic functions. Proposition 3.10 confirms that this is indeed the case for an important subclass of η-like computable linear orderings.

Proposition 3.10 ([Fro12, Har14a])**.** *Suppose that $\mathscr{A}$ is a computable linear ordering satisfying either of the following conditions.*

(1) *$\mathscr{A}$ is strongly η-like.*
(2) *$\mathscr{A}$ is η-like but has no strongly η-like interval.*

Then there exists $\mathbf{0}'$-limitwise monotonic $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ such that $\mathscr{A}$ has order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$.

Roughly speaking this gives us a *lower bound* for the domain of application of our main result below.

Theorem 3.11. *Suppose that $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ is $\mathbf{0}'$-limitwise monotonic and that $\mathscr{B}$ is a computable linear ordering of order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$ containing no interval of order type η. Then there exists computable $\mathscr{L} \cong \mathscr{B}$ such that $\mathscr{L}$ is Π^0_1-rigid.*

Proof. Under the above assumptions, it follows from Lemma 3.6 that we can define a function $\widehat{F} : \mathbb{Q} \times \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that $\widehat{F}$ is computable, and

$$F(q) \ = \ \liminf_{s \to \infty} \widehat{F}(q, s)$$

for all $q \in \mathbb{Q}$. We assume $q_0, q_1, q_2, \ldots$ to be the fixed computable listing of $\mathbb{Q}$ determined by the bijection $Q_{\mathbb{N}} : \mathbb{N} \to \mathbb{Q}$ specified on page 2. For clarity, we usually use $<_{\mathbb{N}}$ and $<_{\mathbb{Q}}$ for the respective standard orderings of these sets.

Our aim is to construct a computable linear ordering $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ with domain $L = \mathbb{N}$ arranged as an infinite dense sum of maximal blocks $\sum\{\, I(n) \ | \ q_n \in \mathbb{Q} \,\}$ such that, for all $n \geq 0$, $F(q_n) = |I(n)|$. Note that this notation—which we use as shorthand for $\sum\{\, I(Q_{\mathbb{N}}^{-1}(q)) \ | \ q \in \mathbb{Q} \,\}$—means that that $I(n)$ is ordered relative to $\{\, I(k) \ | \ k \neq n \,\}$ as q_n is ordered relative to $\{\, q_k \ | \ k \neq n \,\}$. We will proceed by stages $s \geq 0$ defining a finite linear ordering $\mathscr{L}_s = \langle L_s, <^s_{\mathscr{L}} \rangle$ at stage s such that, for some $n_s, r_s \geq 0$, $L_s = \mathbb{N} \restriction n_s + 1$ and such that $\mathscr{L}_s$ is arranged as a finite sum of blocks $\sum\{\, I(n, s) \ | \ q_n \in \mathbb{Q} \ \& \ n \leq r_s \,\}$ where, for all $n \leq r_s$, $I(n, s)$ is the s stage approximation to maximal block $I(n)$. We say that n is the *label* of $I(n, s)$ and use this terminology quite generally in order to distinguish this use of $\mathbb{N}$ from our use of $\mathbb{N}$ as the domain of the linear ordering. The ordering $<^s_{\mathscr{L}}$ is defined by the internal ordering applied within each block and—as indicated by the sum notation above—the ordering between blocks dictated by $<_{\mathbb{Q}}$ over $\{\, q_n \ | \ n \leq r_s \,\}$.

We suppose that $\mathcal{F} = \{f_e\}_{e \in \mathbb{N}}$ is an upwards uniform Δ^0_2 class with associated approximation $\{f_{e,s}\}_{e,s \in \mathbb{N}}$, as defined in Definition 3.2, and such that $\mathcal{F}$ subsumes the class of Π^0_1 functions. The construction aims to satisfy for all $e \in \mathbb{N}$, the following requirements:

$$R_e \quad : \quad f_e \text{ is not a nontrivial automorphism of } \mathscr{L}\,,$$

as also the following structural requirement

$$P \quad : \quad \mathscr{L} \text{ has order type } \tau = \sum\{\, F(q) \ | \ q \in \mathbb{Q} \,\},$$

and the complexity requirement

$$C \quad : \quad \mathscr{L} \text{ is computable.}$$

Note that action taken to satisfy requirements $\{R_e\}_{e \in \mathbb{N}}$ may conflict with the construction's effort to satisfy requirement P due to the rebuilding of blocks entailed by the former. In fact, in order to satisfy P, the construction makes use of the computable approximation $\{\widehat{F}(q_n, s)\}_{n,s \in \mathbb{N}}$ to define, for all $n \geq 0$ and odd stages s, the approximation $I(n, s)$. At even stages s, on the other hand, the construction allows rebuilding of $I(n, s-1)$ for the sake of R requirements. Therefore, to satisfy P, it will suffice to make sure that, for any label n such that $F(q_n) > 1$, there are only finitely many even stages s such that $I(n, s-1)$ is rebuilt at stage s.

The construction also defines $L_s \subset L_{s+1}$, and $<^s_{\mathscr{L}} \ \subset \ <^{s+1}_{\mathscr{L}}$ for all s so ensuring that requirement C is satisfied. It is straightforward to check that the latter condition is satisfied during the construction, and for this reason we use $<_{\mathscr{L}}$ as shorthand for $<^s_{\mathscr{L}}$ throughout.

With the above observations in mind the proof below is aimed at verifying that requirements P and $\{R_e\}_{e \in \mathbb{N}}$ are satisfied. Clearly satisfaction of these requirements entails that $\mathscr{L}$ is indeed a computable linear ordering of order type τ which is Π^0_1-rigid.

Before proceeding to the construction we now give an overview of various items used within the construction itself and its verification.

1) *The Tree of Strategies.*

The set of *tree outcomes* is defined to be $\Sigma = \{\, (n,i) \mid n \in \mathbb{N} \;\&\; i \in \{-1,0,1\}\,\}$ with associated lexicographical ordering $<_\Sigma$, i.e. $(n,i) <_\Sigma (m,j)$ if $n < m$, or $n = m$ and $i < j$. We also define $\mathcal{T} = \Sigma^{<\mathbb{N}}$ and we refer to it as the *tree of strategies.* Each node $\alpha \in \mathcal{T}$ is associated with the strategy for the satisfaction of $R_{|\alpha|}$.

2) *Notation and Terminology for Strings.*

We use standard notation and terminology for strings as found for example in [Soa87]. Accordingly we use $\leq$ and $<$ ($\subseteq$ and $\subset$) to denote respectively nonstrict and strict length lexicographical ordering (inclusion[4]) on $\mathcal{T}$. $\sigma <_{\text{lex}} \tau$ denotes $\sigma < \tau$ but $\sigma \not\subseteq \tau$. λ denotes the strategy of length 0 in $\mathcal{T}$.

3) *The set of minimal block elements* $\{\, m_n \mid n \geq 0\,\}$.

For any label $n \geq 0$, there is a stage t_n such that $I(n,s) = \emptyset$ for all $s < t_n$ and such that $I(n,t) \neq \emptyset$ for all $t \geq t_n$. Moreover, $I(n,t_n)$ contains a single number m (say) such that $m \in I(n,t)$ for all $t \geq t_n$ and such that m is the least (under $<_\mathbb{N}$) number in $I(n,t)$. Accordingly we denote the minimal element of a nontrivially defined block $I(n,t)$ as m_n. Note that numbers may be added to $I(n,t)$ at stages $t > t_n$. However if a number p is removed from the block $I(n,t)$ at such a stage t it is then placed in a newly defined block $I(\hat{n},t)$—such that either $I(\hat{n},t) = \{p\}$ or $I(\hat{n},t) = \{p \; S \; p'\}$ for some $S \in \{<_\mathscr{L}, >_\mathscr{L}\}$ and new $p' >_\mathbb{N} p$—as this block's least number (so that $m_{\hat{n}} = p$). In other words numbers may change blocks at most once. These facts are easily verified from the construction.

4) *Parameters for Stage s.*

• For even s, the *rebuild set* $RS(s)$ contains a label $d \geq 0$ such that the block $I(n,s)$ was rebuilt at stage s, if such a label exists. Otherwise $RS(s) = \emptyset$. Note that at most one such block is rebuilt at any even stage s and that this rebuilding is carried out by the last strategy α to be processed at this stage (since α then records outcome $R(\alpha,s) \in \{\text{udb},\text{ddb}\}$ which terminates the stage).

• $\beta_s \in \mathcal{T}$ denotes the *true path at stage s* in that a strategy γ is processed at stage s if and only if $\gamma \subset \beta_s$ (but β_s itself is not processed).

Notation. We say that stage s is α*-true* if $\alpha \subset \beta_s$.

5) *Parameters for strategy* $\alpha \subset \beta_s$*, of length* $|\alpha| = e$ *and* $s > 0$.

• $R(\alpha,s) \in \{\text{vd},\text{wb},\text{wt},\text{ud},\text{udb},\text{ddb}\}$ is the outcome parameter[5] for α. $R(\alpha,s) =$ vd indicates that α is in its initial state. $R(\alpha,s) =$ wb indicates that α believes that it is not on the true path and terminates stage s. $R(\alpha,s) \in \{\text{wt},\text{ud}\}$ corresponds to different states of α's assessment that f_e is not a nontrivial automorphism of $\mathscr{L}$, and $R(\alpha,s) \in \{\text{udb},\text{ddb}\}$ if α rebuilds a block at stage s, and terminates the stage.

• $l(\alpha,s) \in \mathbb{N}$ points to the last α-true stage $t < s$ such that α has not been initialised

[4]For inclusion, $\subset$ is only used when strictness is important.

[5]The outcome vd corresponds to *void.* For the remaining outcomes, the associations here are as follows. (i) wb - *wait break* (ii) wt - *wait* (iii) ud - *upwards diagonalisation* (iv) udb - *upwards diagonalisation break* (v) ddb - *downward diagonalisation break.*

at any stage $t < r < s$. ($l(\alpha, s) = 0$ if no such stage exists.)

• $b(\alpha, s) \in \{\, n \mid n \geq |\alpha| \,\}$ points to a label bounding all the restraints of higher priority strategies. α only processes blocks $I(n, s-1)$ such that $n >_{\mathbb{N}} b(\alpha, s)$. Thus $b(\alpha, s)$ acts as a *threshold* for α's activity. Note that $b(\alpha, s) \neq b(\alpha, s-1)$ only when $R(\alpha, s-1) = \mathrm{vd}$. (If $R(\alpha, s-1) \neq \mathrm{vd}$ then by construction $b(\alpha, s) = b(\alpha, s-1) = b(\alpha, l(\alpha, s))$.)

• The *injury set* $IS(\alpha, s)$ is the set of labels n such that $I(n, s-1)$ has been rebuilt by some strategy β since $l(\alpha, s)$. (If s is not α-true or $l(\alpha, s) = 0$ then $IS(\alpha, s)$ is not used.) The actual purpose of $IS(\alpha, s)$ is to indicate injury to α by any node $\beta \subset \alpha$ that has carried out rebuilding activity at some stage $l(\alpha, s) < t < s$ (i.e. such that $R(\beta, t) \in \{\mathrm{udb}, \mathrm{ddb}\}$). In this case, for some labels $n \leq_{\mathbb{N}} m$, $\beta^\frown\langle(n, 0)\rangle \subseteq \alpha$ whereas $\beta_t = \beta^\frown\langle(m, 1)\rangle$. This means (by definition of $b(\beta, t)$) that α's restraint $d(\alpha, t-1)$ is not taken into account in the threshold of β—so that β's rebuilding activity might injure the previous activity of α. However, since this injury is recorded in $IS(\alpha, s)$ strategy α is able to take remedial action at stage s.

• $QQ(\alpha, s) \subseteq \{\, n \mid n > b(\alpha, s) \,\}$ is a finite set of labels ordered as a queue with ordering $<^s_{QQ}$. A label n is added to the back of the queue if α guesses that $F(q_n) > 1$. Likewise a label p will be removed from the queue if α guesses that $F(q_p) = 1$.

Notation. We say that label $a >_{\mathbb{N}} b(\alpha, s)$ (or the block labelled by a) is an *apparent diagonalisation candidate* for strategy α at stage s, if α believes that $F(q_a) > 1$ (so that $a \in QQ(\alpha, s)$) and $f_{|\alpha|}(m_a) \neq m_a$.

• $L(\alpha, s)$ is a finite set of apparent diagonalisation candidates chosen from $QQ(\alpha, s)$, provided that $R(\alpha, s) \neq \mathrm{vd}$. ($QQ(\alpha, s) = \emptyset$ by definition when $R(\alpha, s) = \mathrm{vd}$.)

• $a(\alpha, s) \in \{\, n \mid n \geq_{\mathbb{N}} b(\alpha, s) \,\} \cup \{-1\}$, when nontrivially defined (i.e. $\neq -1$), denotes a maximal label of $L(\alpha, s)$ under the queue ordering $<^s_{QQ}$ if $L(\alpha, s) \neq \emptyset$, and otherwise denotes $b(\alpha, s)$. $a(\alpha, s)$ is used by α as a focus of its diagonalisation activity and in defining its restraint.

• The *Diagonalisation Triple* set $DT(\alpha, s)$ contains at most one triple of the form (x, y, n) where n is a label. We say that α has *imposed a diagonal condition* on n in this case. This triple is in effect a snapshot of an *upwards diagonalisation* (see Case 6 of the construction) performed by α and is used to check whether the diagonalisation is still in place. The *diagonalisation restraint* $DR(s)$ is a set, which is nonempty precisely when $DT(\alpha, s) \neq \emptyset$ and in this case contains the (unique) label n on which α has imposed a diagonal condition. The role of $DR(\alpha, s)$ is to protect the rebuilding activity of α from the rebuilding activity of other strategies.

• $d(\alpha, s) \in \{\, n \mid n \geq_{\mathbb{N}} b(\alpha, s) \,\} \cup \{-1\}$ is a label used as a bound for α's restraint to protect α's activity from injury by lower priority strategies. Accordingly $d(\alpha, s) = a(\alpha, s) = b(\alpha, s)$ if $L(\alpha, s) = \emptyset$, whereas $d(\alpha, s)$ is defined to be $\max\{\, d \mid d \in QQ(\alpha, s) \ \&\ d \leq^s_{QQ} a(\alpha, s) \,\} \cup DR(\alpha, s)$ otherwise.

5) *Salient features of the activity of Strategy α.*

Each strategy α such that $|\alpha| = e$ is equipped with a basic module—which, for simplicity we identify with the *activity* of α itself—whose objective is the satisfaction of R_e under the constraint that requirement P must be satisfied by the overall construction. For the satisfaction of R_e the activity of α is directed at either checking that f_e has properties incompatible with a nontrivial automorphism of $\mathscr{L}$ or otherwise rebuilding blocks to prevent f_e from becoming such. Also, in order to contribute to the satisfaction of P, α can only rebuild a block $I(d,s)$ if $d > |\alpha|$, and moreover can rebuild $I(d,s)$ during at most finitely many stages s.

Note. For the sake of the present discussion we call a label n *good* if $F(q_n) > 1$ and we call it *bad* otherwise. Note[6] that, if f_e is a nontrivial automorphism of $\mathscr{L}$, then there are infinitely many good labels a such that f_e maps maximal block $I(a)$ on to some maximal block $I(d)$ such that $d \neq a$.

Strategy α acts nontrivially at stage s if it believes that it is on the true path of the construction. This happens provided (i) that there is at least one α-true stage $t < s$ and (ii) that since the last α-true stage there has been no γ-true stage for any $\gamma <_{\text{lex}} \alpha$. α's activity is defined so as to succeed if assumptions $A1$ and $A2$ below are fulfilled relative to α.

> *A1.* There are infinitely many α-true stages t. Also α is the least strategy of length $|\alpha|$ satisfying this condition.
>
> *A2.* For any label d such that $F(q_d) > 1$, and any strategy $\gamma \subset \alpha$. There are at most finitely many stages s at which γ rebuilds the block $I(d,s)$.

In our present discussion we shall assume that these assumptions are satisfied, and moreover that the stages s involved are large enough so that $\beta_s > \alpha$. Note that under this latter condition we can assume that both (i) and (ii) hold.

α defines a threshold label $b(\alpha,s)$ which is an upper bound on the labels that higher priority strategies are at present trying to protect from being rebuilt. Note that, under our assumption on s, $b(\alpha,s) = b(\alpha, l(\alpha,s))$ where $l(\alpha,s)$ is the last α-true stage. By assessing the evolution of the approximation $\widehat{F}$ since $l(\alpha,s)$, α forms a queue of labels $QQ(\alpha,s) \subseteq \{\, n \mid n >_{\mathbb{N}} b(\alpha,s)\,\}$ that appear to be good. Labels that appear to be bad under this assessment are removed from the queue and, at a later stage t, can only re-enter $QQ(\alpha,t)$ via the back of the queue. In this way, due to the fact that—as $\mathscr{B}$ contains no interval of order type η—there are infinitely many good labels, $QQ(\alpha,t)$ will grow (*inf* wise) over the set of stages $\{\, t \mid t \geq s\,\}$, so that any bad label that enters $QQ(\alpha,t)$ at infinitely many stages t will be pushed back further and further from the front of the queue.

