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THE GROUP OF AUTOMORPHISMS OF THE FIRST WEYL ALGEBRA IN PRIME CHARACTERISTIC AND THE RESTRICTION MAP

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Abstract. Let K be a *perfect* field of characteristic $p > 0$; $A_1 := K\langle x, \partial \mid \partial x - x\partial = 1\rangle$ be the first Weyl algebra; and $Z := K[X := x^p, Y := \partial^p]$ be its centre. It is proved that (i) the restriction map $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \mathrm{Aut}_K(Z)$, $\sigma \mapsto \sigma|_Z$ is a monomorphism with $\mathrm{im(res)} = \Gamma := \{\tau \in \mathrm{Aut}_K(Z) \mid \mathcal{J}(\tau) = 1\}$, where $\mathcal{J}(\tau)$ is the Jacobian of τ, (Note that $\mathrm{Aut}_K(Z) = K^* \ltimes \Gamma$, and if K is *not* perfect then $\mathrm{im(res)} \neq \Gamma$.); (ii) the bijection $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$ is a monomorphism of infinite dimensional algebraic groups which is *not* an isomorphism (even if K is algebraically closed); (iii) an explicit formula for res^{-1} is found via differential operators $\mathcal{D}(Z)$ on Z and negative powers of the Fronenius map F. Proofs are based on the following (non-obvious) equality proved in the paper:

$$\left(\frac{d}{dx} + f\right)^p = \left(\frac{d}{dx}\right)^p + \frac{d^{p-1}f}{dx^{p-1}} + f^p, \quad f \in K[x].$$

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1. Introduction. Let $p > 0$ be a prime number and $\mathbb{F}_p := \mathbb{Z}/\mathbb{Z}p$. Let K be a commutative $\mathbb{F}_p$-algebra and $A_1 := K\langle x, \partial \mid \partial x - x\partial = 1\rangle$ be the first Weyl algebra over K. In order to avoid awkward expressions we sometimes use y instead of ∂; i.e. $y = \partial$. The centre Z of the algebra A_1 is the polynomial algebra $K[X, Y]$ in two variables $X := x^p$ and $Y := \partial^p$. Let $\mathrm{Aut}_K(A_1)$ and $\mathrm{Aut}_K(Z)$ be the groups of K-automorphisms of the algebras A_1 and Z respectively. They contain the subgroups of affine automorphisms $\mathrm{Aff}(A_1) \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2$ and $\mathrm{Aff}(Z) \simeq \mathrm{GL}_2(K)^{op} \ltimes K^2$ respectively. If K is a field of arbitrary characteristic, then the group $\mathrm{Aut}_K(K[X, Y])$ of automorphisms of the polynomial algebra $K[X, Y]$ generated by two of its subgroups, namely $\mathrm{Aff}(K[X, Y])$ and $U(K[X, Y]) := \{\phi_f : X \mapsto X,\ Y \mapsto Y + f \mid f \in K[X]\}$. This was proved by H. W. E. Jung [**5**] for characteristic zero and by W. Van der Kulk [**7**] in general.

If K is a field of characteristic zero J. Dixmier [**4**] proved that the group $\mathrm{Aut}_K(A_1)$ is generated by its subgroups $\mathrm{Aff}(A_1)$ and $U(A_1) := \{\phi_f : x \mapsto x,\ \partial \mapsto \partial + f \mid f \in K[x]\}$. If K is a field of characteristic $p > 0$ L. Makar-Limanov [**8**] proved that the groups $\mathrm{Aut}_K(A_1)$ and $\Gamma := \{\tau \in \mathrm{Aut}_K(K[X, Y]) \mid \mathcal{J}(\tau) = 1\}$ are isomorphic as *abstract* groups in which $\mathcal{J}(\tau)$ is the *Jacobian* of τ. In his paper he used the restriction map

$$\mathrm{res} : \mathrm{Aut}_K(A_1) \to \mathrm{Aut}_K(Z), \quad \sigma \mapsto \sigma|_Z. \tag{1}$$

In this paper, we study this map in detail. Recently, the restriction map (for the nth Weyl algebra) appeared in the papers of Y. Tsuchimoto [**12**], A. Belov-Kanel and M. Kontsevich [**2**] and K. Adjamagbo and A. van den Essen [**1**]. Let us describe some of the results proved in the paper.

THEOREM 1.1. *Let K be a perfect field of characteristic $p > 0$. Then the restriction map* res *is a group monomorphism with* $\mathrm{im}(\mathrm{res}) = \Gamma$.

Note that $\mathrm{Aut}_K(Z) = K^* \ltimes \Gamma$, where $K^* \simeq \{\tau_\lambda : X \mapsto \lambda X,\ Y \mapsto Y \mid \lambda \in K^*\}$.

If K is not perfect, then Theorem 1.1 is *not* true, as one can easily show that the automorphism $\Gamma \ni s_\mu : X \mapsto X + \mu,\ Y \mapsto Y$ does not belong to the image of res provided, $\mu \in K \backslash F(K)$, where $F : a \mapsto a^p$ is the Frobenius map. So, in the case of a perfect field we have another proof of the result of L. Makar-Limanov [**8**]. (In both proofs the results of Jung–Van der Kulk are essential.)

The groups $\mathrm{Aut}_K(A_1)$, $\mathrm{Aut}_K(Z)$ and Γ are infinite dimensional algebraic groups over K in the sense of I. Shafarevich [**10**, **11**] (see also [**9**]).

COROLLARY 1.2. *Let K be a perfect field of characteristic $p > 0$. Then the bijection* $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$, $\sigma \mapsto \sigma|_Z$, *is a monomorphism of algebraic groups over K, which is not an isomorphism of algebraic groups.*

The proofs of Theorem 1.1 and Corollary 1.2 are based on the (non-obvious) formula given next, which allows us to find the inverse map: $\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1)$ (using differential operators $\mathcal{D}(Z)$ on Z; see (14) and Proposition 2.2).

THEOREM 1.3. *Let K be a reduced commutative $\mathbb{F}_p$-algebra and $A_1(K)$ be the first Weyl algebra over K. Then*

$$(\partial + f)^p = \partial^p + \frac{d^{p-1}f}{dx^{p-1}} + f^p$$

for all $f \in K[x]$. In more detail, $(\partial + f)^p = \partial^p - \lambda_{p-1} + f^p$, where $f = \sum_{i=0}^{p-1} \lambda_i x^i \in K[x] = \oplus_{i=0}^{p-1} K[x^p]x^i$, $\lambda_i \in K[x^p]$.

REMARK. We used the fact that $d^{p-1}f/dx^{p-1} = (p-1)!\lambda_{p-1}$ and $(p-1)! \equiv -1 \bmod p$. Theorem 1.3 generalizes the following equality obtained by A. Belov-Kanel and M. Kontsevich [**3**]: if K is a field of characteristic $p > 0$ and $f = dg/dx$ for some polynomial $g \in K[x]$, then $(\partial + f)^p = \partial^p + f^p$.

The group Γ is generated by its two subgroups $U(Z)$ and

$$\Gamma \cap \mathrm{Aff}(Z) = \left\{ \sigma_{A,a} : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto A\begin{pmatrix} X \\ Y \end{pmatrix} + a \,\middle|\, A \in \mathrm{SL}_2(K), a \in K^2 \right\} \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2.$$

Recall that the group $\mathrm{Aut}_K(A_1)$ is generated by its two subgroups $U(A_1)$ and

$$\mathrm{Aff}(A_1) = \left\{ \sigma_{A,a} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} + a \,\middle|\, A \in \mathrm{SL}_2(K), a \in K^2 \right\} \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2.$$

If K is a perfect field of characteristic $p > 0$, then Theorem 1.3 shows that

$$\mathrm{res}(\mathrm{Aff}(A_1)) = \Gamma \cap \mathrm{Aff}(Z) \ \text{ and } \ \mathrm{res}(U(A_1)) = U(Z).$$

In more detail,

$$\mathrm{res}: \mathrm{Aff}(A_1) \to \Gamma \cap \mathrm{Aff}(Z),\ \sigma_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\left(\begin{smallmatrix} e \\ f \end{smallmatrix}\right)} \mapsto \begin{cases} \sigma_{\left(\begin{smallmatrix} a^p & b^p \\ c^p & d^p \end{smallmatrix}\right),\left(\begin{smallmatrix} e^p \\ f^p \end{smallmatrix}\right)}, & \text{if } p > 2, \\ \sigma_{\left(\begin{smallmatrix} a^2 & b^2 \\ c^2 & d^2 \end{smallmatrix}\right),\left(\begin{smallmatrix} e^2+ab \\ f^2+cd \end{smallmatrix}\right)}, & \text{if } p = 2, \end{cases}$$

(see Lemma 3.1 and (11)) and

$$\mathrm{res}: U(A_1) \to U(Z),\quad \phi_f \mapsto \phi_{\theta(f)},$$

where the map $\theta := F + d^{p-1}/dx^{p-1} : K[x] \to K[x^p]$ is a bijection. An explicit formula for the inverse map θ^{-1} is found (Proposition 2.2) via differential operators $\mathcal{D}(Z)$ on Z and negative powers of the Frobenius map F. As a consequence, a formula for the inverse map $\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1)$ is given (see (14)).

2. Proof of Theorem 1.3 and the inverse map θ^{-1}. In this section, a proof of Theorem 1.3 is given, and an inversion formula for a map θ is found, which is a key ingredient in the inversion formula for the restriction map.

Proof of Theorem 1.3. The Weyl algebra $A_1(K) \simeq K \otimes_{\mathbb{F}_p} A_1(\mathbb{F}_p)$ and the Frobenius $F : a \mapsto a^p$ and d^{p-1}/dx^{p-1} behave well under ring extensions, localizations and algebraic closure of the coefficient field. So, without loss of generality we may assume that K is an algebraically closed field of characteristic $p > 0$: the commutative $\mathbb{F}_p$-algebra K is reduced, $\cap_{\mathfrak{p}\in \mathrm{Spec}(K)}\mathfrak{p} = 0$, and $A_1(K)/A_1(K)\mathfrak{p} \simeq A_1(K/\mathfrak{p})$; therefore we may assume that K is a domain; then $A_1(K) \subseteq A_1(\mathrm{Frac}(K)) \subseteq A_1(\overline{\mathrm{Frac}(K)})$, where $\mathrm{Frac}(K)$ is the field of fractions of K, and $\overline{\mathrm{Frac}(K)}$ is its algebraic closure.

First, let us show that the map $L : K[x] \to K[x^p], f \mapsto L(f)$, defined by the rule

$$(\partial + f)^p = \partial^p + L(f) + f^p,$$

is well defined and additive, i.e. $L(f + g) = L(f) + L(g)$. The map

$$K[x] \to \mathrm{Aut}_K(A_1), f \mapsto \sigma_f : x \mapsto x, \partial \mapsto \partial + f$$

is a group homomorphism, i.e. $\sigma_{f+g} = \sigma_f\sigma_g$. Since $\partial^p \in Z(A_1) = K[x^p, \partial^p]$ and $(\partial + f)^p = \sigma(\partial)^p = \sigma(\partial^p) \in Z(A_1)$, the map L is well defined, i.e. $L(f) \in K[x^p]$. Comparing both ends of the series of equalities proves the additivity of the map L:

$$\begin{aligned}\partial^p + L(f+g) + f^p + g^p &= \sigma_{f+g}(\partial)^p = \sigma_{f+g}(\partial^p) = \sigma_f\sigma_g(\partial^p) = \sigma_f(\partial^p + L(g) + g^p)\\ &= \partial^p + L(f) + f^p + L(g) + g^p.\end{aligned}$$

In a view of the decomposition $K[x] = \oplus_{i=0}^{p-1} K[x^p]x^i$ and the additivity of the map L, it suffices to prove the theorem for $f = \lambda x^m$, where $m = 0, 1, \ldots, p-1$ and $\lambda \in K[x^p]$. In addition, we may assume that $\lambda \in K$. This follows directly from the natural $\mathbb{F}_p$-algebra epimorphism

$$A_1(K[t]) \to A_1(K),\quad t \mapsto \lambda,\quad x \mapsto x,\quad \partial \mapsto \partial$$

and the fact that the polynomial algebra $K[t]$ is a domain (hence, reduced). Therefore, it suffices to prove the theorem for $f = \lambda x^m$, where $m = 0, 1, \ldots, p-1$ and $\lambda \in K^*$.