α also builds a list $L(\alpha,s)$ of labels that appear to be good—i.e. $L(\alpha,s) \subseteq QQ(\alpha,s)$—such that also, for any $a \in L(\alpha,s)$, f_e appears to act nontrivially over $I(a)$. α makes this assessment by watching whether or not $f_{e,s}(m_a) \neq m_a$. (Notice that, as $\{f_{e,s}\}_{e,s\in\mathbb{N}}$ is upwards uniform Δ^0_2, if either $f_e(m_a)\uparrow$ or $f_e(m_a)\downarrow \neq m_a$, then there will be a stage s^* such that $f_{e,s}(m_a) \neq m_a$ for all $s \geq s^*$.) If α also makes the assessement that, for some $a \in L(\alpha,s)$ and $d \in QQ(\alpha,s)$, f_e maps $I(a)$ isomorphically onto $I(d)$ then α will rebuild either $I(a,s)$ or $I(a,d)$ and then try to protect this activity from interference by other strategies. Note that α's assessment here is based on whether or not the leftmost binary block in $I(d,s-1)$ is the isomorphic image of the leftmost binary block in $I(a,s-1)$ under $f_{e,s}$. During the rebuilding process α chooses which block to rebuild according to the ordering of $QQ(\alpha,s)$: if $d <^s_{QQ} a$ then α rebuilds $I(a,s)$, whereas if $a <^s_{QQ} d$ then α rebuilds

[6]See the proof of Sublemma 19 on page 34.

$I(d,s)$. Moreover this rebuilding is defined so that, if its outcome is preserved at all later stages, then $I(d)$ is not the isomorphic image of $I(a)$ under f_e.

Note. If $L(\alpha,s) \neq \emptyset$ then $L(\alpha,s)$ satisfies the following condition. Let a be the label in $L(\alpha,s)$ that is furthest from the front of the queue $QQ(\alpha,s)$, i.e. such that $a' \leq^s_{QQ} a$ for all $a' \in L(\alpha,s)$. Then for each $a' \in L(\alpha,s)$ such that $a' <^s_{QQ} a$, α guesses[7] that $f_e(m_{a'})\downarrow$ and that there is some $d' \leq_{\mathbb{N}} b(\alpha,s)$ such that $I(d')$ is the isomorphic image[8] of $I(a')$ under f_e.

Using the above Note we will always be able to define $L(\alpha,s)$ such that $|L(\alpha,s)| \leq b(\alpha,s)+2$ (since otherwise α guesses that there are labels $a' \neq a''$ among the $b(\alpha,s)+2$ labels nearest to the front of the queue $QQ(\alpha,s)$ such that the images of $I(a')$ and $I(a'')$ under f_e coincide). More generally, if $L(\alpha,s) \neq \emptyset$, there is at most one label in $L(\alpha,s)$—namely the label that is furthest from the front of the queue $QQ(\alpha,s)$—that provides evidence at this stage that f_e is not a nontrivial automorphism of $\mathscr{L}$ (perhaps following rebuilding).

$d(\alpha,s)$ is an upper bound for $L(\alpha,s)$ and any label d whose block has been rebuilt by α at some stage $t \leq s$ (and such that the outcome of this rebuilding has not since been invalidated by either the evolution of $\widehat{F}$, changes in $f_{e,s}$, or interference by strategies $\gamma \subset \alpha$). Note that there can be at most one such label d. α uses $d(\alpha,s)$ to indicate to the construction which strategy $\alpha \subset \alpha'$ of length $|\alpha|+1$ it is appropriate to process next. Also, given *any* stage t, and strategy $\alpha < \gamma$, $b(\gamma,t) > d(\alpha,t)$. In other words the definition of $d(\alpha,s)$ is crucial to ensuring that α's activity is protected from interference (i.e. rebuilding) by lower priority strategies.

Now under assumptions A1 and A2 we will be able to show—in accordance with the conditions specified above—that, not only does α's activity lead to the satisfaction of R_e but also, for any label good d, α rebuilds the block $I(d,s)$ during at most finitely many stages. Moreover, we will see that there is some fixed label d such that $d(\alpha,s)$ stabilises at d, or otherwise drops back at infinitely many stages to d. These observations underlie the way in which we will be able to verify that the *true path* (defined on page 31) in $\mathcal{T}$ exists. As a result we will be able to show that, for any good label n there are only finitely many stages s at which $I(n,s)$ can be rebuilt[9] by some/any strategy—so that the order type of $\mathscr{L}$ is indeed $\sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$—and also that R_e is satisfied for all e.

The reader will also find extensive notes in the course of the construction.

The Construction.

<u>*Stage* 0.</u>

Define $L_0 = \{0\}$ and $<^0_{\mathscr{L}} = \emptyset$. Set $\beta_0 = \lambda$; define $I(0,0) = \{0\}$, and let $I(n,0) = \emptyset$,

[7]This guess is in a sense fairly approximate, in order not to introduce further cases into the construction. However it is sufficient for the overall success of the proof.

[8]Although α cannot rebuild $I(d',s)$ in this case we might have considered rebuilding $I(a',s)$. However, α has no control over d' and so we might get the same situation arising for this pair of labels a' and d' at infinitely many stages. This can happen if d' is bad (i.e. $F(d') = 1$) but $\widehat{F}(d',s) > 1$ at infinitely many stages s. In this case α may rebuild $I(a',s)$ at infinitely many stages, despite the fact that a' is good—a situation that we need the construction to preclude.

[9]If n is a bad label some strategy on the true path may rebuild $I(n,s)$ at infinitely many stages. Thus infinite injury along the true path may arise. However this has no overall effect on the construction since $F(q_n) = |I(n)| = 1$ in this case.

for all labels $n > 0$. For each strategy $\gamma \in \mathcal{T}$ set each of γ's permanent parameters—i.e. not *IS* and l—to its trivial initial value, i.e. set $R(\gamma, 0) = \mathrm{vd}$, $h(\gamma, 0) = -1$ and $H(\gamma, 0) = \emptyset$ for each $h \in \{b, a, d\}$ and $H \in \{QQ, L, DT, DR\}$.

Notation. We say that strategy $\gamma \in \mathcal{T}$ is *initialised* at stage s if each of γ's permanent parameters is reset to its initial value at stage s.

At each stage $s+1$ we are given bounds $r_s \geq s$ and $n_s \geq s$ such that

$$r_s \;=\; \max\{\, n \mid I(n, s) \neq \emptyset \,\}$$

(and $r_s = \min\{\, n \mid I(n+1, s) = \emptyset \,\}$, i.e. $I(k, s) \neq \emptyset$, if and only if $k \leq r_s$) and

$$n_s \;=\; \max L_s$$

where L_s is a finite initial segment of $\mathbb{N}$ constituting the domain of the stage s approximation $\mathscr{L}_s = \langle L_s, <_{\mathscr{L}} \rangle$ to $\mathscr{L}$. Note that we use the notation $\mathscr{L}_s = \sum\{\, I(n, s) \mid q_n \in \mathbb{Q} \;\&\; n \leq r_s \,\}$ in order to clearly convey the way in which $\mathscr{L}_s$ is configured.

Notation. Throughout the construction the terms *least*, *greatest*, *minimal* (*min*), *maximal* (*max*) refer to the ordering $<_{\mathbb{N}}$, whereas *left* and *right* refer to $<_{\mathscr{L}}$ so that (as mentioned above) *leftmost* and *rightmost* refer to elements in $I(n, s)$ under $<_{\mathscr{L}}$ with obvious meaning. We say that a number is *new* at any given point in the construction if it has not yet been defined as part of the domain of $\mathscr{L}$. Moreover, when we choose a new number we always mean that we choose the least such. We extend this terminology in the obvious way to finite sets of numbers and apply it also to labels and finite sets of labels.

Note. The definition of the construction will ensure, not only that nontrivial blocks always preserve the same least element, but also that, for any label j and stage t, if $|I(n, t)| \geq 2$, then its least element m_n is either its leftmost or its second leftmost element. Moreover—writing this block $I(n, t)$ as either $\{k_p\}$, $\{k_1 <_{\mathscr{L}} k_p\}$, or $\{k_1 <_{\mathscr{L}} k_2 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_p\}$ depending on whether, respectively, $p = 1$, $p = 2$ or $p > 2$ where p denotes $|I(n, t)|$—we have that $k_i <_{\mathbb{N}} k_{i+1}$ for all $1 \leq i < p$ if $m_n = k_1$ and otherwise, if $m_n = k_2$, that $k_2 <_{\mathbb{N}} k_1$ and (if $p > 2$) $k_1 <_{\mathbb{N}} k_3$ and $k_i <_{\mathbb{N}} k_{i+1}$ for all $3 \leq i < p$. This means that, if $p \geq 2$ then, for all $2 \leq l < p$, the l leftmost numbers in $I(n, t)$ are also the l least numbers in $I(n, t)$.

The construction proceeds in one of two ways according as to whether $s+1$ is even or odd.

Stage $s+1$ with $s+1$ odd

Each block $I(n, s)$ such that $n \leq r_s$ is redefined so that $I(n, s+1)$ has $p_{s+1} = \min\{\widehat{F}(q_n, s), \widehat{F}(q_n, s+1)\}$ elements. Letting p_s denote[10] $|I(n, s)|$, this is done as follows.

■ If $p_{s+1} < p_s$ then $I(n, s+1)$ keeps its $<_{\mathbb{N}}$ least p_{s+1} elements[11] and sheds the rest into some waiting set S.

■ If $p_{s+1} > p_s$ on the other hand, then $I(n, s+1)$ is constructed by adding a set of $p_{s+1} - p_s$ new numbers onto the right hand side of the block with the ordering $<_{\mathscr{L}}$ of these numbers within the block corresponding to $<_{\mathbb{N}}$.

[10] For simplicity we do not explicitly denote the obvious labelling of p_s and p_{s+1} by n.

[11] Thus, if $p_{s+1} \geq 2$, the p_{s+1} leftmost elements are retained in $I(n, s+1)$. On the other hand, if $p_{s+1} = 1$, only the least element m_n is retained in $I(n, s+1)$. (See preceding Note.)

■ If $p_{s+1} = p_s$ then reset $I(n, s+1) = I(n, s)$.

Once this process has been carried out for each $n \leq r_s$ the construction has the set of waiting elements S that have been shed from the blocks to deal with. This is done by putting each of the members of S into a new singleton block and requires searching for the least[12] label $r >_{\mathbb{N}} r_s$ such that within the set $\{\, q_n \mid r_s <_{\mathbb{N}} n \leq_{\mathbb{N}} r \,\}$ there are enough rational numbers to accomodate (under $<_{\mathbb{Q}}$) the set S within the present ordering so that $<_{\mathscr{L}}$ is preserved (when we put each element of S into the singleton block $I(n, s+1)$ for some such q_n). We now set $r_{s+1} = r$ and we notice that we may have some numbers $r_s < n \leq r_{s+1}$ for which $I(n, s+1)$ is not yet defined. In this case in order to define $\mathscr{L}_{s+1}$ we take a further new set of numbers to make each such $I(n, s+1)$ into a singleton block. We now define n_{s+1} to be the greatest number used at this stage. We set $L_{s+1} = \mathbb{N}{\restriction} n_{s+1} + 1$ and we define $\mathscr{L}_{s+1} = \sum \{\, I(n, s+1) \mid q_n \in \mathbb{Q} \;\&\; n \leq r_{s+1} \,\}$. For labels $n > r_{s+1}$ we reset $I(n, s+1) = \emptyset$.

Notation. If $s+1$ is an odd stage and $n \leq r_s$ we say that the block $I(n, s)$ is *reconstructed* at stage $s+1$.

Stage $s+1$ with $s+1$ even.

There are at most s substages. At each such substage $e+1$ (for $e \geq 0$) some strategy $\alpha \in \mathcal{T}$ acts. In so doing, α decides the value of its local parameters and (accordingly), whether to break stage $s+1$ prematurely (i.e. when $R(\alpha, s+1) \in \{\mathrm{wb}, \mathrm{udb}, \mathrm{ddb}\}$), or otherwise which strategy $\alpha^\frown\langle(n, 0)\rangle$ (where $n = d(\alpha, s+1)$) will be *eligible to act next.*

Substage $e+1$. (Under the assumption that stage $s+1$ has not already terminated.)

We suppose that α is the strategy of length e which is eligible to act at this substage. Strategy α begins by some initial parameter resetting before processing at least one of Cases 1-11 and then proceeding to *Ending substage* $e+1$.

Initial Parameter Resetting.

- Define the *injury set* of labels

$$IS(\alpha, s+1) \;=\; \bigcup \{\, RS(t) \mid l(\alpha, s+1) < t \leq s \,\}\,.$$

Remark. $l(\alpha, s+1)$ is the last true stage parameter for α. At any even stage t, if some block $I(n, t)$ was rebuilt (as defined below) at stage t, then n is contained in the rebuild set $RS(t)$.

- Define

$$b(\alpha, s+1) \;=\; \begin{cases} 0 & \text{if } \alpha = \lambda \text{ (i.e. } e = 0), \\ \max\Big(D_{\alpha, s+1} \cup \{|\alpha|\} \Big) & \text{otherwise.} \end{cases}$$

where[13]

$$D_{\alpha, s+1} =_{\mathrm{def}} \{\, d(\beta, s) \mid \beta <_{\mathrm{lex}} \alpha \,\} \cup \{\, d(\gamma, s+1) \mid \gamma \subset \alpha \,\}$$

Remark. Notice that $b(\alpha, s+1)$ is defined in such a way that all blocks restrained by strategies of higher priority are contained in the set $\{\, I(i, s) \mid i \leq_{\mathbb{N}} b(\alpha, s+1) \,\}$.

[12] Notice that the definition of r implies that $r = r_s + 1$ if $S = \emptyset$.

[13] We can also write $D_{\alpha, s+1}$ as the set $\{d(\alpha^-, s+1)\} \cup \{\, d(\beta, s) \mid \beta <_{\mathrm{lex}} \alpha \;\&\; \alpha^- \subset \beta \,\}$ where α^- is the immediate predecessor of α, i.e. $\alpha = \alpha^{-\frown}\langle(d(\alpha^-, s+1), 0)\rangle$.

Note. As $f_e(x)\uparrow$ if and only if $\liminf_{s\to\infty} f_{e,s}(x) = \infty$ the strategy α only needs to keep a record of $f_{e,s+1}(x)$ at α-true stages $s+1$ (and not concern itself with values $f_{e,r}(x)$ for intermediate stages r).

The Cases Processed by α. Strategy α checks first whether Case 1 applies. If not it checks to see whether Case 2 applies. If Case 2(a) applies then no other Case is processed. However if either Case 2(b) applies or Case 2 does not apply, then α performs the "Search" stated on page 15 in order to select and process one of the remaining Cases.

Case 1. $R(\alpha, s) = \text{vd}$ (i.e. *void*).

Set $QQ(\alpha, s+1) = L(\alpha, s+1) = \emptyset$, $d(\alpha, s+1) = a(\alpha, s+1) = b(\alpha, s+1)$, and $R(\alpha, s+1) = \text{wb}$ (causing stage $s+1$ to terminate).

Queues, Lists, and Local Blocks. If Case 1 does not apply there are now several queue, list, and block parameters local to α that need to be redefined. Set $l = l(\alpha, s+1)$, i.e. l is the last α-true stage. Notice that, by definition, the block $I(n, l)$ is defined for all $n \leq_{\mathbb{N}} r_l$. For each $n \leq_{\mathbb{N}} r_l$ we now define

$$F^*(\alpha, q_n, s+1) \;=\; \min\{\, \widehat{F}(q_n, t) \mid l \leq t \leq s \,\}$$

and we define the (at most binary) block $B(\alpha, n, s+1) \subseteq I(n, s)$ to be one of the following.

(1) The leftmost block of two elements in $I(n, s)$ if $F^*(\alpha, q_n, s+1) > 1$.
(2) The singleton block containing the minimal element m_n in $I(n, s)$ otherwise, i.e. if $F^*(\alpha, q_n, s+1) = 1$.

Remark. $b(\alpha, s+1) = b(\alpha, l)$ here as $R(\alpha, s) \neq \text{vd}$.

We are given at this stage a queue

$$QQ(\alpha, s) \;\subseteq\; \{\, n \mid b(\alpha, s+1) <_{\mathbb{N}} n \leq_{\mathbb{N}} r_l \,\}$$

with associated queue ordering $\leq^s_{QQ}$ and a list of apparent diagonalisation candidates[14] $L(\alpha, s) \subseteq QQ(\alpha, s)$. We now redefine the queue as follows. Let R denote the *rogue set* $\{\, n \mid n \in QQ(\alpha, s) \;\&\; |B(\alpha, n, s+1)| = 1 \,\}$. In other words R contains the labels of blocks in $QQ(\alpha, s)$ that now appear to be singleton blocks. Let G denote the *good set*

$$\{\, n \mid n \notin QQ(\alpha, s) \;\&\; b(\alpha, s+1) <_{\mathbb{N}} n \leq_{\mathbb{N}} r_l \;\&\; |B(\alpha, n, s+1)| = 2 \,\}.$$

I.e. G is the set of labels in $\{\, n \mid b(\alpha, s+1) <_{\mathbb{N}} n \leq_{\mathbb{N}} r_l \,\}$ that now appear to represent nonsingleton blocks. Now define $QQ(\alpha, s+1)$ to be $QQ(\alpha, s)$ with the rogue set R removed, and the good set G ordered by $<_{\mathbb{N}}$ added to the back of the queue. Note that this definition implies that the $\leq^s_{QQ}$ ordering of $QQ(\alpha, s) \setminus R$ is preserved within $QQ(\alpha, s+1)$ whereas $a <^{s+1}_{QQ} b$ if $a \in QQ(\alpha, s) \setminus R$ and $b \in G$ or if $a, b \in G$ and $a <_{\mathbb{N}} b$.

Notation. For $a, b \in QQ(\alpha, s)$ we say that a has *lesser QQ-rank* than b if $a <^s_{QQ} b$ and we extend this terminology in the standard manner. In other words the head of the queue has *least* QQ-rank and the back of the queue has *greatest* QQ-rank.

[14] More precisely, the set $L(\alpha, s) = L(\alpha, l)$ where $l = l(\alpha, s+1)$ and the labels in $L(\alpha, l)$ were observed by α to be apparent diagonalisation candidates at stage l.

Case 2. At least one of the following conditions holds.

• $L(\alpha, s) \not\subseteq QQ(\alpha, s+1)$. I.e. there is at least one block labelled by a member of $L(\alpha, s)$ that now appears to α to be a singleton.

• $IS(\alpha, s+1) \cap L(\alpha, s) \neq \emptyset$. I.e. some block labelled by a member of $L(\alpha, s)$ has been rebuilt since the last α-true stage.

• For some $d \in L(\alpha, s)$, $f_{e,s+1}(m_d) = m_d$.

Then define $L^*(\alpha, s)$ to be the set of labels:

$$\begin{aligned}\{\, b \mid b \in L(\alpha, s) \;\&\; (\forall d \in L(\alpha, s))[\, d \leq^s_{QQ} b \;\Rightarrow\; & d \in QQ(\alpha, s+1) \\ & \&\; d \notin IS(\alpha, s+1) \\ & \&\; f_{e,s+1}(m_d) \neq m_d \,]\,\}.\end{aligned}$$

There are now two cases as follows.

Case 2(*a*). $L^*(\alpha, s) = \emptyset$. Then define $L(\alpha, s+1) = \emptyset$ and $a(\alpha, s+1) = b(\alpha, s+1)$. Set $R(\alpha, s+1) = \text{wt}$.

Case 2(*b*). Otherwise perform the "Search" below and process whichever of Cases 3-9 or 11 is thus designated.

Remark. Note that, if $L^*(\alpha, s) \neq \emptyset$ then, for some $a^* \in L(\alpha, s)$, $L^*(\alpha, s) = \{\, d \mid d \in L(\alpha, s) \;\&\; d \leq^{s+1}_{QQ} a^* \,\}$. Also, by definition, $L^*(\alpha, s) \subseteq QQ(\alpha, s+1)$, $IS(\alpha, s+1) \cap L^*(\alpha, s) = \emptyset$ and $f_{e,s+1}(m_d) \neq m_d$, for all $d \in L^*(\alpha, s)$.

$\square$ *Notes. Strategy α wants to keep $d \in L(\alpha, s+1)$ only if (i) it appears that $|I(d)| > 1$, (ii) the approximation to $I(d)$ has not been rebuilt by another strategy since the last α-true stage, and (iii) it appears that $f_e(m_d) \neq m_d$. Use of Case* 2 *ensures that these conditions are satisfied by every $d \in L(\alpha, s+1)$.*

Notation. If $d \in L(\alpha, s) \setminus L^*(\alpha, s)$ we say that d is *removed from* $L(\alpha, s+1)$ *via Case* 2. We also say (during the Verification) that *Case* 2 *applies* in this case.

Search. If Case 2 did not apply set $L^*(\alpha, s) = L(\alpha, s)$. α tests whether there exists a label $a \in L^*(\alpha, s)$ satisfying one of Cases 3-9 below or otherwise $a \in QQ(\alpha, s+1)$ satisfying Case 10. If there is such a, α chooses a of least QQ-rank and processes the first Case to apply to a. If there is no such a, α processes Case 11. Note that by definition α will only process some a via Case 10 if (i) $L^*(\alpha, s) = L(\alpha, s)$ and (ii) Cases 3-9 fail for *all* $a \in L^*(\alpha, s)$.

Case 3. $a \in L^*(\alpha, s)$ and $f_{e,s+1}(m_a) \neq f_{e,l(\alpha,s+1)}(m_a)$. There are two cases.

Case 3(*a*). $R(\alpha, s) \in \{\text{udb}, \text{ud}\}$ and $a = a(\alpha, s)$ $\big(= a(\alpha, l(\alpha, s+1))\big)$. Then process a via Case 5.

Case 3(*b*). Otherwise. Set $a(\alpha, s+1) = a$, $R(\alpha, s+1) = \text{wt}$, and define $L(\alpha, s+1) = \{\, n \mid n \in L^*(\alpha, s) \;\&\; n \leq^{s+1}_{QQ} a \,\}$,

$\square$ *Notes. α guesses that $f_e(m_a)\uparrow$. However α also wants to preserve any remaining valid diagonal condition that it has imposed for the sake of a at a previous stage, and hence redirects the processing to Case* 5 *if necessary. (In the case when α processes some label a infinitely often via Case* 3*, this way of proceeding is necessary to prevent α from rebuilding a single block $I(d, s)$ at infinitely many stages s.)*

Notation. For any labels k, l we use $f_{e,s+1} : B(\alpha, k, s+1) \cong B(\alpha, l, s+1)$ to denote that $B(\alpha, l, s+1)$ is the isomorphic image of $B(\alpha, k, s+1)$ under $f_{e,s+1}$.

Case 4. $a \in L^*(\alpha, s)$ and, for some $d \in QQ(\alpha, s+1)$, $f_{e,s+1}(m_a) \in B(\alpha, d, s+1)$ but $f_{e,s+1} : B(\alpha, a, s+1) \ncong B(\alpha, d, s+1)$. There are two cases.

Case 4(a). $R(\alpha, s) \in \{\text{udb}, \text{ud}\}$ and $a = a(\alpha, s)$. Then process a via Case 5.

Case 4(b). Otherwise. Set $a(\alpha, s+1) = a$, $R(\alpha, s+1) = \text{wt}$, and define $L(\alpha, s+1) = \{\, n \mid n \in L^*(\alpha, s) \ \&\ n \leq_{QQ}^{s+1} a \,\}$,

□ *Notes. α guesses that the blocks labelled by a and d witness that f_e is not an automorphism. Again α wants to preserve any remaining valid diagonal condition that it has imposed for the sake of a at a previous stage, and redirects the processing to Case 5 if necessary.*

Case 5. $a \in L^*(\alpha, s)$ and both $a = a(\alpha, s)$—so that $L^*(\alpha, s) = L(\alpha, s)$—and $R(\alpha, s) \in \{\text{ud}, \text{udb}\}$. Accordingly $DT(\alpha, s) = \{(x, m_d, d)\}$ and $DR(\alpha, s) = \{d\}$ for some $x \in L_s$ and label d.

□ *Notes. This means that, for some stage $\hat{s} < s+1$, a received attention via Case 6(b) where, by definition $x \in B(\alpha, a, \hat{s})$, $f_{e,\hat{s}}(x) = m_d$ and $I(d, \hat{s})$ was rebuilt at stage $\hat{s}$. The construction ensures that we also have $B(\alpha, a, s+1) = B(\alpha, a, \hat{s})$ in this Case so that $x \in B(\alpha, a, s+1)$.*

Proceed by carrying out the following.

(i) Check that $d \in QQ(\alpha, s+1)$.
(ii) Check that $f_{e,s+1}(x) = m_d$.
(iii) Check that $d \notin IS(\alpha, s+1)$. (If $d \in IS(\alpha, s+1)$ then the block labelled by d has been rebuilt by some different strategy γ since the last α-true stage.)

If all three tests succeed, reset $DT(\alpha, s+1) = DT(\alpha, s)$ and $DR(\alpha, s+1) = DR(\alpha, s)$. Otherwise set $DT(\alpha, s+1) = DR(\alpha, s+1) = \emptyset$.

Reset $a(\alpha, s+1) = a$ and[15] $L(\alpha, s+1) = L(\alpha, s)$. There are now two cases.

Case 5(a). $DR(\alpha, s+1) = \emptyset$. Then set $R(\alpha, s+1) = \text{wt}$.

Case 5(b). Otherwise set $R(\alpha, s+1) = \text{ud}$.

□ *Notes. If $DR(\alpha, s+1) \neq \emptyset$ then the previous (most recent) upwards diagonalisation performed by strategy α is preserved and is restrained against injury from lower priority strategies by the definition of $d(\alpha, s+1)$ (on page 19).*

Case 6. $a \in L^*(\alpha, s)$ and for some $d \in QQ(\alpha, s+1)$, such that $a <_{QQ}^{s+1} d$, it is the case that: $f_{e,s+1} : B(\alpha, a, s+1) \cong B(\alpha, d, s+1)$. There are two cases.

Case 6(a). $a \neq a(\alpha, s)$.

Define $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = \{\, n \mid n \in L^*(\alpha, s) \ \&\ n \leq_{QQ}^{s+1} a \,\}$ and set $R(\alpha, s+1) = \text{wt}$.

□ *Notes. If $DR(\alpha, s) \neq \emptyset$ then $R(\alpha, s) \in \{\text{ud}, \text{udb}\}$. So if $a = a(\alpha, s)$, then a will be processed via Case 5 and* not *Case 6. Accordingly the purpose of Case 6(a) is to ensure that, even when the conditions of Case 6 apply, if $DR(\alpha, s) \neq \emptyset$, then $DR(\alpha, s+1) = \emptyset$. (See "*Ending substage $e+1$*".)*

[15] $L(\alpha, s) = \{\, n \mid n \in L(\alpha, s) \ \&\ n \leq_{QQ}^{s+1} a \,\}$ in this Case. (The same observation applies to Case 6(b) below.)

Case $6(b)$. Otherwise, i.e. $a = a(\alpha, s)$, so that $L^*(\alpha, s) = L(\alpha, s)$. Then proceed via the following rebuilding process before resetting the local parameters.

Diagonal Rebuilding of $I(d, s)$. Suppose that $I(a, s) = \{x_1 <_{\mathscr{L}} x_2 <_{\mathscr{L}} \cdots <_{\mathscr{L}} x_l\}$ and $I(d, s) = \{y_1 <_{\mathscr{L}} y_2 <_{\mathscr{L}} \cdots <_{\mathscr{L}} y_p\}$. (I.e. $|I(a, s)| = l \geq 2$ and $|I(d, s)| = p \geq 2$, $B(\alpha, a, s+1) = \{x_1 <_{\mathscr{L}} x_2\}$, $B(\alpha, d, s+1) = \{y_1 <_{\mathscr{L}} y_2\}$ and $f_{e,s+1}(x_k) = y_k$ for $1 \leq k \leq 2$.) Choose the least $r > r_s$ such that the set $\{\, q_n \mid r_s <_{\mathbb{N}} n \leq_{\mathbb{N}} r \,\}$ contains a subset U of cardinality $p - 1$ satisfying the following conditions.

(i) Each $q \in U$ is ordered as q_d relative to the set $\{\, q_n \mid n \leq_{\mathbb{N}} r_s \,\} \setminus \{q_d\}$
(ii) $U \cup \{q_d\}$ has ordering $q_{i_1} <_{\mathbb{Q}} q_{i_2} <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{i_p}$, where $d = i_1$ if $y_1 = m_d$ and $d = i_2$ if $y_2 = m_d$. (The purpose of this condition is to preserve m_d in $I(d, s+1)$.)

Let $\hat{m} = n_s + 1$. (I.e. $\hat{m}$ is new.) Define

$$I(d, s+1) \;=\begin{cases} \{\hat{m} <_{\mathscr{L}} m_d\} & \text{if } y_1 = m_d, \\ \{m_d <_{\mathscr{L}} \hat{m}\} & \text{if } y_2 = m_d. \end{cases}$$

Now, using the set $\{q_{i_1}, \ldots, q_{i_p}\} \setminus \{q_d\}$, for each $1 \leq k \leq p$ such that $i_k \neq d$ define $I(i_k, s+1) = \{y_k\}$, i.e. the singleton block containing y_k.

Remark. Note here that $y_k = m_{i_k}$ for $1 \leq k \leq p$—so that y_k remains in $I(i_k, t)$ for all $t \geq s + 1$.

To finish the rebuilding process let V be the subset of rational numbers in $\{\, q_n \mid r_s <_{\mathbb{N}} n \leq_{\mathbb{N}} r \,\}$ that have not been used so far. Letting $\hat{n} = |V|$ use the set of new numbers $n_s+2, \ldots, n_s+\hat{n}+1$ to define $I(n, s+1)$ as a singleton block for each such n. Finally define $r_{s+1} = r$, $n_{s+1} = n_s + \hat{n} + 1$ and the *rebuild set* $RS(s+1) = \{d\}$.

Using the notation $B(\alpha, a, s+1) = \{x_1 <_{\mathscr{L}} x_2\}$ from above, define $DT(\alpha, s+1) = \{(x_j, m_d, d)\}$ where $1 \leq j \leq 2$ is the index such that $f_{e,s+1}(x_j) = m_d$. Define the diagonal restraint $DR(\alpha, s+1) = \{d\}$.

Define $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = L(\alpha, s)$. Set $R(\alpha, s+1) = \text{udb}$ (causing stage stage $s+1$ to terminate).

$\square$ *Notes. During the process of* "Ending Stage $s+1$" *below we reset $I(a, s+1) = I(a, s)$. Thus the leftmost binary block in $I(a, s+1)$ is $\{x_1 <_{\mathscr{L}} x_2\}$ whereas the construction in this case means that $I(d, s+1) = \{\hat{m} <_{\mathscr{L}} m_d\}$ if $f_{e,s+1}(x_1) = m_d$ and $I(d, s+1) = \{m_d <_{\mathscr{L}} \hat{m}\}$ if $f_{e,s+1}(x_2) = m_d$. Hence $I(d, s+1)$ cannot be the isomorphic image of $I(a, s+1)$ under $f_{e,s+1}$. Moreover—letting $1 \leq j \leq 2$ be such that $f_{e,s+1}(x_j) = m_d$—if, for all $t \geq s+1$, $f_{e,t}(x_j) = f_{e,s+1}(x_j)$, and the leftmost binary blocks of both $I(a, t)$ and $I(d, t)$ are preserved (i.e. are neither reconstructed nor rebuilt), then $f_e(m_a) \in I(d)$ but $I(d)$ is not the isomorphic image of $I(a)$ under f_e.*

Case 7. $a \in L^*(\alpha, s)$ and, for some $d \in QQ(\alpha, s+1)$ such that $d <_{QQ}^{s+1} a$, $f_{e,s+1} : B(\alpha, a, s+1) \cong B(\alpha, d, s+1)$.

Then rebuild $I(a, s)$ by swapping the roles of labels d and a in the *Diagonal Rebuilding* of Case 6. This means—using the same notation as in Case 6—that we define

$$I(a, s+1) \;=\begin{cases} \{\hat{m} <_{\mathscr{L}} m_a\} & \text{if } x_1 = m_a, \\ \{m_a <_{\mathscr{L}} \hat{m}\} & \text{if } x_2 = m_a. \end{cases}$$

Finish the building process in a similar way to Case 6 and so define r_{s+1} and n_{s+1}. Also define the rebuild set $RS(s+1) = \{a\}$.

Define $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = \{\, n \mid n \in L^*(\alpha, s) \;\&\; n \leq_{QQ}^{s+1} a \,\}$. Set $R(\alpha, s+1) = \mathrm{ddb}$ (causing stage stage $s+1$ to terminate).

□ *Notes. During the process of* "Ending Stage $s+1$" *we will reset* $I(d, s+1) = I(d, s)$. *Thus the leftmost binary block in* $I(d, s+1)$ *is* $\{y_1 <_{\mathscr{L}} y_2\}$ *whereas the construction in this case means that* $I(a, s+1) = \{\hat{m} <_{\mathscr{L}} m_a\}$ *if* $f_{e,s+1}(m_a) = y_1$ *and* $I(a, s+1) = \{m_a <_{\mathscr{L}} \hat{m}\}$ *if* $f_{e,s+1}(m_a) = y_2$. *Hence,* $I(d, s+1)$ *cannot be the isomorphic image of* $I(a, s+1)$ *under* $f_{e,s+1}$. *Moreover if, for all* $t \geq s+1$, $f_{e,t}(m_a) = f_{e,s+1}(m_a)$, *and the leftmost binary blocks of both* $I(a, t)$ *and* $I(d, t)$ *are preserved, then* $f_e(m_a) \in I(d)$ *but* $I(d)$ *is not the isomorphic image of* $I(a)$ *under* f_e. *Note that while processing* "Ending substage $e+1$" *we will set* $DT(\alpha, s+1) = DR(\alpha, s+1) = \emptyset$ *in this case as* $R(\alpha, s+1) = \mathrm{ddb}$. *Nevertheless, as* $d <_{QQ}^{s+1} a$, $d(\alpha, s+1)$ *is defined as an upper bound for* d, *as well as for* a, *so that both of these labels are protected from rebuilding by lower priority strategies.*

<u>Case 8.</u> $a \in L^*(\alpha, s)$ and (i) for some $d \leq_{\mathbb{N}} b(\alpha, s+1)$, $f_{e,s+1}(m_a) \in B(\alpha, d, s+1)$ and (ii) for some $\hat{a} \in L^*(\alpha, s)$ such that $\hat{a} <_{QQ}^{s+1} a$, $f_{e,s+1}(m_{\hat{a}}) \in B(\alpha, d, s+1)$.