The result is obvious for $m = 0$. So, we fix the natural number m such that $1 \leq m \leq p-1$. Then

$$l_m(\lambda) := L(\lambda x^m) = \sum_{k=0}^{m-1} l_{mk}(\lambda) x^{kp}$$

is a sum of *additive* polynomials $l_{mk}(\lambda)$ in λ of degree $\leq p-1$ (by the very definition of $L(\lambda x^m)$ and its additivity). Recall that a polynomial $l(t) \in K[t]$ is additive if $l(\lambda+\mu) = l(\lambda) + l(\mu)$ for all $\lambda, \mu \in K$. By Lemma 20.3.A **[6]**, each additive polynomial $l(t)$ is a p-polynomial, i.e. a linear combination of the monomials t^{p^r} and $r \geq 0$. Hence, $l_m(\lambda) = a_m \lambda$ for some polynomial $a_m = \sum_{k=0}^{m-1} a_{mk} x^{kp}$, where $a_{mk} \in K$, i.e.

$$(\partial + \lambda x^m)^p = \partial^p + \lambda \sum_{k=0}^{m-1} a_{mk} x^{kp} + (\lambda x^m)^p.$$

Applying the K-automorphism $\gamma : x \mapsto \mu x$, $\partial \mapsto \mu^{-1}\partial$, $\mu \in K^*$, of the Weyl algebra A_1 to the equality above, we have

$$\begin{aligned}
\text{LHS} &= (\mu^{-1}\partial + \lambda\mu^m x^m)^p = \mu^{-p}(\partial + \lambda\mu^{m+1}x^m)^p \\
&= \mu^{-p}\Big(\partial^p + \lambda\mu^{m+1}\sum_{k=0}^{m-1} a_{mk}x^{kp} + (\lambda\mu^{m+1}x^m)^p\Big), \\
\text{RHS} &= \mu^{-p}\partial^p + \lambda\sum_{k=0}^{m-1} a_{mk}\mu^{kp}x^{kp} + (\lambda\mu^m x^m)^p.
\end{aligned}$$

Equating the coefficients of x^{kp} gives $\lambda a_{mk}\mu^{m+1-p} = \lambda a_{mk}\mu^{kp}$. If $a_{mk} \neq 0$ then $\mu^{m+1-p} = \mu^{kp}$ for all $\mu \in K^*$, i.e. $m+1-p = kp$. The maximum of $m+1-p$ is 0 at $m = p-1$, the minimum of kp is 0 at $k = 0$. Therefore, $a_{mk} = 0$ for all $(m, k) \neq (p-1, 0)$.

For $(m, k) = (p-1, 0)$, let $a := a_{p-1,0}$. Then

$$(\partial + \lambda x^{p-1})^p = \partial^p + \lambda a + (\lambda x^{p-1})^p.$$

In order to find the coefficient $a \in K$, consider the left A_1-module

$$V := A_1/(A_1x^p + A_1\partial) \simeq K[x]/K[x^p] = \oplus_{i=0}^{p-1} K\overline{x}^i,$$

where $\overline{x}^i := x^i + A_1x^p + A_1\partial$. An easy induction on i gives the equalities

$$(\partial + \lambda x^{p-1})^i \overline{x}^{p-1} = (p-1)(p-2)\cdots(p-i)\overline{x}^{p-1-i}, \quad i = 1, 2, \ldots, p-1.$$

Now,

$$\begin{aligned}
(\partial + \lambda x^{p-1})^p \overline{x}^{p-1} &= (\partial + \lambda x^{p-1})(\partial + \lambda x^{p-1})^{p-1}\overline{x}^{p-1} = (\partial + \lambda x^{p-1})(p-1)!\overline{1} \\
&= (p-1)!\lambda\overline{x}^{p-1}.
\end{aligned}$$

On the other hand,

$$(\partial^p + \lambda a + (\lambda x^{p-1})^p)\overline{x}^{p-1} = \lambda a\overline{x}^{p-1},$$

and so $a = (p-1)! \equiv -1 \mod p$. This finishes the proof of Theorem 1.3. $\square$

2.1. The map θ and its inverse. Let K be a commutative $\mathbb{F}_p$-algebra. The polynomial algebra $K[x] = \oplus_{i\geq 0}Kx^i$ is a positively graded algebra and a positively filtered algebra $K[x] = \cup_{i\geq 0}K[x]_{\leq i}$, where $K[x]_{\leq i} := \oplus_{j=0}^{i}Kx^j = \{f \in K[x] \mid \deg(f) \leq i\}$. Similarly, the polynomial algebra $K[x^p]$ in the variable x^p is a positively graded algebra $K[x^p] = \oplus_{i\geq 0}Kx^{pi}$ and a positively filtered algebra $K[x^p] = \cup_{i\geq 0}K[x^p]_{\leq i}$, where $K[x^p]_{\leq i} := \oplus_{j=0}^{i}Kx^{pj} = \{f \in K[x^p] \mid \deg_{x^p}(f) \leq i\}$. The associated graded algebras gr $K[x]$ and gr $K[x^p]$ are canonically isomorphic to $K[x]$ and $K[x^p]$ respectively. For a polynomial $f = \sum_{i=0}^{d}\lambda_i x^i \in K[x]$ (resp. $g = \sum_{i=0}^{d}\mu_i x^{pi} \in K[x^p]$) of degree d, $\lambda_d x^d$ (resp. $\mu_d x^{pd}$) is called the leading term of f (resp. g) denoted by $l(f)$ (resp. $l(g)$). Consider the $\mathbb{F}_p$-linear map (see Theorem 1.3)

$$\theta : F + \frac{d^{p-1}}{dx^{p-1}} : K[x] \to K[x^p], \quad f \mapsto f^p + \frac{d^{p-1}f}{dx^{p-1}}, \tag{2}$$

where $F : f \mapsto f^p$ is the Frobenius ($\mathbb{F}_p$-algebra monomorphism). In more detail,

$$\theta : K[x] = \oplus_{i=0}^{p-1}K[x^p]x^i \to K[x^p] = \oplus_{i=0}^{p-1}K[x^{p^2}]x^{pi}, \quad \sum_{i=0}^{p-1}a_i x^i \mapsto \sum_{i=0}^{p-1}a_i^p x^{pi} - a_{p-1},$$

where $a_i \in K[x^p]$. This means that the map θ respects the filtrations of the algebras $K[x]$ and $K[x^p]$ and $\theta(K[x]_{\leq j}) \subseteq K[x^p]_{\leq j}$ for all $j \geq 0$, and so the associated graded map $\mathrm{gr}(\theta) : K[x] \to K[x^p]$ coincides with the Frobenius F:

$$\mathrm{gr}(\theta) = F. \tag{3}$$

LEMMA 2.1. *Let K be a perfect field of characteristic $p > 0$. Then*

(1) $\mathrm{gr}(\theta) = F : K[x] \to K[x^p]$ *is an isomorphism of $\mathbb{F}_p$-algebras;*

(2) $\theta : K[x] \to K[x^p]$ *is an isomorphism of vector spaces over $\mathbb{F}_p$ such that $\theta(K[x]_{\leq i}) = K[x^p]_{\leq i}$, $i \geq 0$; and*

(3) *for each $f \in K[x]$, $l(\theta(f)) = l(f)^p$.*

Proof. Statement 1 is obvious, since K is a perfect field of characteristic $p > 0$ ($F(K) = K$). Statements 2 and 3 follow from statement 1. □

REMARK. The problem of finding the inverse map res^{-1} of the group isomorphism $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$, $\sigma \mapsto \sigma|_Z$ is essentially equivalent to the problem of finding θ^{-1} (see (14)).