Then define $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = \{\, n \mid n \in L^*(\alpha, s) \;\&\; n \leq_{QQ}^{s+1} a \,\}$. Set $R(\alpha, s+1) = \mathrm{wt}$.

□ *Notes. The only case in which, for any label* $a \in L^*(\alpha, s)$, α *is not free to put in place a diagonalisation—appropriate to the satisfaction of* R_e*—relative to the action of* $f_{e,s+1}$ *over* $B(\alpha, a, s+1)$, *is when* $f_{e,s+1}(m_a) \in B(\alpha, d, s+1)$ *for some* $d \leq_{\mathbb{N}} b(\alpha, s+1)$. *However if* $b(\alpha, s+1) + 2$ *labels in* $L^*(\alpha, s)$ *satisfy this property then Case* 8 *will apply. Note that* α *guesses that* f_e *is not an automorphism in this case relative to the action of the latter over the blocks labelled by* a *and* $\hat{a}$.

<u>Case 9.</u> $a \in L^*(\alpha, s)$ and

$$f_{e,s+1}(m_a) \notin \bigcup \{\, B(\alpha, d, s+1) \mid d \leq_{\mathbb{N}} b(\alpha, s+1) \vee d \in QQ(\alpha, s+1) \,\}.$$

Then define $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = \{\, n \mid n \in L^*(\alpha, s) \;\&\; n \leq_{QQ}^{s+1} a \,\}$. Set $R(\alpha, s+1) = \mathrm{wt}$.

□ *Notes.* α *guesses that* f_e *is not an automorphism in that* $f_e(m_a) \in I(d)$ *for some label* d *such that, either* $|I(d)| = 1$, *or* $|I(d)| > 1$ *but* $f_e(m_a)$ *does not belong to the leftmost binary block in* $I(d)$ *(whereas* α *guesses that* $|I(a)| > 1$ *with* m_a *being, by definition, in the leftmost binary block in* $I(a)$*). Note the separation of the case* $d \leq_{\mathbb{N}} b(\alpha, s+1)$ *since* α *has no* control *(i.e. cannot rebuild) a block labelled by such* d.

<u>Case 10.</u> $L^*(\alpha, s) = L(\alpha, s)$, $a \in QQ(\alpha, s+1)$, $f_{e,s+1}(m_a) \neq m_a$ and, if it is the case that $L(\alpha, s) \neq \emptyset$, $a(\alpha, s) <_{QQ}^{s+1} a$ and $f_{e,s+1}(m_{a(\alpha,s)}) <_{\mathbb{N}} a$.

Define $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = L(\alpha, s) \cup \{a\}$. Set $R(\alpha, s+1) = \mathrm{wt}$.

□ *Notes. Strategy* α *guesses that* $|I(a)| > 1$ *and that* $f_e(m_a) \neq m_a$. *Thus the label* a *is an apparent diagonalisation candidate that* α *wants to collect into its list* $L(\alpha, s+1)$ *in order to protect* $I(a, t)$ *(and any future rebuilding for the sake of* a *via Cases* 6 *or* 7*) from rebuilding by lower priority strategies at stages* $t \geq s+1$. *(Reminder. By definition* $a(\alpha, s)$ *is the label of greatest* QQ*-rank in* $L(\alpha, s)$.*) Note that that the condition* "$f_{e,s+1}(m_{a(\alpha,s)}) <_{\mathbb{N}} a$" *is important in ensuring that, for any label* d, α *only rebuilds the block labelled by* d *finitely often. (In the verification this is used in Sublemma 12 via Sublemma 7.)*

Case 11. None of Cases 1, 2(a) or 3-10 applies. Note that $L^*(\alpha, s) = L(\alpha, s)$ in this case.

Then define $L(\alpha, s+1) = L(\alpha, s)$, $a(\alpha, s+1) = a(\alpha, s)$ and $R(\alpha, s+1) = \mathrm{wt}$. (Thus $a(\alpha, s+1) = b(\alpha, s+1)$ if $L(\alpha, s+1) = \emptyset$, and $a(\alpha, s+1)$ is the label of maximal QQ-rank in $L(\alpha, s+1)$ otherwise.)

Ending substage $e+1$. If $R(\alpha, s+1) \notin \{\mathrm{ud}, \mathrm{udb}\}$—i.e. if either $L(\alpha, s+1) = \emptyset$, or $L(\alpha, s+1) \neq \emptyset$ and $a(\alpha, s+1)$ was not processed via Case 5(b) or Case 6(b)—then set $DT(\alpha, s+1) = DR(\alpha, s+1) = \emptyset$.

The Restraint. Define $d(\alpha, s+1)$ as follows.

- If $a(\alpha, s+1) = b(\alpha, s+1)$ (i.e. $L(\alpha, s+1) = \emptyset$) define $d(\alpha, s+1) = a(\alpha, s+1)$.
- Otherwise (so $a(\alpha, s+1) >_{\mathbb{N}} b(\alpha, s+1)$) define

$$d(\alpha, s+1) \;=\; \max\{\, n \mid n \in QQ(\alpha, s+1) \;\&\; n \leq_{QQ}^{s+1} a(\alpha, s+1) \,\} \cup DR(\alpha, s+1)\,.$$

□ *Notes. If $a \in L(\alpha, s+1)$ then α does not want strategies of lower priority interfering with the block labelled by a or with blocks labelled by $n \in QQ(\alpha, s+1)$ of lesser QQ-rank than a, in order to prevent its activity being overwritten by such strategies. Also the use of $DR(\alpha, s+1)$ in the definition of $d(\alpha, s+1)$ implies that α's diagonalisation activity via Case 6 is protected from interference by lower priority strategies. Note that, from a more general point of view, this use of $d(\alpha, s+1)$ will help ensure that the overall construction can only rebuild a block that we need to be nonsingleton (to ensure that $\mathscr{L}$ has the correct order type) finitely many times.*

If $R(\alpha, s+1) \in \{\mathrm{wb}, \mathrm{udb}, \mathrm{ddb}\}$ or if $e+1 = s$, set $\beta_{s+1} = \alpha^\frown\langle\big(d(\alpha, s+1), i\big)\rangle$, where

$$i \;=\; \begin{cases} -1 & \text{if } R(\alpha, s+1) = \mathrm{wb} \\ \;\;1 & \text{if } R(\alpha, s+1) \in \{\mathrm{udb}, \mathrm{ddb}\} \\ \;\;0 & \text{otherwise,} \end{cases}$$

and go to *Ending stage* $s+1$. Otherwise let $\alpha^\frown\langle\big(d(\alpha, s+1), 0\big)\rangle$ be eligible to act next and go to substage $e+2$.

Ending Stage $s+1$. Supposing that α was the last strategy to be processed there are two cases as follows.

- $\beta_{s+1} = \alpha^\frown\langle\big(d(\alpha, s+1), 1\big)\rangle$, i.e. r_{s+1} and n_{s+1} have already been defined, $RS(s+1) = \{d\}$ where $I(d, s)$ was the block rebuilt by strategy α either via Case 6 or Case 7, and, for $r_s <_{\mathbb{N}} n \leq_{\mathbb{N}} r_{s+1}$, $I(n, s+1)$ are the newly defined blocks.
- $\beta_{s+1} = \alpha^\frown\langle\big(d(\alpha, s+1), i\big)\rangle$ for some $i \in \{-1, 0\}$. In this case reset $r_{s+1} = r_s$, $n_{s+1} = n_s$ and set $RS(s+1) = \emptyset$.

In both of these cases, for all labels $n \leq r_s$ such that $n \notin RS(s+1)$ redefine $I(n, s+1) = I(n, s)$. Now define $L_{s+1} = \mathbb{N}{\restriction}n_{s+1} + 1$ and

$$\mathscr{L}_{s+1} = \sum\{\, I(n, s+1) \mid q_n \in \mathbb{Q} \;\&\; n \leq r_{s+1} \,\}\,.$$

For labels $n > r_{s+1}$ reset $I(n, s+1) = \emptyset$.

Initialise all strategies $\beta_{s+1} \leq \gamma$. (Note in particular that this means that $R(\gamma, s+1) = \mathrm{vd}$ and $DR(\gamma, s+1) = \emptyset$ for every strategy $\gamma \in \mathcal{T}$ such that $\beta_{s+1} \leq \gamma$.) For every $\gamma \in \mathcal{T}$ such that $\gamma <_{\mathrm{lex}} \beta_{s+1}$, for each of γ's permanent parameters z reset

$z(\gamma, s+1) = z(\gamma, s)$.

Proceed to stage $s+2$.

$\square$ *Notes. Suppose that* $RS(s+1) = \{d\}$*—i.e. that* $I(d, s+1)$ *was rebuilt at stage* $s+1$*— and that* α *was the last strategy processed. Then it may be the case that* $d \in L(\gamma, s+1) \cup DR(\gamma, s+1)$ *for some* $\gamma <_{\text{lex}} \beta_{s+1}$*. (In fact, supposing that* $\beta_{s+1} = \alpha^\frown\langle(n,1)\rangle$*, we see that* $\alpha^\frown\langle(m,0)\rangle \subset \gamma$ *for some label* $m \leq_{\mathbb{N}} n$ *in this case.) If* $d \in DR(\gamma, s+1)$*, this means that* γ *imposed a diagonal condition (see page 9) on* d *at some earlier stage* t *via Case* 6*(b), whereas this condition has been overwritten by* α*'s action at stage* $s+1$*. Accordingly* γ*'s diagonal condition for* d *will no longer be valid (and in fact may already have been invalidated since the last* γ*-true stage). However—supposing, without loss of generality, that* γ *is not subsequently initialised—there is no conflict here as, if* γ *is revisited at a later stage* u*, then* d *is removed from* $DR(\gamma, u)$ *via Case* 5 *(perhaps redirected via Case* 3 *or* 4*) or via the fact that some other Case applies (so that* $DR(\gamma, u) = \emptyset$*). On the other hand if* $d \in L(\gamma, s+1)$ *then, defining stage* u *as above, Case* 2 *applies at stage* u *and* d *is removed from* $L(\gamma, u)$*. Note that this mechanism helps to ensure that, if* $L(\gamma, u) = L(\gamma, l(\gamma, u))$ *and* $DR(\gamma, u) = DR(\gamma, l(\gamma, u)) = \{d'\}$*, for some label* d'*, then there is still a valid diagonal condition—signalling that* $f_{|\gamma|}$ *does not seem to be an automorphism—on* d'*.*

Verification.

The verification proceeds via a number of intermediate results dealt with after the following introductory definitions and notes.

Definition. For $e \geq 0$, δ_e is defined to be the least (under $<_{\text{lex}}$) strategy α such that $|\alpha| = e$, $\{\, t \mid \beta_t < \alpha \,\}$ is finite, and $\{\, s \mid \alpha \subset \beta_s \,\}$ is infinite, if such α exists. Otherwise δ_e is undefined.

Definition. Define the set of *good labels* to be $GL = \{\, n \mid F(q_n) > 1 \,\}$.

The first part of the verification is aimed at showing that δ_e is defined for all e and that the rebuilding activity of the construction tends to infinity (*inf* wise) over GL. To this end we choose some $e \geq 0$ and start working under the following assumption.

Assumption 1. δ_e is defined. Moreover if $e > 0$ then, for all $0 < d \leq e$, $\delta_d = \delta_{d-1}{}^\frown\langle(n_d, 0)\rangle$ for some $n_d \in \mathbb{N}$.

Definitions of s_α *and* $b(\alpha)$. Using the shorthand $\alpha = \delta_e$ we deduce from Assumption 1 that there is a stage $s_\alpha > |\alpha| = e$ such that $\alpha \subset \beta_{s_\alpha}$ (i.e. s_α is α-true) and such that, for all $s \geq s_\alpha$, $\alpha < \beta_s$, $b(\alpha, s) = b(\alpha, s_\alpha)$ and $R(\alpha, s) \neq \text{wb}$. Accordingly we define $b(\alpha) = \lim_{s\to\infty} b(\alpha, s) = b(\alpha, s_\alpha)$.

Sublemma 1. *Suppose that* $s \geq s_\alpha$ *is a stage such that* $RS(s) = \{d\}$, $d \leq_{\mathbb{N}} d(\alpha, s)$ *and* $\beta_s \neq \alpha^\frown\langle(d(\alpha, s), 1)\rangle$*. (I.e. it is not the case that* α *itself rebuilt* $I(d, s)$ *at stage* s*.) Then, for some* $m \leq_{\mathbb{N}} n$ *and strategy* γ *such that* $\gamma^\frown\langle(m,0)\rangle \subset \alpha$, $\beta_s = \gamma^\frown\langle(n,1)\rangle$*.*

Remark. In the above $n = d(\gamma, s) \geq_{\mathbb{N}} d$ by definition.

Proof. By definition, we know that $\beta_s = \gamma^\frown\langle(n', 1)\rangle$ for some label n'. Suppose firstly that s is α-true. Then, under the present hypotheses, $\alpha^\frown\langle(d(\alpha, s), 0)\rangle \subset \beta_s$. But in this case $d(\alpha, s) \leq_{\mathbb{N}} b(\gamma, s) <_{\mathbb{N}} d$. Similarly if s is not α-true and $\alpha <_{\text{lex}} \gamma$, $d(\alpha, s) \leq_{\mathbb{N}} b(\gamma, s) <_{\mathbb{N}} d$ by automatic resetting of $d(\alpha, s)$. It follows that $\gamma^\frown\langle(m,0)\rangle \subset \alpha$ for some $m \leq_{\mathbb{N}} n$. $\square$

Definition. For any strategy γ, define $\hat{t}(\gamma, s)$ as follows. $\hat{t}(\gamma, 0) = 0$ and $\hat{t}(\gamma, s+1) = \hat{t}(\gamma, s)$ if $s+1$ is not γ-true whereas, if $s+1$ is γ-true,

$$\hat{t}(\gamma, s+1) \;=\; \begin{cases} d & \text{if } d \in GL \text{ and } R(\gamma, s+1) \in \{\text{ddb}, \text{udb}\} \text{ and} \\ & I(d,s) \text{ is the block rebuilt by } \gamma\,, \\ s+1 & \text{otherwise.} \end{cases}$$

Definition. For all α-true stages $s+1 \geq s_\alpha$, define

$$t(\alpha, s+1) \;=\;$$
$$\min\,\{\, \hat{t}(\gamma, u) \mid \gamma \subset \alpha \;\&\; l(\alpha, s+1) < u \leq s \;\&\; \hat{t}(\gamma, u) >_{\mathbb{N}} b(\alpha, s+1) \,\} \cup \{s+1\}\,.$$

Also for any stage $s+1$ that is not α-true, let $t(\alpha, s+1) = t(\alpha, s)$.

Note. Sublemma 1 shows that injury to α's activity at stages $s > s_\alpha$ can only emanate from the activity of strategies $\gamma \subset \alpha$. Thus $t(\alpha, s)$ is an indicator of the injury suffered by α. Moreover $t(\alpha, s)$ being defined only over GL, indicates the level of *true* injury to α's activity in the sense that, if $n \notin GL$, then $F(q_n) = 1$ so any rebuilding of the block $I(n, s)$ simply reduces this block to the singleton $\{m_n\}$, i.e. to $I(n)$ itself.

We are now in a position to state our second assumption and Inductive Hypothesis.

Assumption 2. $\liminf_{s\to\infty} \hat{t}(\delta_d, s) = \infty$ for all $0 \leq d < e$.

Note. Assumption 2 obviously implies that $\liminf_{s\to\infty} t(\alpha, s) = \infty$.

Inductive Hypothesis. This is the conjunction of Assumptions 1-2.

For all Sublemmas 1-14 we work under the Inductive Hypothesis. Note that we continue using the shorthand $\alpha = \delta_e$ from above in what follows.

Notation. If $s+1$ is an α-true stage, then we say that the label $a \in L^*(\alpha, s)$ $(\subseteq L(\alpha, s))$ *requires attention* at stage $s+1$ via Case i for some $3 \leq i \leq 9$ if Case i is applicable to a. We say that a *receives attention* at stage $s+1$ via Case i when α in fact processes a via Case i. We use the shorthand Case $3 \to 5$ (Case $4 \to 5$) when Case 5 applies via Case 3 (Case 4).

Note 1. If $s+1$ is an α-true stage, and $a \in L^*(\alpha, s)$ receives attention at stage $s+1$, then $a(\alpha, s+1) = a$ and $L(\alpha, s+1) = \{\, d \mid d \in L^*(\alpha, s) \;\&\; d \leq^{s+1}_{QQ} a \,\} = \{\, d \mid d \in L(\alpha, s) \;\&\; d \leq^{s}_{QQ} a \,\}$ by definition of the construction.

Note 2. For all $s \geq s_\alpha$ either $a(\alpha, s) = b(\alpha)$ or $a(\alpha, s) \in L(\alpha, s) \subseteq QQ(\alpha, s)$.

Definition. Define α's construction queue to be

$$QQ(\alpha) \;=\; \{\, n \mid \exists t(\forall s \geq t)[\, n \in QQ(\alpha, s)\,]\,\}$$

with ordering[16] $\leq_{QQ} = \lim_{s\to\infty} \leq^s_{QQ}$ and we refer to the QQ-rank of labels in this queue in a similar way to that used above for $QQ(\alpha, s)$.

The next result follows easily from the definition of the construction.

Sublemma 2. $GL \cap \{\, n \mid b(\alpha) <_{\mathbb{N}} n \,\} = QQ(\alpha)$.

Definition. Define $L(\alpha) = \{\, a \mid \exists t(\forall s \geq t)[\, a \in L(\alpha, s)\,]\,\}$.

[16]For any $a, b \in QQ(\alpha)$ there exists a stage $s_{a,b}$ such that, for some $R \in \{\leq, \geq\}$, $a \; R^s_{QQ} \; b$ for all $s \geq s_{a,b}$.

Sublemma 3. *For any label $a >_{\mathbb{N}} b(\alpha)$, and $s \geq s_\alpha$, if $a \notin QQ(\alpha, s)$ then $a \notin L(\alpha, s)$. Thus $L(\alpha) \subseteq QQ(\alpha)$.*

Proof. This follows from the fact that $L(\alpha, s) \subseteq QQ(\alpha, s)$ for any stage s. □

Sublemma 4. *Suppose that s, t are α-true stages such that $s_\alpha \leq s < t$. Then s, t satisfy the following condition. For any label a, if $a \in L(\alpha, r)$ for all $s \leq r \leq t$, then $a \leq^t_{QQ} a(\alpha, t)$ and*

$$\{\, d \mid d \in L(\alpha, t) \ \&\ d \leq^t_{QQ} a \,\} \;=\; \{\, d \mid d \in L(\alpha, s) \ \&\ d \leq^s_{QQ} a \,\}.$$

Proof. Fix α-true $s_\alpha \leq s$ and suppose that $s < t$ is an α-true stage such that the statement of Sublemma 4 holds for all α-true stages $s < p < t$. Suppose also that $a \in L(\alpha, r)$ for all $s \leq r \leq t$. Then by hypothesis (and automatic resetting) $a \leq^{t-1}_{QQ} a(\alpha, t-1)$ and

$$\{\, d \mid d \in L(\alpha, t-1) \ \&\ d \leq^{t-1}_{QQ} a \,\} \;=\; \{\, d \mid d \in L(\alpha, s) \ \&\ d \leq^s_{QQ} a \,\}.$$

Suppose that some label d is added (via Case 10) at stage t. Then by definition $L(\alpha, t-1) \subseteq L(\alpha, t) \subseteq QQ(\alpha, t)$—otherwise one of Cases 2-9 would have been applied—and $a(\alpha, t-1) <^t_{QQ} d$. However, as also $a \in L(\alpha, t-1)$, $a \in L(\alpha, t)$ and $a \leq^t_{QQ} a(\alpha, t-1)$, by definition of $<^t_{QQ}$. Thus $a <^t_{QQ} d = a(\alpha, t)$.

Now suppose that there is some $d \in L(\alpha, t-1)$ such that $d <^{t-1}_{QQ} a$ and $d \notin QQ(\alpha, t)$. Then d is removed via Case 2 and—as $a \notin \{\, b \mid b \in L(\alpha, t-1) \ \&\ b <^{t-1}_{QQ} d \,\}$—$a$ is removed from $L(\alpha, t)$. Hence it must be the case, for all $d \in L(\alpha, t-1)$ such that $d <^{t-1}_{QQ} a$, that $d \in QQ(\alpha, t)$, so that $d <^t_{QQ} a$ (again by definition of $<^t_{QQ}$). Finally suppose that, for some such d, $d \notin L(\alpha, t)$. Then either d is removed via Case 2 or some $d' \in L(\alpha, t-1)$ such that $d' <^t_{QQ} d$ receives attention via one of Cases 3-9. However in both these cases a is removed from $L(\alpha, t)$. Thus there is no such d.

We conclude by induction over α-true stages $t > s$ that Sublemma 4 is true for s and hence—as our choice of s was arbitrary—for all α-true $s_\alpha \leq s < t$. □

Note 3. By Sublemma 4, if $a \in L(\alpha)$, then there exists a stage r_a such that, for all $s \geq r_a$, $\{\, d \mid d \in L(\alpha, s) \ \&\ d \leq^s_{QQ} a \,\} = \{\, d \mid d \in L(\alpha, r_a) \ \&\ d \leq^{r_a}_{QQ} a \,\}$.

Notation. Let $a \in QQ(\alpha)$. We say that a has *stabilised* in $QQ(\alpha)$ at stage $\hat{s}$ if, for all $b \in QQ(\alpha, \hat{s})$ such that $b \leq^{\hat{s}}_{QQ} a$, $b \in QQ(\alpha, s)$ for all $s \geq \hat{s}$. In other words, if the front of the queue up to a has already settled down at stage $\hat{s}$. We also say that $a \in L(\alpha)$ has *stabilised* in $L(\alpha)$ at stage s' if (i) a has stabilised in $QQ(\alpha, s')$ and (ii) $a \in L(\alpha, s)$ for all $s \geq s'$. Notice that, on the strength of Sublemma 4 and Note 3 this means that, for each $b \in QQ(\alpha, s')$ such that $b \leq^{s'}_{QQ} a$, $b \in L(\alpha, s')$ if and only if $b \in L(\alpha, s)$ for all $s \geq s'$.

Remark. For all $s \geq s_\alpha$, and parameter $X \in \{QQ, L\}$, $X(\alpha, s+1) = X(\alpha, s)$ by automatic resetting if $s+1$ is not α-true.

Definition. We define

$$a(\alpha) \;=\; \begin{cases} b(\alpha) & \text{if } L(\alpha) = \emptyset \\ \max_{QQ} L(\alpha) & \text{if } L(\alpha) \neq \emptyset \text{ and } L(\alpha) \text{ is finite,} \\ \uparrow & \text{otherwise,} \end{cases}$$

where $\max_{QQ} L(\alpha)$ denotes the label of maximal QQ-rank in $L(\alpha)$.

Sublemma 5. *If there exists $a \in L(\alpha)$ such that a requires attention at infinitely many α-true stages, then $a(\alpha)\downarrow = a$ so that $L(\alpha) \subseteq \{ d \mid d \in QQ(\alpha) \ \&\ d \leq_{QQ} a \}$.*

Proof. Let $s_a \geq s_\alpha$ be a stage such that a has stabilised in[17] $QQ(\alpha)$ and $L(\alpha)$. Then at every α-true stage $s \geq s_a$ at which a requires attention, a receives attention[18] so that $a(\alpha, s) = a$ and a is the label of maximal QQ-rank in $L(\alpha, s)$. It follows that $a(\alpha)\downarrow = a$ and $L(\alpha) \subseteq \{ d \mid d \in QQ(\alpha) \ \&\ d \leq_{QQ} a \}$. □

Sublemma 6. *There is at most one label $a \in L(\alpha)$ that requires attention at infinitely many stages. Moreover, if $a \in L(\alpha)$ and $f_e(m_a)\uparrow$, then $a(\alpha)\downarrow = a$.*

Proof. The first sentence of Sublemma 6 is an immediate corollary of Sublemma 5. The second sentence follows from Sublemma 5 in conjunction with the fact that, if $a \in L(\alpha)$ and $f_e(m_a)\uparrow$, then a will require attention via Case 3 or Case 3 $\to$ 5 at infinitely many stages. □

Sublemma 7. *For any α-true stage s and $a, b \in L(\alpha, s)$ such that $b <^s_{QQ} a$, $f_{e,s}(m_b) <_{\mathbb{N}} a$.*

Proof. Note that if $a \in L(\alpha, l(\alpha, s)) \cap L(\alpha, s)$ then, for all $d \in L(\alpha, s)$, such that $d <^s_{QQ} a$, $d \in L(\alpha, l(\alpha, s))$ and $d <^{l(\alpha,s)}_{QQ} a$ by Sublemma 4. Also $f_{e,s}(m_d) = f_{e,l(\alpha,s)}(m_d)$ otherwise some $d' \in L(\alpha, l(\alpha, s))$ such that $d' <^s_{QQ} a$ would require attention at stage s causing a to be removed from $L(\alpha, s)$. On the other hand, if $a \in L(\alpha, s) \setminus L(\alpha, l(\alpha, s))$, then $L(\alpha, s) = L(\alpha, l(\alpha, s)) \cup \{a\}$ and, $f_{e,s}(m_d) = f_{e,l(\alpha,s)}(m_d)$ for all $d \in L(\alpha, l(\alpha, s))$ since these conditions are necessary for Case 10 to apply to a, for similar reasons to those applied in the case $a \in L(\alpha, l(\alpha, s)) \cap L(\alpha, s)$. Moreover[19] $f_{e,s}(m_{a(\alpha,l(\alpha,s))}) <_{\mathbb{N}} a$ by definition of Case 10.

Thus Sublemma 7 follows by a straightforward argument by induction over α-true stages. □

Sublemma 8. *Suppose that stages $s_\alpha \leq s < t$ are α-true stages such that $DR(\alpha, r) \neq \emptyset$ for all (α-true) $s \leq r \leq t$. Then $a(\alpha, t) = a(\alpha, s)$ and $DR(\alpha, t) = DR(\alpha, s)$ and α carries out no rebuilding at stage t.*

Proof. Fix α-true $s_\alpha \leq s$ and suppose that $s < t$ is an α-true stage such that the statement of Sublemma 8 holds for all α-true stages $s < p < t$. Then $a(\alpha, l(\alpha, t)) = a(\alpha, s)$ and $DR(\alpha, l(\alpha, t)) = DR(\alpha, s) \neq \emptyset$. As $DR(\alpha, t) \neq \emptyset$ we know that $a = a(\alpha, l(\alpha, t))$ due to a receiving attention via one of the Cases 3 $\to$ 5, 4 $\to$ 5, 5, or 6(b). However, as $DR(\alpha, l(\alpha, t)) \neq \emptyset$ we see that a receives attention via one of the first three Cases so that $DR(\alpha, t) = DR(\alpha, l(\alpha, t))$. Thus $a(\alpha, t) = a(\alpha, s)$, $DR(\alpha, t) = DR(\alpha, s)$ and no block is rebuilt by α at stage t.

We conclude by induction over α-true stages $t > s$ that Sublemma 8 is true for s and hence—as our choice of s was arbitrary—for all α-true $s_\alpha \leq s < t$. □

Sublemma 9. *Suppose that stages $s_\alpha \leq r < t$ are α-true stages such that $DR(\alpha, r) \neq \emptyset$ and $DR(\alpha, t) \neq \emptyset$ but $DR(\alpha, t) \neq DR(\alpha, r)$. Then there exists an α-true stage $r < s < t$ such that $DR(\alpha, s) = \emptyset$ and $a(\alpha, s) = a(\alpha, t)$.*

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[17]For clarity we always state this condition in full even though, if at stage s, a has stabilised in $L(\alpha)$, then it has already stabilised in $QQ(\alpha)$ by definition.

[18]Note that this is the reason for the fact that Case 2 redirects the construction via Cases 3-9 (or 11) when $L^*(\alpha, s) \neq \emptyset$.

[19]$a(\alpha, l(\alpha, s)) = a(\alpha, s-1)$ by automatic resetting. (The value $f_{e,s}(m_{a(\alpha,s-1)})$ is used in Case 10.)

Proof. Suppose that $DR(\alpha, p) \neq \emptyset$ for all (α-true) stages $r < p < t$. Then it follows from Sublemma 8 that $DR(\alpha, t) = DR(\alpha, s)$ in contradiction with our present hypothesis. Therefore there exists an α-true stage $r < \hat{s} < t$ such that $DR(\alpha, \hat{s}) = \emptyset$. Suppose that $\hat{s}$ is the greatest such stage. Let s be the next α-true stage (so $\hat{s} < s \leq t$). By definition of $\hat{s}$, $DR(\alpha, s) \neq \emptyset$ and $a = a(\alpha, s)$ received attention via Case 6(b). However this means, by definition of Case 6(b), that $a(\alpha, s) = a(\alpha, \hat{s})$. Moreover, Sublemma 8 applies to stages $s < t$, again by definition of $\hat{s}$. Thus $a(\alpha, t) = a(\alpha, s)$; i.e. $a(\alpha, t) = a(\alpha, \hat{s})$. □

Note 4. As the set of α-true stages is infinite, for every label d, there is a stage r_d such that $B(\alpha, d, s)$ is defined for all $s \geq r_d$.

Sublemma 10. *For any $n \in L$ there exists a label d_n such that, for some stage t^* either condition (i) or (ii) holds.*

(i) *$n \in B(\alpha, d_n, s)$ for all α-true stages $s \geq t^*$.*
(ii) *$n \in I(d_n, s) \setminus B(\alpha, d_n, s)$ for all α-true stages $s \geq t^*$.*

Proof. Let $\hat{s}$ be a stage such that $n_{\hat{s}} \geq_{\mathbb{N}} n$. I.e. $n \in I(d, \hat{s})$ for some label d. There are 2 cases.

Case 1. $n \in I(d, s)$ for all $s \geq \hat{s}$. Then the block containing the elements $\{ m \mid m \leq_{\mathscr{L}} n \} \cap I(d, \hat{s})$ is preserved in $I(d, s)$ for all $s \geq \hat{s}$. Let $s^* \geq \hat{s}$ be a stage such that $B(\alpha, d, s^*)$ is defined. Then either (i) or (ii) holds with $d_n = d$ and $t^* = s^*$.

Case 2. Otherwise. I.e. for some $s > \hat{s}$, $n \notin I(d, s)$. Let s' be the least such stage. (Thus either s' is odd and $I(d, s)$ is reconstructed at stage s' or s' is even and $I(d, s')$ is rebuilt at stage s'.) Then, as $n \neq m_d$ in this case, by construction there is some new label d^* such that $I(d^*, s') = \{n\}$ so that $n = m_{d^*}$. Let $s^* \geq s'$ be an α-true stage such that $B(\alpha, d^*, s^*)$ is defined. Then (i) holds with $d_n = d^*$ and $t^* = s^*$. □

Sublemma 11. *Suppose that $a \in L(\alpha)$ requires attention infinitely often and that $f_e(m_a)\downarrow$. Then $a(\alpha) = lim_{s \to \infty} a(\alpha, s) = a$, lim $inf_{s \to \infty} d(\alpha, s)$ is defined, and the set $\{ s \mid a(\alpha, s) = a \ \& \ R(\alpha, s) \notin \{\text{ddb}, \text{udb}\} \}$ is infinite. Also lim $inf_{s \to \infty} \hat{t}(\alpha, s) = \infty$.*

Proof. The fact that $a(\alpha) = a$ follows from Sublemma 5. Let $n_a = f_e(m_a)$. Let $s_a > s_\alpha$ be an α-true stage such that a has stabilised in $QQ(\alpha)$ and $L(\alpha)$ and, both $t(\alpha, s) >_{\mathbb{N}} \max \{ b \mid b \in QQ(\alpha) \ \& \ b \leq_{QQ} a \}$ and $f_{e,s}(m_a) = n_a$, for all[20] $s \geq l(\alpha, s_a)$.

Stability Note 1. These conditions mean that $a \in L(\alpha, s)$, for all $s \geq s_a$ so that, for any $d \in QQ(\alpha)$ such that $d \leq_{QQ} a$, $d \leq_{\mathbb{N}} d(\alpha, s)$, and hence $I(d, s)$ is protected from rebuilding by lower priority strategies, whereas the fact that $t(\alpha, s) >_{\mathbb{N}} d$ implies that $I(d, s)$ is not rebuilt by any strategy $\gamma \subset \alpha$. Also as $s_a \geq s_\alpha$ no strategy $\gamma <_{\text{lex}} \alpha$ is visited by the construction at or after stage s_a. We can thus assume that $|I(d, s)| \geq 2$ and that $B(\alpha, d, s) = B(\alpha, d, s_a)$ for every $d \in QQ(\alpha)$ such that $d <_{QQ} a$ and all α-true $s \geq s_a$. On the other hand, for any such s, $|I(a, s)| \geq 2$ and it is only the case that $B(\alpha, a, s) \neq B(\alpha, a, s_a)$ if a receives attention from α via Case 7 at some α-true stage $s_a < t \leq s$.

[20]The condition that $f_{e,s}(m_a) = n_a$ for all $s \geq l(\alpha, s_a)$ ensures that a does not require attention via Case 3 at stage s_a.