The inversion formula for θ^{-1} (Proposition 2.2) is given via certain differential operators. We recall some facts of differential operators that are needed in the proof of Proposition 2.2.

Let K be a field of characteristic $p > 0$ and $\mathcal{D}(K[x]) = \oplus_{i\geq 0}K[x]\partial^{[i]}$ be the ring of differential operators on the polynomial algebra $K[x]$, where $\partial^{[i]} := \frac{\partial^i}{i!}$. The algebra $K[x]$ is a left $\mathcal{D}(K[x])$-module (in the usual sense):

$$\partial^{[i]}(x^j) = \binom{j}{i}x^{j-i} \quad \text{for all} \quad i, j \geq 0.$$

In particular,

$$\partial^{[pi]}(x^{pj}) = \binom{pj}{pi}x^{p(j-i)} = \binom{j}{i}x^{p(j-i)} \quad \text{for all} \quad i, j \geq 0.$$

The subalgebra $K[x^p]=\oplus_{i=0}^{p-1}K[x^{p^2}]x^{pi}$ of $K[x]$ is $x^p\partial^{[p]}$-invariant, and for each $i=0,1,\dots,p-1$, $K[x^{p^2}]x^{pi}$ is the eigenspace of the element $x^p\partial^{[p]}$ that corresponds to the eigenvalue i. Let $J(i):=\{0,1,\dots,p-1\}\backslash\{i\}$. Then

$$\pi_i:=\partial^{[pi]}\frac{\prod_{j\in J(i)}(x^p\partial^{[p]}-j)}{\prod_{j\in J(i)}(i-j)}:K[x^p]\to K[x^{p^2}],\quad \sum_{i=0}^{p-1}a_ix^{pi}\mapsto a_i, \tag{4}$$

where all $a_i\in K[x^{p^2}]$ (since the map $\frac{\prod_{j\in J(i)}(x^p\partial^{[p]}-j)}{\prod_{j\in J(i)}(i-j)}:K[x^p]\to K[x^p]$ is the projection onto the summand $K[x^{p^2}]x^{pi}$ in the decomposition $K[x]=\oplus_{i=0}^{p-1}K[x^{p^2}]x^{pi}$ and $\partial^{[pi]}(a_ix^{pi})=a_i$).

Let K be a perfect field of characteristic $p>0$. Consider the $\mathbb{F}_p$-linear map

$$\partial^{[(p-1)p]}F^{-1}:K[x^{p^2}]\to K[x^{p^2}],\quad \sum_{i\geq 0}a_ix^{p^2i}\mapsto \sum_{i\geq 0}a_{p-1+pi}^{\frac{1}{p}}x^{p^2i}, \tag{5}$$

where $a_i\in K$. By induction on a natural number n, we have

$$\left(\partial^{[(p-1)p]}F^{-1}\right)^n\left(\sum_{i\geq 0}a_ix^{p^2i}\right)=\sum_{i\geq 0}a_{(p-1)(1+p+\cdots+p^{n-1})+p^ni}^{p^{-n}}x^{p^2i},\quad n\geq 1. \tag{6}$$

This shows that the map $\partial^{[(p-1)p]}F^{-1}$ is a *locally nilpotent* map. This means that $K[x^{p^2}]=\cup_{n\geq 1}\ker(\partial^{[(p-1)p]}F^{-1})^n$; i.e. for each element $a\in K[x^{p^2}]$, $(\partial^{[(p-1)p]}F^{-1})^n(a)=0$ for all $n\gg 0$. Hence, the map $1-\partial^{[(p-1)p]}F^{-1}$ is invertible, and its inverse is given by the rule

$$\left(1-\partial^{[(p-1)p]}F^{-1}\right)^{-1}=\sum_{j\geq 0}\left(\partial^{[(p-1)p]}F^{-1}\right)^j. \tag{7}$$

The proposition given next gives an explicit formula for θ^{-1}.

PROPOSITION 2.2. *Let K be a perfect field of characteristic $p>0$. Then the inverse map $\theta^{-1}:K[x^p]=\oplus_{i=0}^{p-1}K[x^{p^2}]x^{pi}\to K[x]=\oplus_{i=0}^{p-1}K[x^p]x^i$, $\sum_{i=0}^{p-1}\mu_ix^{pi}\mapsto\sum_{i=0}^{p-1}\lambda_ix^i$, $\mu_i\in K[x^{p^2}]$, $\lambda_i\in K[x^p]$, is given by the rule*

(1) *for $i=0,1,\dots,p-2$, $\lambda_i=\mu_i^{\frac{1}{p}}+F^{-1}\pi_iF^{-1}\sum_{j\geq 0}(\partial^{[(p-1)p]}F^{-1})^j(\mu_{p-1})$ and*

(2) $\lambda_{p-1}=(\sum_{i=0}^{p-2}x^{pi}\pi_iF^{-1}\sum_{j\geq 0}(\partial^{[(p-1)p]}F^{-1})^j+x^{p(p-1)}\sum_{j\geq 1}(\partial^{[(p-1)p]}F^{-1})^j)(\mu_{p-1})$, *where π_i is defined in* (4).