Stability Note 2. Suppose—with[21] Sublemma 12 also in mind, i.e. dropping the assumption that $f_{e,s}(m_a) = n_a$ for all $s \geq l(\alpha, s_a)$ in the present Note—that there exists $r^* \geq s_a$ such that[22] $f_{e,s}(m_a) \notin \bigcup\{ B(\alpha, d, s_a) \mid d \in QQ(\alpha) \ \& \ d <_{QQ} a \}$, for all α-true $s \geq r^*$. Then $B(\alpha, a, s) = B(\alpha, a, r^*)$ for all such s by the last sentence of Stability Note 1. Suppose also that there exists α-true $\hat{r} \geq r^*$ such that $a(\alpha, \hat{r}) = a$ and $DR(\alpha, \hat{r})$ is nonempty, i.e. for some label $a <^{\hat{r}}_{QQ} d$, $DR(\alpha, \hat{r}) = \{d\}$ due to $DT(\alpha, \hat{r}) = \{(x, m_d, d)\}$ with $x \in B(\alpha, a, \hat{r}) = B(\alpha, a, r^*)$ and $f_{e,\hat{r}}(x) = m_d$. Suppose furthermore that $f_{e,s}(x) = m_d$, $d \in QQ(\alpha, s)$ and $t(\alpha, s) >_{\mathbb{N}} d$ for all α-true $s \geq \hat{r}$. Then, under these conditions, by application of the same argument as in Stability Note 1, we see that a will require attention via Case 5 at all subsequent α-true stages and that all three tests of the latter will succeed. This entails that $d(\alpha, s) \geq_{\mathbb{N}} d$, that $DR(\alpha, s) = DR(\alpha, \hat{r}) = \{d\}$, and that $B(\alpha, d, s) = B(\alpha, d, \hat{r})$, for all α-true stages $s \geq \hat{r}$. (Also we will have that a receives attention via one of Cases $3 \to 5$, $4 \to 5$ or 5 at every such stage $s > \hat{r}$. Note that in this case, whereas $x \in B(\alpha, a, \hat{r})$ and $f_e(x) \in B(\alpha, d, \hat{r})$, we have that $f_e : B(\alpha, a, \hat{r}) \ncong B(\alpha, d, \hat{r})$.)

By Sublemma 10 there is a label d such that, for some stage $s' \geq s_\alpha$ either $n_a \in B(\alpha, d, s)$ for all α-true $s \geq s'$ or $n_a \in I(d, s) \setminus B(\alpha, d, s)$ for all $s \geq s'$. Accordingly we assume that $s_a > s'$ for the least such s'.

Note 5. Suppose $s \geq s_a$ is an α-true stage such that $a(\alpha, s) = a$ and a receives attention via Case 6(b). Then, by the above assumption on s_a, if b is the label such that α rebuilds $I(b, s)$, then $f_e(m_a) = n_a \in I(b, s-1) \cap I(b, s)$ so that $n_a = m_b$. I.e. $d = b$ and $DR(\alpha, s) = \{d\}$.

There are seven cases as follows.

Remark. Note that Cases C-G exhaust the possibilities arising from the construction under the hypotheses of Sublemma 11 and that Cases A-B are used in order to simplify the arguments used in the latter.

Case A. $DR(\alpha, s) \neq \emptyset$ for all stages $s \geq s_a$. Then, by Sublemma 8, $a(\alpha, s) = a(\alpha, s_a)$, $DR(\alpha, s) = DR(\alpha, s_a)$ and α rebuilds no block, at any stage $s \geq s_a$. Clearly also $a(\alpha, s_a) = a$ (as by hypothesis a receives attention at infinitely many stages). Thus $d(\alpha, s) = \max\{ d \mid d \in QQ(\alpha) \ \& \ d \leq_{QQ} a \} \cup DR(\alpha, s_a)$, whereas $R(\alpha, s) = \mathrm{ud}$ and, if s is α-true, $\hat{t}(\alpha, s) = s$, for all such s.

Case B. Otherwise, but $\{ s \mid a(\alpha, s) = a \ \& \ DR(\alpha, s) \neq \emptyset \}$ is infinite and $d \in GL$. Let $s_d \geq s_a$ be an α-true stage such that d has stabilised in $QQ(\alpha)$ and such that $t(\alpha, s) >_{\mathbb{N}} d$ for all $s \geq s_d$. There are two subcases.

Case B1. $DR(\alpha, s) \neq \emptyset$ for all stages $s \geq s_d$. This is Case A with s_d replacing s_a.

Case B2. Otherwise, i.e. $DR(\alpha, s) = \emptyset$ for some $s \geq s_d$. In this case let $s^* \geq s_d$ be a stage such that $DR(\alpha, s^*) = \emptyset$. Let $\hat{s} > s^*$ be the least α-true stage such that $a(\alpha, \hat{s}) = a$ and $DR(\alpha, \hat{s}) \neq \emptyset$. This means that a received attention via Case 6(b). Thus, by Note 5, $n_a = m_d \in I(d, \hat{s})$ and $DR(\alpha, \hat{s}) = \{d\}$. However, as $\hat{s} \geq s_d$ (and $s_d \geq s_a$), by Stability Note 2 it follows that $d \in DR(\alpha, s)$, for all $s \geq \hat{s}$. So this is Case A again with $\hat{s}$ replacing s_a.

Note 6. In Cases C-G we assume that neither Case A nor Case B applies.

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[21] In Sublemma 12, $f_e(m_a)\uparrow$.

[22] By Stability Note 1 $B(\alpha, d, s) = B(\alpha, d, s_a)$ for all $d <_{QQ} a$ and $s \geq s_a$.

Case C. $n_a \in B(\alpha, d, s_a)$ and $d \leq b(\alpha)$. Then, by our assumption in Note 6, we can choose $\hat{s} \geq s_a$ to be an α-true stage such that $DR(\alpha, \hat{s}) = \emptyset$. Let s^* be the next α-true stage at which a requires attention. Then a can only require (and so receive) attention via Case 8 under these conditions. Moreover, by definition of s_a, for all $b \in L(\alpha)$ (i.e. $b \in L(\alpha)$ and $b \leq_{QQ} a$), $f_e(m_b)\downarrow$ and $f_{e,s}(m_b) = f_e(m_b)$ for all α-true stages $s \geq l(\alpha, s_a)$. (Otherwise some such b would require attention via Case 3 at some stage $s \geq s_a$ forcing $a(\alpha, s) <^s_{QQ} a$ in contradiction with the definition of s_a.) By Sublemma 10 we know that there exists a stage $s' \geq s_\alpha$ and, for each such b, a label d_b such that either (i) $f_e(m_b) \in B(\alpha, d_b, s)$ for all $s \geq s'$, or $f_e(m_b) \in I(d_b, s) \setminus B(\alpha, d_b, s)$ for all $s \geq s'$. We can thus also assume that $s_a \geq s'$ for the least such s'. Now, as it is not the case that $a(\alpha, s) <_{QQ} a$ for any stage $s \geq s_a$ we deduce that no label $b \in L(\alpha)$ such that $b <_{QQ} a$ requires attention via Case 9 at stage s_a. Thus for each such b, $f_e(m_b) \in B(\alpha, d_b, s_a)$. Also $d_b \leq_{\mathbb{N}} b(\alpha)$ since otherwise b would require—and one such b would receive—attention via one of Cases 4-7, $4 \to 5$ or 9 at stage s_a. Therefore we see that, for every α-true stage $s \geq s^*$, a requires attention via Case 8 relative to some fixed $\hat{a} \in L(\alpha)$ such that $\hat{a} <_{QQ} a$. Hence $a(\alpha, s) = a$, $d(\alpha, s) = \max\{\, d \mid d \in QQ(\alpha)\ \&\ d \leq_{QQ} a \,\}$, $R(\alpha, s) = \mathrm{wt}$ and, if s is α-true, $\hat{t}(\alpha, s) = s$, for all stages $s \geq s^*$.

Case D. $n_a \in B(\alpha, d, s_a)$, $d \in QQ(\alpha)$ and $d \leq_{QQ} a$. By our assumption in Note 6 (and the fact that $d \in GL$) we can suppose that s_a is large enough such that $DR(\alpha, s) = \emptyset$ for all α-true stages $s \geq s_a$ such that $a(\alpha, s) = a$. Also, as $d \leq_{QQ} a$, d has already stabilised in $QQ(\alpha)$ at stage s_a and $t(\alpha, s) >_{\mathbb{N}} d$ for all $s \geq s_a$ (by definition of s_a). There are two subcases.

Case D1. $f_{e,s} : B(\alpha, a, s) \cong B(\alpha, d, s)$ for some α-true stage $s > s_a$. Let s^* be the least such stage. Note firstly that $d \neq a$ in this case, since otherwise $f_{e,s^*}(m_a) = m_a$, so that a would be removed from $L(\alpha, s^*)$ via Case 2 contradicting the fact that $s^* \geq s_a$. Thus $d <_{QQ} a$ and a receives attention via Case 7 at stage s^*. But then, by the fact that $s^* \geq s_a$ and Stability Note 1, $B(\alpha, y, s) = B(\alpha, y, s^*)$ for $y \in \{a, d\}$ and all $s > s^*$. Hence $f_{e,s} : B(\alpha, a, s) \ncong B(\alpha, d, s)$ and a receives attention via Case 4 at every such stage s.

Case D2. Otherwise $f_{e,s} : B(\alpha, a, s) \ncong B(\alpha, d, s)$ for all α-true stages $s > s_a$, so that a receives attention via[23] Case 4 at every such stage.

Letting $s^* = s_a$ if Case D2 applies, we thus see that, in both Case D1 and D2, $a(\alpha, s) = a$, $d(\alpha, s) = \max\{\, d \mid d \in QQ(\alpha)\ \&\ d \leq_{QQ} a \,\}$, $R(\alpha, s) = \mathrm{wt}$ and, if s is α-true, $\hat{t}(\alpha, s) = s$, for all $s > s^*$.

Case E. $n_a \in B(\alpha, d, s_a)$, $d \in QQ(\alpha)$ and $a <_{QQ} d$. By our assumption in Note 6 we can suppose that s_a is large enough such that $DR(\alpha, s) = \emptyset$ and that a does not require attention via Case $4 \to 5$, 5 or 6(b) at any α-true stage $s \geq s_a$ such that $a(\alpha, s) = a$. Let $s_d \geq s_a$ be an α-true stage such that d has stabilised in $QQ(\alpha)$ at stage s_d. Suppose that $f_{e,s_d} : B(\alpha, a, s_d) \cong B(\alpha, d, s_d)$. Then, as a does not receive attention via Case 6(b), at stage s_d it must be the case that a receives attention via Case 6(a). Let s^* be the next α-true stage. Then $f_{e,s^*} : B(\alpha, a, s^*) \ncong B(\alpha, d, s^*)$, since otherwise a would receive attention via Case 6(b) (contradicting the fact that $s_d \geq s_a$). Hence we can assume, without loss of generality, that $f_{e,s_d} : B(\alpha, a, s_d) \ncong B(\alpha, d, s_d)$. But then we also easily deduce, by application of

[23] In Case D, for the sake of simplicity, we have not ruled out the possibility of a receiving attention via Case $4 \to 5$ or 5 at stage s_a.

the same argument (via induction over α-true stages) that, for *all* α-true stages $s \geq s_d$, $f_{e,s} : B(\alpha, a, s) \not\cong B(\alpha, d, s)$ and a receives attention via Case 4 at stage s. Hence $d(\alpha, s) = \max\{ d \mid d \in QQ(\alpha) \ \& \ d \leq_{QQ} a \}$, $R(\alpha, s) = \mathrm{wt}$ and, if s is α-true, $\hat{t}(\alpha, s) = s$, for all $s \geq s_d$.

Case F. $n_a \in B(\alpha, d, s_a)$ but $d \notin \{ b \mid b \leq_{\mathbb{N}} b(\alpha) \} \cup QQ(\alpha)$. Note that this means that $d >_{\mathbb{N}} b(\alpha)$ and that $d \notin GL$, i.e. that $F(q_d) = 1$ so that, by construction $I(d) = \{m_d\}$ and also, by definition of this case, that $n_a = m_d$. Notice that it may be the case that $S_d = \{ s \mid s \geq s_a \ \& \ d \in QQ(\alpha, s) \}$ is infinite. However, for every $s \in S_d$, $a <^s_{QQ} d$ by definition of s_a. Therefore, at every α-true stage $s \geq s_a$, a requires attention via at least one of Cases 4-6, $4 \to 5$, or 9. Thus $a(\alpha, s) = a$ (and, by Stability Note 1, $B(\alpha, a, s) = B(\alpha, a, s_a)$) for all $s \geq s_a$. Suppose that $s \geq s_a$ is an α-true stage such that a requires attention via Case 6(b). Then it follows from Note 5 that $DR(\alpha, s) = \{d\}$. Now, by our assumption in Note 6 we can choose $\hat{s} \geq s_a$ such that $DR(\alpha, \hat{s}) = \emptyset$. We therefore see that, for any stage $s > \hat{s}$, if $DR(\alpha, s) \neq \emptyset$, then $DR(\alpha, s) = \{d\}$ so that, as $d \notin QQ(\alpha)$, there are infinitely many α-true stages s such that[24] $DR(\alpha, s) = \emptyset$. Notice that, at any such stage[25] s, $R(\alpha, s) \notin \{\mathrm{ddb}, \mathrm{udb}\}$ and also that $d(\alpha, s) = d(\alpha, \hat{s}) = \max\{ d \mid d \in QQ(\alpha) \ \& \ d \leq_{QQ} a \}$. Thus $\{ s \mid a(\alpha, s) = a \ \& \ R(\alpha, s) \notin \{\mathrm{ddb}, \mathrm{udb}\} \}$ is infinite and $\liminf_{s\to\infty} d(\alpha, s) = d(\alpha, \hat{s})$. Moreover $\hat{t}(\alpha, s) = s$ for all α-true stages $s \geq s_a$ due to the fact that $d \notin GL$ and that, if α rebuilds $I(b, s)$ at any such stage s, then $b = d$ as noted above.

Case G. Otherwise. I.e. $n_a \in I(d, s_a) \setminus B(\alpha, d, s_a)$. In this case, by definition of s_a, $n_a \in I(d, s) \setminus B(\alpha, d, s)$ for all α-true stages $s \geq s_a$. By our assumption in Note 6 we can choose (least) α-true $\hat{s} \geq s_a$ such that $DR(\alpha, \hat{s}) = \emptyset$. Let s^* be the next α-true stage. Then a receives attention via Case 9 at all α-true stages[26] $s \geq s^*$. Thus $a(\alpha, s) = a$, $d(\alpha, s) = \max\{ d \mid d \in QQ(\alpha) \ \& \ d \leq_{QQ} a \}$, $R(\alpha, s) = \mathrm{wt}$ and, if s is α-true, $\hat{t}(\alpha, s) = s$, for all $s \geq s^*$. □

Sublemma 12. *Suppose that $a \in L(\alpha)$ requires attention infinitely often and that $f_e(m_a)\uparrow$. Then $a(\alpha) = a$, lim $\inf_{s\to\infty} d(\alpha, s)$ is defined, and the set $\{ s \mid a(\alpha, s) = a \ \& \ R(\alpha, s) \notin \{\mathrm{ddb}, \mathrm{udb}\} \}$ is infinite. Also lim $\inf_{s\to\infty} \hat{t}(\alpha, s) = \infty$.*

Proof. The fact that $a(\alpha) = a$ follows from Sublemma 5. Let $s_a \geq s_\alpha$ be an α-true stage such that a has stabilised in $QQ(\alpha)$ and $L(\alpha)$, and $t(\alpha, s) >_{\mathbb{N}} \max\{ b \mid b \in QQ(\alpha) \ \& \ b \leq_{QQ} a \}$ for all $s \geq s_a$. Apply *Stability Notes 1-2* as stated on page 24. Notice that, under our present assumption that $f_e(m_a)\uparrow$, we know that $\liminf_{s\to\infty} f_{e,s}(m_a) = \infty$ by definition of the approximation $\{f_{e,s}\}_{e,s\in\mathbb{N}}$.

Note 7. Consider any α-true stage $s \geq s_a$ and label d such that $a(\alpha, s) \neq a$ (i.e. $a <^s_{QQ} a(\alpha, s)$), and α rebuilds block $I(d, s)$ at stage s. Then $a(\alpha, s) \leq^s_{QQ} d$ by definition of Cases 6 and 7. Moreover, by Sublemma 7, $f_{e,s}(m_a) <_{\mathbb{N}} a(\alpha, s)$. Now suppose that $b \in QQ(\alpha)$ and (using the fact that $\liminf_{s\to\infty} f_{e,s}(m_a) = \infty$) let $r_b \geq s_a$ be a stage such that b has stabilised in $QQ(\alpha)$ and $f_{e,s}(m_a) >_{\mathbb{N}} \max\{ d' \mid d' \in QQ(\alpha) \ \& \ d' \leq_{QQ} b \}$ for all $s \geq r_b$. Then, at every α-true stage $s \geq r_b$,

[24] If $DR(\alpha, t) \neq \emptyset$ at some stage $t > \hat{s}$, then letting s be the next α-true stage at which $d \notin QQ(\alpha, s)$, d is removed from $DR(\alpha, s)$ due to the fact that a receives attention via Case 5 or $4 \to 5$, and that Check (i) fails while Case 5 is being processed, at stage s.

[25] $R(\alpha, s) \neq \mathrm{ddb}$ as $a <^s_{QQ} d$, whereas $R(\alpha, s) \neq \mathrm{udb}$ as $DR(\alpha, s) = \emptyset$.