Proof. Let $g=\sum_{i=0}^{p-1}\mu_ix^{pi}\in K[x^p]$, $\mu_i\in K[x^{p^2}]$; $f=\sum_{i=0}^{p-1}\lambda_ix^i\in K[x]$, $\lambda_i\in K[x^p]$; and $\lambda_{p-1}=\sum_{i=0}^{p-1}a_ix^{pi}$, $a_i\in K[x^{p^2}]$. Then $\theta^{-1}(g)=f$ iff $g=\theta(f)$ iff $F^{-1}(g)=F^{-1}\theta(f)$ iff

$$\sum_{i=0}^{p-1}F^{-1}(\mu_i)x^i=F^{-1}(F(f)-\lambda_{p-1})=f-F^{-1}(\lambda_{p-1})=\sum_{i=0}^{p-1}(\lambda_i-F^{-1}(a_i))x^i$$

iff

$$\lambda_i=F^{-1}(\mu_i+a_i),\quad i=0,1,\dots,p-1. \tag{8}$$

For $i = p-1$, (8) can be rewritten as follows:

$$\sum_{i=0}^{p-2} a_i x^{pi} + a_{p-1}x^{p(p-1)} = F^{-1}(\mu_{p-1} + a_{p-1}). \tag{9}$$

For each $i = 0, 1, \dots, p-2$, applying the map π_i (see (4)) to (9) gives the equality $a_i = \pi_i F^{-1}(\mu_{p-1} + a_{p-1})$, and so the equalities (8) can be rewritten as follows:

$$\lambda_i = F^{-1}(\mu_i + \pi_i F^{-1}(\mu_{p-1} + a_{p-1})), \quad i = 0, 1, \dots, p-2. \tag{10}$$

Applying $\partial^{[(p-1)p]}$ to (9) yields $a_{p-1} = \partial^{[(p-1)p]}F^{-1}(\mu_{p-1} + a_{p-1})$, and so $(1-\Delta)a_{p-1} = \Delta(\mu_{p-1})$, where $\Delta := \partial^{[(p-1)p]}F^{-1}$. By (7), $a_{p-1} = \sum_{j\geq 1}\Delta^j(\mu_{p-1})$. Putting this expression in (10) yields

$$\lambda_i = F^{-1}(\mu_i) + F^{-1}\pi_i F^{-1}\sum_{j\geq 0}\Delta^j(\mu_{p-1}), \quad i = 0, 1, \dots, p-2.$$

This proves statement 1. Finally,

$$\begin{aligned}
\lambda_{p-1} &= \sum_{i=0}^{p-1} a_i x^{pi} = \sum_{i=0}^{p-2} a_i x^{pi} + a_{p-1}x^{p(p-1)} \\
&= \sum_{i=0}^{p-2} x^{pi}\pi_i F^{-1}(\mu_{p-1} + a_{p-1}) + x^{p(p-1)}\sum_{j\geq 1}\Delta^j(\mu_{p-1}) \\
&= \sum_{i=0}^{p-2} x^{pi}\pi_i F^{-1}\sum_{j\geq 0}\Delta^j(\mu_{p-1}) + x^{p(p-1)}\sum_{j\geq 1}\Delta^j(\mu_{p-1}) \\
&= \left(\sum_{i=0}^{p-2} x^{pi}\pi_i F^{-1}\sum_{j\geq 0}\left(\partial^{[(p-1)p]}F^{-1}\right)^j + x^{p(p-1)}\sum_{j\geq 1}\left(\partial^{[(p-1)p]}F^{-1}\right)^j\right)(\mu_{p-1}). \qquad \square
\end{aligned}$$

3. The restriction map and its inverse. In this section, Theorems 1.1 and 3.4 and Corollary 1.2 are proved. An inversion formula for the restriction map res : $\mathrm{Aut}_K(A_1) \to \Gamma$ is found (see (14)).

3.1. The group of affine automorphisms. Let K be a perfect field of characteristic $p > 0$. Each element a of the Weyl algebra $A_1 = \oplus_{i,j\in\mathbb{N}} Kx^i y^j$ is a unique sum $a = \sum \lambda_{ij} x^i y^j$, where all but finitely many scalars $\lambda_{ij} \in K$ are equal to zero. The number $\deg(a) := \max\{i + j \mid \lambda_{ij} \neq 0\}$ is called the degree of a, $\deg(0) := -\infty$. Note that $\deg(ab) = \deg(a) + \deg(b)$, $\deg(a+b) \leq \max\{\deg(a), \deg(b)\}$ and $\deg(\lambda a) = \deg(a)$ for all $\lambda \in K^*$. For each $\sigma \in \mathrm{Aut}_K(A_1)$,

$$\deg(\sigma) := \max\{\deg(\sigma(x)), \deg(\sigma(y))\}$$

is called the *degree* of σ. The set (which is obviously a subgroup of $\mathrm{Aut}_K(A_1)$) $\mathrm{Aff}(A_1) = \{\sigma \in \mathrm{Aut}_K(A_1) \mid \deg(\sigma) = 1\}$ is called the group of affine automorphisms of the Weyl algebra A_1. Clearly,

$$\mathrm{Aff}(A_1) = \left\{ \sigma_{A,a} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} + a \,\middle|\, A \in \mathrm{SL}_2(K),\, a \in K^2 \right\}, \quad \sigma_{A,a}\sigma_{B,b} = \sigma_{BA,Ba+b}.$$

For each group G, let G^{op} be its *opposite* group. ($G^{op} = G$ as set, but the product ab in G^{op} is equal to ba in G.) The map $G \to G^{op}$, $g \mapsto g^{-1}$, is a group automorphism. The group $\mathrm{Aff}(A_1)$ is the semi-direct product $\mathrm{SL}_2(K)^{op} \ltimes K^2$ of its subgroups $\mathrm{SL}_2(K)^{op} = \{\sigma_{A,0} \mid A \in \mathrm{SL}_2(K)\}$ and $K^2 \simeq \{\sigma_{1,a} \mid a \in K^2\}$, where K^2 is the normal subgroup of $\mathrm{Aff}(A_1)$ since $\sigma_{A,0}\sigma_{1,a}\sigma_{A,0}^{-1} = \sigma_{1,A^{-1}a}$. It is obvious that the group $\mathrm{Aff}(A_1)$ is generated by the automorphisms

$$s : x \mapsto y,\ y \mapsto -x;\quad t_\mu : x \mapsto \mu x,\ y \mapsto \mu^{-1}y;\quad \phi_{\lambda x^i} : x \mapsto x,\ y \mapsto y + \lambda x^i,$$

where $\lambda \in K$, $\mu \in K^*$ and $i = 0, 1$.