[26] Note that a may require (and receive) attention via Case 5 at stage $\hat{s}$.

such that $a(\alpha,s) \neq a$, $a(\alpha,s) \not\leq^s_{QQ} b$ (as $f_{e,s}(m_a) <_{\mathbb{N}} a(\alpha,s)$). I.e. $b <^s_{QQ} a(\alpha,s)$ so that, if α rebuilds block $I(d,s)$ at stage s, then $d \neq b$. Thus, for s restricted to the set $\{\, s \mid s \text{ is } \alpha\text{-true and } a(\alpha,s) \neq a \,\}$, $\hat{t}(\alpha,s)$ tends to infinity "*inf* wise". It therefore only remains to show that this condition also holds over the set $\{\, s \mid s \text{ is } \alpha\text{-true and } a(\alpha,s) = a \,\}$, to prove that $\lim\inf_{s\to\infty}\hat{t}(\alpha,s) = \infty$.

By Stability Note 1, $B(\alpha,d,s) = B(\alpha,d,s_a)$ for all $d <_{QQ} a$ and $s \geq s_a$. Also, as $\lim\inf_{s\to\infty} f_{e,s}(m_a) = \infty$, we can suppose that s_a is large enough such that, for all $s \geq s_a$, $f_{e,s}(m_a) \notin \bigcup\{\, B(\alpha,d,s_a) \mid d \in QQ(\alpha) \;\&\; d <_{QQ} a \,\}$. Thus, if $s \geq s_a$ is a stage such that $a(\alpha,s) = a$ and α rebuilds a block at stage s then this is because a receives attention via Case 6(b). Moreover, as a can no longer receive attention via Case 7, $B(\alpha,a,s) = B(\alpha,a,s_a)$ for all $s \geq s_a$, again by application of Stability Note 1.

Let m $(\in L_{s_a})$ be such that $B(\alpha,a,s_a) = \{m_a \; S \; m\}$ for some $S \in \{>_{\mathscr{L}}, <_{\mathscr{L}}\}$. There are two cases to consider.

Case A. $f_e(m)\!\downarrow$. Suppose in this case that $s^* \geq s_a$ is such that $f_{e,s}(m) = f_e(m)$ for all $s \geq s^*$. Suppose also that s^* is α-true.

By Sublemma 10 there is a label d such that, for some stage $s' \geq s_\alpha$ either $f_e(m) \in B(\alpha,d,s)$ for all $s \geq s'$, or $f_e(m) \in I(d,s) \setminus B(\alpha,d,s)$ for all $s \geq s'$. Accordingly we assume that $s^* > s'$ for the least such s'. A similar observation to that of Note 5 now applies.

Note 8. Suppose that stage $s \geq s^*$ and label b are such that α rebuilds $I(b,s)$ at stage s. Then $f_e(m) \in I(b,s-1) \cap I(b,s)$ by our further assumption on s^*. I.e. $f_e(m) = m_b$, $d = b$ and $DR(\alpha,s) = \{d\}$.

There are 3 subcases to consider.

Case A1. $d \notin GL$. (I.e. $I(d) = \{f_e(m)\}$.) Note that $d \notin QQ(\alpha)$ but that there may be infinitely many stages s such that $d \in QQ(\alpha,s)$. Now, by Note 8, if a receives attention via Case 6(b) at some α-true stage $s \geq s^*$, then $I(d,s)$ is rebuilt so that $\hat{t}(\alpha,s) = s$ because $d \notin GL$. Therefore $\hat{t}(\alpha,s) = s$ at all α-true stages $s \geq s^*$ such that $a(\alpha,s) = a$. Moreover, as $d \notin GL$ there are infinitely many α-true stages such that $a(\alpha,s) = a$ and $DR(\alpha,s) = \emptyset$. Indeed suppose firstly that the set $\widehat{S} = \{\, s \mid s > s^* \;\&\; a(\alpha,s) \neq a \,\}$ is infinite. Then the set $S^* = \{\, s \mid s > s^* \;\&\; s \text{ is } \alpha\text{-true} \;\&\; a(\alpha,s) = a \;\&\; a(\alpha,s) \neq a(\alpha,l(\alpha,s)) \,\}$ is also infinite. Now, a does not receive attention via Case 7 at any stage $s \in S^*$ as $s^* \geq s_a$. Also a does not receive attention via Case $3 \to 5$, $4 \to 5$, 5, or 6(b) at any stage $s \in S^*$. Thus $DR(\alpha,s) = \emptyset$ for all $s \in S^*$. Secondly suppose that $\widehat{S}$ is finite, so that there is some stage $\hat{s} \geq s^*$ such that $a(\alpha,s) = a$ for all $s \geq \hat{s}$. However there are infinitely many α-true stages $s > s^*$ such that $d \notin QQ(\alpha,s)$ (as $d \notin GL$) so that, at any such s, Case 6(b) does not apply whereas, if $DR(\alpha,l(\alpha,s)) \neq \emptyset$, then $DR(\alpha,s) = \emptyset$ due to a receiving attention via one of Cases $3 \to 5$, $4 \to 5$, or 5 (and Check (i) of Case 5 failing).

Hence we see—for both the case $\widehat{S}$ infinite and the case $\widehat{S}$ finite—that there are infinitely many α-true stages s, with $a(\alpha,s) = a$, such that $R(\alpha,s) \notin \{\mathrm{ddb}, \mathrm{udb}\}$ and $d(\alpha,s) = \max\{\, d \mid d \in QQ(\alpha) \;\&\; d \leq_{QQ} a \,\}$.

Case A2. $d \in GL$ and $d \in DR(\alpha,s)$—i.e. $DR(\alpha,s) = \{d\}$—at infinitely many α-true stages s such that $a(\alpha,s) = a$. Suppose in this case that s^* is large enough such that d has stabilised in $QQ(\alpha)$ at stage s^* and $t(\alpha, s) >_{\mathbb{N}} d$ for all $s \geq s^*$. Let $\hat{s} \geq s^*$ be an α-true stage such that $a(\alpha, \hat{s}) = a$ and $d \in DR(\alpha, \hat{s})$. Then as $\hat{s} \geq s^*$, by Stability Note 2 (on page 25), we see that $d \in DR(\alpha, s)$ for all $s \geq \hat{s}$ and that, at every α-true stage $s > \hat{s}$, a requires attention via Case 5 and receives attention via Case 3 $\to$ 5, Case 4 $\to$ 5, or Case 5. Thus $a(\alpha, s) = a$, $d(\alpha, s) = \max\{\, d \mid d \in QQ(\alpha)\ \&\ d \leq_{QQ} a \,\} \cup \{d\}$, $R(\alpha, s) = \mathrm{ud}$ and, if s is α-true, $\hat{t}(\alpha, s) = s$, for all $s > \hat{s}$.

Case A3. $d \in GL$ and $d \in DR(\alpha, s)$ for only finitely many stages s. Suppose in this case that s^* is large enough such that $d \notin DR(\alpha, s)$ for all $s \geq s^*$. Then by Note 8 and the definition of s^*, a receives attention via Case 6 at no stage $s \geq s^*$. Suppose that $DR(\alpha, s^*) \neq \emptyset$ and suppose also that b is the label such that $DR(\alpha, s^*) = \{b\}$ (i.e. $b \neq d$ so that $f_e(m) \neq m_b$). Let $\hat{s} > s^*$ be an α-true stage such that $a(\alpha, \hat{s}) = a$ and $f_{e,\hat{s}}(m_a) > m_b$. If $a(\alpha, l(\alpha, \hat{s})) \neq a$, or if $a(\alpha, l(\alpha, \hat{s})) = a$ but $DR(\alpha, l(\alpha, \hat{s})) = \emptyset$, then $DR(\alpha, \hat{s}) = \emptyset$. If not, then $a(\alpha, l(\alpha, \hat{s})) = a$ and $DR(\alpha, l(\alpha, \hat{s})) = \{b\}$ (as a does not receive attention via Case 6(b) at any stage $s \geq s^*$), so that a will receive attention at stage $\hat{s}$ via one of Cases 3 $\to$ 5, 4 $\to$ 5 or 5 causing $DR(\alpha, \hat{s}) = \emptyset$ due to the failure of Check (ii) when Case 5 is processed (as neither $f_{e,\hat{s}}(m_a) = m_b$ nor $f_{e,\hat{s}}(m) = m_b$). Thus, letting $\hat{s} = s^*$ in the case when $DR(\alpha, s^*) = \emptyset$ it follows that $DR(\alpha, s) = \emptyset$ for all α-true stage $s \geq \hat{s}$ such that $a(\alpha, s) = a$. But then, at every such stage $R(\alpha, s) \notin \{\mathrm{ddb}, \mathrm{udb}\}$, $\hat{t}(\alpha, s) = s$, and $d(\alpha, s) = \max\{\, d \mid d \in QQ(\alpha)\ \&\ d \leq_{QQ} a \,\}$.

From the outcomes of all 3 subcases and Note 7 we conclude that, in Case A, $\liminf_{s\to\infty} \hat{t}(\alpha, s) = \infty$. (Note also that, in each subcase, $\liminf_{s\to\infty} d(\alpha, s) = d(\alpha, s')$ for some fixed α-true stage s'.)

Case B. $f_e(m)\uparrow$. In this case, for any label d there are at most finitely many stages s such that either $f_{e,s}(m_a) = m_d$ or $f_{e,s}(m) = m_d$. This means that there are only finitely many α-true s such that a receives attention via Case 6 and α rebuilds $I(d, s)$. It follows that there exists a stage $\hat{s}_d$ such that for all α-true stages $s \geq \hat{s}_d$ with $a(\alpha, s) = a$, $\hat{t}(\alpha, s) >_{\mathbb{N}} d$. From this and Note 7 we conclude that $\liminf_{s\to\infty} \hat{t}(\alpha, s) = \infty$. Now consider any α-true stage $s \geq s_a$ such that $a(\alpha, s) = a$ and $DR(\alpha, s) \neq \emptyset$, and suppose that b is the label such that $DR(\alpha, s) = \{b\}$. Choose α-true stage $t > s$ such that $f_{e,t}(m_a) >_{\mathbb{N}} m_b$ and $f_{e,t}(m) >_{\mathbb{N}} m_b$ and such that $a(\alpha, t) = a$. Suppose also that $DR(\alpha, l(\alpha, t)) \neq \emptyset$. Then, if $a(\alpha, l(\alpha, t)) \neq a$, a does not require attention via Case 6(b) or any of the Cases involving Case 5, so that $DR(\alpha, t) = \emptyset$. Suppose otherwise, i.e. that $a(\alpha, l(\alpha, t)) = a$. Then if $DR(\alpha, l(\alpha, t)) \neq DR(\alpha, s)$, by Lemma 9 there exists an α-true stage $s < r < l(\alpha, t)$ such that $a(\alpha, r) = a$ and $DR(\alpha, r) = \emptyset$. On the other hand, if $DR(\alpha, l(\alpha, t)) = DR(\alpha, s) = \{b\}$ ($= DR(\alpha, t-1)$ by automatic resetting), then a will receive attention at stage t via one of Cases 3 $\to$ 5, 4 $\to$ 5 or 5 causing $DR(\alpha, t) = \emptyset$ due to the failure of Check (ii) when Case 5 is processed. We thus deduce that $DR(\alpha, s) = \emptyset$ for infinitely many α-true stages $s \geq s_a$ such that $a(\alpha, s) = a$. But then, at every such stage $R(\alpha, s) \notin \{\mathrm{ddb}, \mathrm{udb}\}$, and $d(\alpha, s) = \max\{\, d \mid d \in QQ(\alpha)\ \&\ d \leq_{QQ} a \,\}$. (Thus $\liminf_{s\to\infty} d(\alpha, s) = d(\alpha, s')$ for some fixed s'.) $\square$

Remark. If $f_e(m_a)\uparrow$ as in Sublemma 12 it may be the case that, for some $d \leq_{\mathbb{N}} b(\alpha)$ such that $d \notin GL$ (i.e. $F(q_d) = 1$), $P_d = \{\, s \mid s \text{ is } \alpha\text{-true } \& \ |B(\alpha, d, s)| = 2 \,\}$ is infinite, and moreover that for infinitely many stages $s \in P_d$, $f_{e,s}(m_a) \in B(\alpha, d, s)$. (This can happen because new elements from $\mathbb{N}$ are used whenever

$I(d,s)$ is reconstructed or rebuilt.) Then there might be infinitely many such stages s, at which a $(= a(\alpha))$ does *not* require attention. Suppose that we did not impose the condition "$f_{e,s+1} <_{\mathbb{N}} a$"—which we call (C) for present purposes—in Case 10 of the construction. Then we could get some $a' \in QQ(\alpha)$ such that $a <_{QQ} a'$ (so that $a' \notin L(\alpha)$, i.e. $a' \notin L(\alpha,s)$ infinitely often) but that for infinitely many α-true stages s, $a' \in L(\alpha,s)$ and a' receives attention via Case 6 or Case 7. Note that this can happen since when a' drops out of $L(\alpha,s)$, the block that it labels is no longer protected from interference by lower priority strategies. (Our inductive hypothesis imposes a finiteness condition on interference by higher priority strategies only.) Thus we could have that, for infinitely many such stages s, $I(a',s)$ is rebuilt via Case 7 or that, for some fixed $d' \in QQ(\alpha)$ such that $a' <_{QQ} d'$, $I(d',s)$ is rebuilt via Case 6. Letting b' be the least such label for which this happens, we would thus get that $\liminf_{s\to\infty}\hat{t}(\alpha,s) \leq_{\mathbb{N}} b'$. This would invalidate our inductive hypothesis for $e+1$ and thus our overall induction argument would break down. However, as we have seen, condition (C) in Case 10 means that $a' \in L(\alpha,s)$ for only finitely many stages s since, for some s^*, $f_{e,s}(m_a) >_{\mathbb{N}} a'$ for all $s \geq s^*$.

Sublemma 13. *Suppose that $L(\alpha)$ is finite and that there is no $a \in L(\alpha)$ such that a requires attention infinitely often. Then either $L(\alpha) = \emptyset$ and $\liminf_{s\to\infty} d(\alpha,s) = a(\alpha) = b(\alpha)$, or $L(\alpha) \neq \emptyset$, so that $a(\alpha) >_{\mathbb{N}} b(\alpha)$, and $\liminf_{s\to\infty} d(\alpha,s) = \max\{\, d \mid d \in QQ(\alpha)\ \&\ d \leq_{QQ} a(\alpha)\,\}$. Also, the set $\{\, s \mid a(\alpha,s) = a(\alpha)\ \&\ R(\alpha,s) \notin \{\mathrm{ddb}, \mathrm{udb}\}\,\}$ is infinite. Moreover $\liminf_{s\to\infty}\hat{t}(\alpha,s) = \infty$.*

Proof. Suppose that s^* is a stage such that, for all $a \in L(\alpha)$, a has stabilised in $QQ(\alpha)$ and $L(\alpha)$ and a does not require attention at any $s \geq s^*$. Consider any α-true stage $s \geq s^*$. If $L(\alpha,s) \neq L(\alpha)$ let a be the label of least (s-stage) QQ-rank in $L(\alpha,s)$ such that $a(\alpha) <^s_{QQ} a$. As $a \notin L(\alpha)$ there is a least α-true stage $r > s$ such that $a \notin L(\alpha,r)$—i.e. when a is removed from $L(\alpha,r)$ via Case 2. But in this case it follows from Sublemma 4 that $L(\alpha,r) = L(\alpha)$. Also $DR(\alpha,r) = \emptyset$.

We know therefore that, either there is some $t^* \geq s^*$ such that $L(\alpha,s) = L(\alpha)$ for all $s \geq t^*$ and Case 11 applies, $DR(\alpha,s) = \emptyset$ and $R(\alpha,s) = \mathrm{wt}$ at every α-true stage $s \geq t^*$, or there are infinitely many α-true stages s such that $L(\alpha,s) = L(\alpha)$, $DR(\alpha,s) = \emptyset$ and $R(\alpha,s) = \mathrm{wt}$ due to Case 2 being applied at stage s. This proves all but the last sentence of the statement of Sublemma 12.

Now consider any $a \in QQ(\alpha)$ such that $a \notin L(\alpha)$. Suppose that there are infinitely many stages s such that $a \in L(\alpha,s)$ and suppose also that a is the label of least QQ-rank to satisfy this property. Then, by definition $f_{e,s}(m_a) \neq m_a$ for infinitely many stages so that, for some stage $r_a \geq s^*$, $f_{e,s}(m_a) \neq m_a$ for all $s \geq r_a$. Now we can also suppose that r_a is α-true and $a \in L(\alpha,r_a)$, and moreover that r_a is large enough so that, for all $s \geq r_a$, there exists no $a(\alpha) <^s_{QQ} d <^s_{QQ} a$ such that $d \in L(\alpha,s)$. Then a will remain in $L(\alpha,t)$ for all $t \geq r_a$. I.e. $a \in L(\alpha)$ contradicting our present definition of $L(\alpha)$. We therefore conclude that, for every $a \in QQ(\alpha)$ such that $a \notin L(\alpha)$ the set $\{\, s \mid a \in L(\alpha,s)\,\}$ is finite.

Choose any $d >_{\mathbb{N}} b(\alpha)$ such that $d \in GL$—so that $d \in QQ(\alpha)$. Let $t_d \geq s^*$ be a stage such that d has stabilised in $QQ(\alpha)$ and such that, for all $s \geq t_d$ and $d' \in QQ(\alpha) \setminus L(\alpha)$ such that $d' \leq_{QQ} d$, $d' \notin L(\alpha,s)$ for all $s \geq t_d$. Then, at any α-true stage $s \geq t_d$ such that $a(\alpha,s) \neq a(\alpha)$, $d <^s_{QQ} a(\alpha,s)$. Also if $\hat{d}$ is a label such that $I(\hat{d},s)$ is rebuilt at stage s, $a(\alpha,s) \leq^s_{QQ} \hat{d}$ so that, if $\hat{d} \in GL$, $d <_{QQ} \hat{d}$.