Recall that the centre Z of the Weyl algebra A_1 is the polynomial algebra $K[X, Y]$ in $X := x^p$ and $Y := y^p$ variables. Let $\deg(z)$ be the total degree in X and Y of a polynomial $z \in Z$. For each automorphism $\sigma \in \mathrm{Aut}_K(Z)$,

$$\deg(\sigma) := \max\{\deg(\sigma(X)), \deg(\sigma(Y))\}$$

is called the *degree* of σ.

$$\begin{aligned} \mathrm{Aff}(Z) &:= \{\sigma \in \mathrm{Aut}_K(Z) \mid \deg(\sigma) = 1\} \\ &= \left\{ \sigma_{A,a} : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto A\begin{pmatrix} X \\ Y \end{pmatrix} + a \,\middle|\, A \in \mathrm{GL}_2(K),\, a \in K^2 \right\} \end{aligned}$$

is the group of affine automorphisms of Z, $\sigma_{A,a}\sigma_{B,b} = \sigma_{BA,Ba+b}$. The group $\mathrm{Aff}(A_1)$ is the semi-direct product $\mathrm{GL}_2(K)^{op} \ltimes K^2$ of its subgroups $\mathrm{GL}_2(K)^{op} = \{\sigma_{A,0} \mid A \in \mathrm{GL}_2(K)\}$ and $K^2 \simeq \{\sigma_{1,a} \mid a \in K^2\}$, where K^2 is a normal subgroup of $\mathrm{Aff}(Z)$ since $\sigma_{A,0}\sigma_{1,a}\sigma_{A,0}^{-1} = \sigma_{1,A^{-1}a}$.

A group G is called an *exact* product of its subgroups G_1 and G_2 denoted by $G = G_1 \times_{ex} G_2$ if each element $g \in G$ is a unique product $g = g_1g_2$ for some elements $g_1 \in G_1$ and $g_2 \in G_2$. Then $\mathrm{GL}_2(K)^{op} = K^* \times_{ex} \mathrm{SL}_2(K)^{op}$, where $K^* \simeq \{\gamma_\mu : X \mapsto \mu X,\ Y \mapsto Y \mid \mu \in K^*\}$, $\gamma_\mu\gamma_\nu = \gamma_{\mu\nu}$. Clearly, $\mathrm{Aff}(Z) = (K^* \times_{ex} \mathrm{SL}_2(K)^{op}) \ltimes K^2$, and so the group $\mathrm{Aff}(Z)$ is generated by the following automorphisms (where $\lambda \in K$, $\mu \in K^*$ and $i = 0, 1$):

$$\begin{aligned} &s : X \mapsto Y,\ Y \mapsto -X;\ \ t_\mu : X \mapsto \mu X,\ Y \mapsto \mu^{-1}Y;\ \phi_{\lambda X^i} : X \mapsto X, \\ &\quad Y \mapsto Y + \lambda X^i;\ \text{ and }\ \gamma_\mu. \end{aligned}$$

The automorphisms t_μ and γ_ν commute.

LEMMA 3.1. *Let K be a perfect field of characteristic $p > 0$. Then the restriction map* $\mathrm{res}_{aff} : \mathrm{Aff}(A_1) \to \mathrm{Aff}(Z)$, $\sigma \mapsto \sigma|_Z$, *is a group monomorphism with* $\mathrm{im}(\mathrm{res}_{aff}) = \mathrm{SL}_2(K)^{op} \ltimes K^2$.

Proof. Since $\mathrm{res}_{aff}(s) = s$, $\mathrm{res}_{aff}(t_\mu) = t_{\mu^p}$; for $i = 0, 1$, $\mathrm{res}_{aff}(\phi_{\lambda x^i}) = \phi_{\lambda^p X^i}$ if $p > 2$ and $\mathrm{res}_{aff}(\phi_{\lambda x^i}) = \phi_{\lambda^2 X^i + \delta_{i,1}\lambda}$ if $p = 2$, where $\delta_{i,1}$ is the Kronecker delta (Theorem 1.3);

i.e.

$$\mathrm{res}_{\mathit{aff}}\left(\sigma_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right),\left(\begin{smallmatrix} e \\ f\end{smallmatrix}\right)}\right) = \begin{cases} \sigma_{\left(\begin{smallmatrix} a^p & b^p \\ c^p & d^p\end{smallmatrix}\right),\left(\begin{smallmatrix} e^p \\ f^p\end{smallmatrix}\right)}, & \text{if } p > 2, \\ \sigma_{\left(\begin{smallmatrix} a^2 & b^2 \\ c^2 & d^2\end{smallmatrix}\right),\left(\begin{smallmatrix} e^2+ab \\ f^2+cd\end{smallmatrix}\right)}, & \text{if } p = 2. \end{cases} \tag{11}$$

The result is obvious. □

LEMMA 3.2. *The automorphisms of the algebra Z: s, t_μ, $\phi_{\lambda X^i}$ and γ_μ satisfy the following relations:*

(1) $st_\mu = t_{\mu^{-1}}s$ *and* $s\gamma_\mu = \gamma_\mu t_{\mu^{-1}}s$;
(2) $\phi_{\lambda X^i} t_\mu = t_\mu \phi_{\lambda\mu^{-i-1}X^i}$ *and* $\phi_{\lambda X^i}\gamma_\mu = \gamma_\mu \phi_{\lambda\mu^{-i}X^i}$; *and*
(3) $s^2 = t_{-1}$, $s^{-1} = t_{-1}s : X \mapsto -Y,\ Y \mapsto X$.

Proof. Straightforward. □

The map

$$K[X] \to \mathrm{Aut}(Z),\ \ f \mapsto \phi_f : X \mapsto X, \quad Y \mapsto Y + f,$$

is a group monomorphism ($\phi_{f+g} = \phi_f\phi_g$). For $\sigma \in \mathrm{Aut}(Z)$, $\mathcal{J}(\sigma) := \det\begin{pmatrix} \frac{\partial\sigma(X)}{\partial X} & \frac{\partial\sigma(X)}{\partial Y} \\ \frac{\partial\sigma(Y)}{\partial X} & \frac{\partial\sigma(Y)}{\partial Y}\end{pmatrix}$ is the Jacobian of σ. It follows from the equality (which is a direct consequence of the chain rule) $\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau))$ that $\mathcal{J}(\sigma) \in K^*$ (since $1 = \mathcal{J}(\sigma\sigma^{-1}) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\sigma^{-1}))$ in $K[X, Y]$), and so the kernel $\Gamma := \{\sigma \in \mathrm{Aut}_K(Z) \mid \mathcal{J}(\sigma) = 1\}$ of the group epimorphism $\mathcal{J} : \mathrm{Aut}(Z) \to K^*$, $\sigma \mapsto \mathcal{J}(\sigma)$, is a normal subgroup of $\mathrm{Aut}_K(Z)$. Hence,

$$\mathrm{Aut}_K(Z) = K^* \ltimes \Gamma \tag{12}$$

is the semi-direct product of its subgroups Γ and $K^* \simeq \{\gamma_\mu \mid \mu \in K^*\}$.

COROLLARY 3.3. *Let K be a field of characteristic $p > 0$. Then*

(1) *each automorphism $\sigma \in \mathrm{Aut}_K(Z)$ is a product $\sigma = \gamma_\mu t_\nu \phi_{f_1} s\phi_{f_2} s \ldots \phi_{f_{n-1}} s\phi_{f_n}$ for some $\mu, \nu \in K^*$ and $f_i \in K[x]$, and*
(2) *each automorphism $\sigma \in \Gamma$ is a product $\sigma = t_\nu \phi_{f_1} s\phi_{f_2} s \ldots \phi_{f_{n-1}} s\phi_{f_n}$ for some $\nu \in K^*$ and $f_i \in K[x]$.*

Proof. (1) Statement 1 follows at once from Lemma 3.2 and the fact that the group $\mathrm{Aut}_K(Z)$ is generated by $\mathrm{Aff}(Z)$ and $\phi_{\lambda X^i}$, $\lambda \in K$, $i \in \mathbb{N}$.

(2) Statement 2 follows from statement 1: $\sigma = \gamma_\mu t_\nu \phi_{f_1} s\phi_{f_2} s \ldots \phi_{f_{n-1}} s\phi_{f_n} \in \Gamma$ iff

$$1 = \mathcal{J}(\sigma) = \mathcal{J}(\gamma_\mu t_\nu \phi_{f_1} s\phi_{f_2} s \ldots \phi_{f_{n-1}} s\phi_{f_n}) = \mathcal{J}(\gamma_\mu)\gamma_\mu(1) = \mu$$

iff $\sigma = t_\nu \phi_{f_1} s\phi_{f_2} s \ldots \phi_{f_{n-1}} s\phi_{f_n}$. □

Proof of Theorem 1.1. Step 1: res *is a monomorphism*. It is obvious that

$$\deg\ \mathrm{res}(\sigma) = \deg\ \sigma, \quad \sigma \in \mathrm{Aut}_K(A_1). \tag{13}$$

The map res is a group homomorphism; so we have to show that $\mathrm{res}(\sigma) = \mathrm{id}_Z$ implies $\sigma = \mathrm{id}_{A_1}$, where id_Z and id_{A_1} are the identity maps on Z and A_1 respectively. By (13), $\mathrm{res}(\sigma) = \mathrm{id}_Z$ implies $\deg(\sigma) = 1$. Then, by (11), $\sigma = \mathrm{id}_{A_1}$.

Step 2: $\Gamma \subseteq \mathrm{im}(\mathrm{res})$. By Corollary 3.3.(2), each automorphism $\sigma \in \Gamma$ is a product, $\sigma = t_\nu \phi_{f_1} s \dots \phi_{f_{n-1}} s \phi_{f_n}$. Since $\mathrm{res}(t_{\nu^{\frac{1}{p}}}) = t_\nu$, $\mathrm{res}(\phi_{\theta^{-1}(f_i)}) = \phi_{f_i}$ and $\mathrm{res}(s) = s$, we have $\sigma = \mathrm{res}(t_{\nu^{\frac{1}{p}}} \phi_{\theta^{-1}(f_1)} s \dots \phi_{\theta^{-1}(f_{n-1})} s \phi_{\theta^{-1}(f_n)})$, and so $\Gamma \subseteq \mathrm{im}(\mathrm{res})$.

Step 3: $\Gamma = \mathrm{im}(\mathrm{res})$. Let $\sigma \in \mathrm{im}(\mathrm{res})$. By Corollary 3.3.(1),

$$\mathrm{res}(\sigma) = \gamma_\mu t_\nu \phi_{f_1} s \dots \phi_{f_{n-1}} s \phi_{f_n} = \gamma_\mu \mathrm{res}(\tau)$$

for some $\tau \in \mathrm{Aut}_K(A_1)$, such that $\mathrm{res}(\tau) \in \Gamma$, by Step 2. Then $\mathrm{res}(\sigma\tau^{-1}) = \gamma_\mu$. By (13), $\deg(\sigma\tau^{-1}) = \deg\, \mathrm{res}(\sigma\tau^{-1}) = \deg\, \gamma_\mu = 1$, and so $\sigma\tau^{-1} \in \mathrm{Aff}(A_1)$. By Lemma 3.1, $\gamma_\mu = 1$, and so $\sigma = \tau$; hence $\mathrm{res}(\sigma) = \mathrm{res}(\tau) \in \Gamma$. This means that $\Gamma = \mathrm{im}(\mathrm{res})$. □

If K is a *perfect* field of characteristic $p > 0$ we obtain the result of L. Makar-Limanov.

THEOREM 3.4. *Let K be a perfect field of characteristic $p > 0$. Then the group* $\mathrm{Aut}_K(A_1)$ *is generated by* $\mathrm{Aff}(A_1) \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2$ *and the automorphisms* $\phi_{\lambda x^i}$, $\lambda \in K^*$, $i = 2, 3, \dots$.

Proof. By Theorem 1.1, the map $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$ is the isomorphism of groups. By Corollary 3.3.(2), each element $\gamma \in \Gamma$ is a product,

$$\gamma = t_\nu \phi_{f_1} s \dots \phi_{f_{n-1}} s \phi_{f_n} = \mathrm{res}\big(t_{\nu^{\frac{1}{p}}} \phi_{\theta^{-1}(f_1)} s \dots \phi_{\theta^{-1}(f_{n-1})} s \phi_{\theta^{-1}(f_n)}\big).$$

Now, it is obvious that the group $\mathrm{Aut}_K(A_1)$ is generated by $\mathrm{Aff}(A_1)$ and the automorphisms $\phi_{\lambda x^i}$, $\lambda \in K^*$, $i = 2, 3, \dots$. □

3.2. The inverse map $\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1)$**.** By Corollary 3.3.(2), each element $\gamma \in \Gamma$ is a product $\gamma = t_\nu \phi_{f_1} s \dots \phi_{f_{n-1}} s \phi_{f_n}$. By Proposition 2.2, the inverse map for res is given by the rule

$$\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1),\ \ \gamma = t_\nu \phi_{f_1} s \dots \phi_{f_{n-1}} s \phi_{f_n} \mapsto t_{\nu^{\frac{1}{p}}} \phi_{\theta^{-1}(f_1)} s \dots \phi_{\theta^{-1}(f_{n-1})} s \phi_{\theta^{-1}(f_n)}. \tag{14}$$