It follows that, for any $d \in GL$, $\hat{t}(\alpha, s) = d$ for only finitely many stages s. Thus $\liminf_{s\to\infty} \hat{t}(\alpha, s) = \infty$. □

Sublemma 14. *$L(\alpha)$ is finite. In fact $|L(\alpha)| \leq b(\alpha) + 2$.*

Proof. Note firstly that $|\{ d \mid d \leq_{\mathbb{N}} b(\alpha) \}| = b(\alpha) + 1$. Let $l = b(\alpha) + 2$ and suppose that $|L(\alpha)| \geq l$. Let $D = \{a_1 <_{QQ} a_2 <_{QQ} \cdots <_{QQ} a_l\}$ be the set of l labels of least QQ-rank in $L(\alpha)$. Choose α-true $s_D > s_\alpha$ such that a_l has stabilised in $QQ(\alpha)$ and $L(\alpha)$. Note that this means that, for all $1 \leq i < l$, the following conditions hold.

(i) $f_e(m_{a_i})\downarrow$ and $f_{e,s}(m_{a_i}) = f_e(m_{a_i})$ for all (α-true) $s \geq l(\alpha, s_D)$.
(ii) There exists some label $d_i \leq_{\mathbb{N}} b(\alpha)$ such that $f_e(m_{a_i}) \in B(\alpha, d_i, s)$ for all α-true stages $s \geq s_D$.
(iii) For any $1 \leq j < l$ such that $j \neq i$, $d_j \neq d_i$.

Indeed otherwise there would exist some least α-true stage $s \geq s_D$ and label $a \in D \setminus \{a_l\}$ such that a receives attention via[27] Case 3 due to failure of (i), via one of Cases 4, 6(a), 7, or 9 due to failure of (ii), or via Case 8 due to failure of (iii). This would entail $a(\alpha, s) <^s_{QQ} a_l$ in contradiction with our assumption that a_l has stabilised in $L(\alpha)$ at stage s_D.

Now consider any α-true stage $s \geq s_D$. Then if $f_{e,s}(m_{a_l}) \in B(\alpha, d, s)$ for some label $d \leq_{\mathbb{N}} b(\alpha)$, a requires attention via (at least) Case 8 at stage s (as $l > b(\alpha)+1$). On the other hand, if $f_{e,s}(m_{a_l}) \notin B(\alpha, d, s)$ for any label $d \leq b(\alpha)$ then a requires attention attention via one of Cases 3-7, $3 \to 5$, $4 \to 5$ or 9 at stage s. Thus a receives attention at stage s and $a(\alpha, s) = a_l$. It follows that $a_l = a(\alpha)$ and $L(\alpha) = D$.

We conclude therefore that it is always the case that $|L(\alpha)| \leq b(\alpha) + 2$. □

We remind the reader that up to this point in the verification we have been working under the Inductive Hypothesis stated on page 21.

Sublemma 15. *Under the Inductive Hypothesis the following is true. δ_{e+1} is defined and, for all $0 < d \leq e + 1$, $\delta_d = \delta_{d-1}{}^\frown\langle(n_d, 0)\rangle$ for some $n_d \in \mathbb{N}$. Also lim $\inf_{s\to\infty} \hat{t}(\delta_d, s) = \infty$ for all $0 \leq d < e + 1$.*

Proof. Working under the Inductive Hypothesis we saw in Sublemma 14 that $L(\alpha)$ is finite. Thus Sublemmas 11-13 exhaust all the possible outcomes of the activity of strategy $\alpha = \delta_e$. However, in each case we showed that $d(\alpha) = \liminf_{s\to\infty} d(\alpha, s)$ is defined. We also showed that $\{ s \mid R(\alpha, s) = \mathrm{wb} \}$ is finite and that $\{ s \mid R(\alpha, s) \notin \{\mathrm{ddb}, \mathrm{udb}\} \}$ is infinite. Thus $\delta_{e+1} = \alpha^\frown\langle(d(\alpha), 0)\rangle$ is defined. We showed moreover that $\liminf_{s\to\infty} \hat{t}(\alpha, s) = \infty$. Clearly Sublemma 15 follows from these results and the definition of the Inductive Hypothesis itself. □

Sublemma 16. *For all $e \geq 0$, δ_e is defined and lim $\inf_{s\to\infty} \hat{t}(\delta_e, s) = \infty$.*

Proof. This follows directly by induction over indices $e \geq 0$ using the definition of the Inductive Hypothesis, the fact that the Inductive Hypothesis applies trivially for $e = 0$, and application of Sublemma 15. □

Notation. We call $\delta = \bigcup_{e\in\omega} \delta_e$ the *true path* of the construction.

[27]Our assumption that a_l has stabilised in $L(\alpha)$ at stage s_D implies that a does not require attention via Case 5 or 6(b) at such a stage s as $a_l \leq^s_{QQ} a(\alpha, l(\alpha, s))$, so that $a \neq a(\alpha, l(\alpha, s))$.

Definition. For all labels $n \geq 0$, define $I(n)$ to be the block consisting of the elements $\{\, x \mid \exists t(\forall s \geq t)[\, x \in I(n,s)\,]\,\} \subseteq L$. Note that by construction this means that, for some stage r_n, $I(n)$ is the leftmost block of elements in $I(n,s)$ for all $s \geq r_n$. Define $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ by setting $G(q_n) = |I(n)|$ for all $n \geq 0$.

Note 9. It follows from Sublemma 10 (or by inspection of the construction) that for every $x \in L$ there exists label n such that $x \in I(n)$. Also, by density of $\mathbb{Q}$, $\{\, I(n) \mid n \geq 0\,\}$ is precisely the set of maximal blocks in $\mathscr{L}$. Hence $\mathscr{L}$ has order type $\sum\{\, G(q) \mid q \in \mathbb{Q}\,\}$. Moreover, due to reconstruction carried out at odd stages we see that if, for all labels $n \in GL = \{\, n \mid F(q_n) > 1\,\}$, the set of even stages s at which the block $I(n,s)$ can be rebuilt is finite, then $G = F$.

Sublemma 17. *$G = F$. I.e. $\mathscr{L}$ has order type $\sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$.*

Proof. Bearing Note 9 in mind, we show that, for every label $n \in GL$, there are only finitely many (even) stages s at which $I(n,s)$ is rebuilt. So consider any such label n. Choose stage s_n such that $\beta_s > \delta_n$ and such that $\hat{t}(\delta_m, s) >_{\mathbb{N}} n$ for every $m \leq n$ and all $s \geq s_n$. (Note that $n \leq_{\mathbb{N}} b(\delta_n)$ by definition of the latter.) Suppose that there exists strategy γ and even stage $s \geq s_n$ such that γ rebuilds $I(n,s)$ at stage s. Then by definition of s_n it is not the case that $\gamma <_{\text{lex}} \delta_n$. Also by definition of $b(\gamma,s)$ it is not the case that $\delta_n \subseteq \gamma$ or $\delta_n <_{\text{lex}} \gamma$. Thus it can only be the case that $\gamma \subset \delta_n$. However this last case is ruled out by the fact that $\hat{t}(\delta_m, s) >_{\mathbb{N}} n$ for all $m < n$. Thus no such strategy γ and stage s exist and so we can conclude that $I(n,s)$ is only rebuilt at stages $s < s_n$. $\square$

Sublemma 18. *Suppose that index $e \geq 0$ is such that $f_e : L \to L$ is an injective function satisfying conditions* (1) *and* (2).

(1) *For all labels b there exists a label d such that:*
$$f_e(\{\, x \mid x \in I(b)\,\}) = \{\, x \mid x \in I(d)\,\} \tag{18.1}$$
(2) *There exist infinitely many pairs of labels b, d satisfying* (18.1) *such that $b \in GL$ and $b \neq d$.*

Then there exist labels b, d satisfying (18.1) *such that $f_e : I(b) \ncong I(d)$.*

Proof. Note firstly that, by injectivity of f_e, in (18.1) we also know that $|I(b)| = |I(d)|$. Also, for any labels b, b' such that $f_e(\{\, x \mid x \in I(b)\,\}) = f_e(\{\, x \mid x \in I(b')\,\})$, $b = b'$. Let $\alpha = \delta_e$ and suppose (as before) that s_α is an α-true stage such that $\alpha < \beta_s$ for all $s \geq s_\alpha$. Notice that by condition (2) there exist infinitely many $b \in QQ(\alpha)$ $(= GL \cap \{\, d \mid d >_{\mathbb{N}} b(\alpha)\,\})$ such that $f_e(m_b) \neq m_b$. Let $DC(\alpha) = \{\, b \mid b \in QQ(\alpha) \;\&\; f_e(m_b) \neq m_b\,\}$. (We think of $DC(\alpha)$ as the set of *Diagonalisation Candidates* for α.) Let b_0 be the label in $DC(\alpha)$ of least QQ-rank. If $b_0 \in L(\alpha)$ define $a_0 = b_0$. Otherwise, let $t_{b_0} > s_\alpha$ be an α-true stage such that b_0 has stabilised in $QQ(\alpha)$ and such that, for every $d \in QQ(\alpha)$ such that $d \leq_{QQ} b_0$, $f_{e,s}(m_d) = f_e(m_d)$, for all $s \geq t_{b_0}$. Then there can be no α-true stage $s' \geq t_{b_0}$ such that $b_0 \in L(\alpha, s')$ since the definition of t_{b_0} would entail that, as a result, $b_0 \in L(\alpha, s)$ for all $s \geq s'$ contradicting our present hypothesis. But then also there can be no α-true stage $s'' \geq t_{b_0}$ such that $L(\alpha, s'') = \emptyset$ since, at the next α-true stage s''', b_0 would be inserted into $L(\alpha, s''')$ via Case 10. Hence it is necessarily the case that $L(\alpha, s) \neq \emptyset$ for all $s \geq t_{b_0}$. Let a_0 be the label in $L(\alpha, t_{b_0})$ of least (t_{b_0}-stage) QQ-rank. Then $a_0 \in L(\alpha, s)$ for all $s \geq t_{b_0}$. Indeed suppose otherwise and let $s^* > t_{b_0}$ be the least (α-true) stage such that $a_0 \notin L(\alpha, s^*)$. Then, by application of Sublemma 4 we know that $L(\alpha, s) \cap \{ d \mid d <^s_{QQ} a_0 \} = \emptyset$ for all $t_{b_0} \leq s < s^*$. We thus see that removal of a_0 from $L(\alpha, s^*)$ is due to application of Case 2 at stage s^* and that, as a result, $L(\alpha, s^*) = \emptyset$ contradicting our assumption. So in the present case we set $a_0 = b_0$, to get once again that $a_0 \in L(\alpha)$ (with a_0 the label of least QQ-rank in $L(\alpha)$).

Notation. We use $B(d)$ below to denote the leftmost binary block in $I(d)$ if $d \in GL$ and otherwise (the singleton block) $I(d)$ itself if $d \notin GL$.

Let d_0 be the label such that $f_e(\{ x \mid x \in I(a_0) \}) = \{ x \mid x \in I(d_0) \}$. Then if $f_e(m_{a_0}) \in B(d_0)$ and $d_0 \leq b(\alpha)$, let b_1 be the label of least QQ-rank in $DC(\alpha)$ such that $a_0 <_{QQ} b_1$ and $f_e(m_{a_0}) <_{\mathbb{N}} b_1$. Apply the same argument applied relative to b_0 now relative to b_1 to find the label a_1 of least QQ-rank such that $a_0 <_{QQ} a_1$ and $a_1 \in L(\alpha)$. Continue this procedure until finding a_n with corresponding d_n such that $f_e(\{ x \mid x \in I(a_n) \}) = \{ x \mid x \in I(d_n) \}$ for which, either $d_n >_{\mathbb{N}} b(\alpha)$, or $d_n \leq_{\mathbb{N}} b(\alpha)$ but $f_e(m_{a_n}) \notin B(d_n)$. Notice that, under this procedure, either $n = 0$ or $n > 0$ and $a_0 <_{QQ} \cdots <_{QQ} a_n$. Note also that such n exists by injectivity of f_e and application of Conditions (1) and (2). (However it may be the case that $d_n = a_n$.)

Now, if $d_n \leq_{\mathbb{N}} b(\alpha)$, then the fact that $f(m_{a_n}) \in I(d_n) \setminus B(d_n)$ implies that $f_e : I(a_n) \not\cong I(d_n)$. So suppose otherwise, i.e. $d_n >_{\mathbb{N}} b(\alpha)$ so that $d_n \in QQ(\alpha)$, and let $\hat{s} > s_\alpha$ be an α-true stage such that a_n has stabilised in $QQ(\alpha)$ and $L(\alpha)$, d_n has stabilised in $QQ(\alpha)$, and $f_{e,s}(m_{a_n}) = f_e(m_{a_n})$ for all $s \geq l(\alpha, \hat{s})$. Suppose also that $\hat{s}$ is large enough such that, for all $s \geq \hat{s}$, $I(a_n) \subseteq I(a_n, s)$ and $I(d_n) \subseteq (d_n, s)$.

Remark. Note the use of Sublemma 17 here since the latter tells us that $F(q_p) = |I(p)|$ for any label p so that the equation $QQ(\alpha) = \{ n \mid n >_{\mathbb{N}} b(\alpha) \ \& \ |I(n)| > 1 \}$ is valid.

Suppose that $f_e : B(\alpha, a_n, \hat{s}) \cong B(\alpha, d_n, \hat{s})$. (Notice that $a_n \neq d_n$ since the latter would imply $f_e(m_{a_n}) = m_{a_n}$.) Then a_n requires attention at stage $\hat{s}$ via Case 6 or Case 7 and, taking Note 10 below into account we can suppose, without loss of generality, that a_n receives attention via one of Case 6(b) or Case 7 at stage $\hat{s}$.

Note 10. Let $\hat{t}$ be the first α-true stage after $\hat{s}$ and suppose that a_n does not receive attention via Case 6(b) or Case 7 at stage $\hat{s}$. Then this is because a_n receives attention via one of the Cases 3, 3 $\rightarrow$ 5, 5 or 6(a). However in each case $DR(\alpha, \hat{s})$ is reset to $\emptyset$ (as if $DR(\alpha, \hat{s} - 1) \neq \emptyset$ then $DR(\alpha, \hat{s} - 1) = \{d\}$ for some $d \neq d_n$ and Check (ii) of Case 5 will fail at stage $\hat{s}$). Hence $a(\alpha, \hat{s}) = a_n$ and a_n receives attention at stage $\hat{t}$ via the appropriate choice of Case 6(b) or 7.

Now, if Case 6(b) applies at stage $\hat{s}$, then $|I(d_n, \hat{s})| = 2$ but $I(d_n, \hat{s})$ contains some new element so that $I(d_n) \not\subseteq I(d_n, \hat{s})$. On the other hand, if Case 7 applies, then $|I(a_n, \hat{s})| = 2$ but $I(a_n, \hat{s})$ contains some new element so that $I(a_n) \not\subseteq I(a_n, \hat{s})$. Since in both cases this contradicts the definition of $\hat{s}$, it must be the case that $f_e : B(\alpha, a_n, \hat{s}) \not\cong B(\alpha, d_n, \hat{s})$. Moreover we can now see that the definition of $\hat{s}$ implies that $B(\alpha, a_n, \hat{s}) = B(a_n)$ and $B(\alpha, d_n, \hat{s}) = B(d_n)$, i.e. that these are the leftmost binary blocks in $I(a_n)$ and $I(d_n)$ respectively. It thus follows—just as in the case in which $d_n \leq_{\mathbb{N}} b(\alpha)$ that we considered above—that[28] $f_e(\{ x \mid x \in I(a_n) \}) = \{ x \mid x \in I(d_n) \}$ but that $f_e : I(a_n) \not\cong I(d_n)$. □

[28]Notice that, by definition of the construction, $a(\alpha) = a_n$.

Sublemma 19. *Requirement R_e is satisfied for all $e \geq 0$.*

Proof. Suppose that, for some index $e \geq 0$, f_e is a nontrivial automorphism of $\mathscr{L}$. Note that f_e satisfies all the conditions of Sublemma 18.

Remark. To see that condition (2) holds for f_e observe that, as f_e is a nontrivial automorphism, we can choose labels $a \neq d$ such that $f : I(a) \cong I(d)$. Then, as $\mathscr{L}$ contains no interval of order type η (and as f_e is an automorphism) there exist infinitely many pairs of labels b, d_b such that $|I(b)| > 1$, $I(b)$ lies between $I(a)$ and $I(d)$ in $\mathscr{L}$, $d_b \neq b$ and $f : I(b) \cong I(d_b)$.

Sublemma 18 however tells us that there exists a pair of labels a, d such that $f_e(\{\, x \mid x \in I(a)\,\}) = \{\, x \mid x \in I(d)\,\}$ but for which $f_e : I(a) \ncong I(d)$, in contradiction with our assumption that f_e is an automorphism. Thus there is no such index e. □

This concludes the proof of Theorem 3.11. □

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