Proof of Corollary 1.2. The group $\mathrm{Aut}_K(A_1)$ (resp. $\mathrm{Aut}_K(Z)$) are infinite-dimensional algebraic groups over K (and over $\mathbb{F}_p$), where the coefficients of the polynomials $\sigma(x)$ and $\sigma(y)$, where $\sigma \in \mathrm{Aut}_K(A_1)$ (resp. of $\tau(X)$ and $\tau(Y)$ in which $\tau \in \mathrm{Aut}_K(Z)$), are coordinate functions (see **[10]** and **[11]**). The group Γ is a closed subgroup of $\mathrm{Aut}_K(Z)$. By the very definition, the map $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$ is a polynomial map (i.e. a morphism of algebraic varieties). By (14) and Proposition 2.2, res^{-1} is not a polynomial map over K (and over $\mathbb{F}_p$ either). □

4. The image of the restriction map res_n.

Let K be a field of characteristic $p > 0$ and $A_n = K\langle x_1, \dots, x_{2n}\rangle$ be the nth Weyl algebra over K: for $i, j = 1, \dots, n$,

$$[x_i, x_j] = 0,\ \ [x_{n+i}, x_{n+j}] = 0,\ \ [x_{n+i}, x_j] = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. The centre Z_n of the algebra A_n is the polynomial algebra $K[X_1, \dots, X_{2n}]$ in $2n$ variables, where $X_i := x_i^p$. The groups of K-automorphisms $\mathrm{Aut}_K(A_n)$ and $\mathrm{Aut}_K(Z_n)$ contain the affine subgroups

$\mathrm{Aff}(A_n) = \mathrm{Sp}_{2n}(K)^{op} \ltimes K^n$ and $\mathrm{Aff}(Z_n) = \mathrm{GL}_n(K)^{op} \ltimes K^n$ respectively. Clearly, $\mathrm{Aff}(A_n) = \{\sigma \in \mathrm{Aut}_K(A_n) \mid \deg(\sigma) = 1\}$ and $\mathrm{Aff}(Z_n) = \{\tau \in \mathrm{Aut}_K(Z_n) \mid \deg(\tau) = 1\}$, where $\deg(\sigma)$ (resp. $\deg(\tau)$) is the (total) degree of σ (resp. τ), defined in the obvious way. The kernel Γ_n of the group epimorphism $\mathcal{J} : \mathrm{Aut}_K(Z_n) \to K^*$, $\tau \mapsto \mathcal{J}(\tau) := \det((\partial\tau(X_i))/(\partial X_j))$ is the normal subgroup $\Gamma_n := \{\tau \in \mathrm{Aut}_K(Z_n) \mid \mathcal{J}(\tau) = 1\}$, and $\mathrm{Aut}_K(Z_n) = K^* \ltimes \Gamma_n$ is the semi-direct product of $K^* \simeq \{\gamma_\mu \mid \gamma_\mu(X_1) = \mu X_1, \gamma_\mu(X_j) = X_j, j = 2, \dots, 2n; \mu \in K^*\}$ and Γ_n.

By considering leading terms of the polynomials $\sigma(X_i)$, it follows as in the case of $n = 1$ that the restriction map

$$\mathrm{res}_n : \mathrm{Aut}_K(A_n) \to \mathrm{Aut}_K(Z_n), \quad \sigma \mapsto \sigma|_{Z_n},$$

is a group monomorphism. If K is a perfect field, then

$$\mathrm{res}_n(\mathrm{Aff}(A_n)) = \mathrm{Sp}_{2n}(K)^{op} \ltimes K^{2n} \subset \mathrm{Aff}(Z_n) = \mathrm{GL}_{2n}(K)^{op} \ltimes K^{2n}.$$

This follows from the fact that for any element of $\mathrm{Aff}(A_n)$, $\sigma_{A,a} : x \mapsto Ax + a$, where $A = (a_{ij}) \in \mathrm{Sp}_{2n}(K)$ and $a = (a_i) \in K^{2n}$,

$$\mathrm{res}_n(\sigma_{A,a}) = \begin{cases} \sigma_{(a_{ij}^p),(a_i^p)} & \text{if } p > 2, \\ \sigma_{(a_{ij}^2),(a_i^2 + \sum_{j=1}^n a_{ij}a_{i,n+j})} & \text{if } p = 2, \end{cases} \tag{15}$$

which can be proved in the same fashion as (11). Since $\mathrm{Sp}_{2n}(K) \subseteq \mathrm{SL}_{2n}(K)$,

$$\mathrm{res}_n(\mathrm{Aff}(A_n)) \subseteq \mathrm{SL}_{2n}(K)^{op} \ltimes K^{2n} \subset \Gamma_n.$$

(Any symplectic matrix $S \in \mathrm{Sp}_{2n}(K)$ has the from $S = TJT^{-1}$ for some matrix $T \in \mathrm{GL}_{2n}(K)$, where $J = \mathrm{diag}(\binom{0\ 1}{-1\ 0}, \dots, \binom{0\ 1}{-1\ 0})$, n times; hence $\det(S) = 1$.)

Question 1. For an algebraically closed field K of characteristic $p > 0$, is $\mathrm{im}(\mathrm{res}_n) \subseteq \Gamma_n$?

Question 2. For an algebraically closed field K of characteristic $p > 0$, is the injection

$$\mathrm{Aff}(Z_n)/\mathrm{res}_n(\mathrm{Aff}(A_n)) \simeq \mathrm{GL}_{2n}(K)^{op}/\mathrm{Sp}_{2n}(K)^{op} \to \mathrm{Aut}_K(Z_n)/\mathrm{im}(\mathrm{res}_n)$$

a bijection?

The next corollary follows from Theorem 1.3.

COROLLARY 4.1. *Let K be a reduced commutative $\mathbb{F}_p$-algebra, $A_n(K)$ be the Weyl algebra and $\partial_i := x_{n+i}$. Then*

$$(\partial_i + f)^p = \partial_i^p + \frac{\partial^{p-1} f}{\partial x_i^{p-1}} + f^p$$

for all polynomials $f \in K[x_1, \dots, x_n]$.

Proof. Without loss of generality we may assume that $i = 1$. Since $K[x_2, \dots, x_n]$ is a reduced commutative $\mathbb{F}_p$-algebra and $\partial_1 + f \in A_1(K[x_2, \dots, x_n])$, the result follows from Theorem 1.3. □

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