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Persistently Exciting Tube MPC

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***Abstract*— This paper presents a new approach to deal with the dual problem of system identification and regulation. The main feature consists of breaking the control input to the system into a regulator part and a persistently exciting part. The former is used to regulate the plant using a robust MPC formulation, in which the latter is treated as a bounded additive disturbance. The identification process is executed by a simple recursive least squares algorithm. In order to guarantee sufficient excitation for the identification, an additional non-convex constraint is enforced over the persistently exciting part.**

I. Introduction

The performance and optimality of a model predictive controller are limited by the accuracy of the model used to make predictions [1], [2]. If the model is a poor representation of reality, the resulting control actions will not be optimal for the latter, causing unexpected behaviour. Moreover, properties desirable for model predictive control (MPC) formulations, such as stability and feasibility [3], [4], often require the computation of invariant sets, which are model dependent [5]. Adaptive MPC, as a way to cope with model uncertainty, has been receiving increasing amounts of attention from researchers in the last ten years, but it remains a largely open problem [6]. Adaptive MPC attempts to couple closed-loop system identification with regulation through a MPC controller; an inherent difficulty of such design is that the two objectives are incompatible. This is referred to as the *dual control problem*: while the controller tries to steady the system, the identifier needs to excite it [7].

A key challenge for adaptive MPC is how to maintain the stability and feasibility guarantees, particularly when hard constraints are considered. In [8] the model of an unconstrained plant is updated through a modified recursive least squares (RLS) algorithm and a fuzzy supervisor attempts to modify the controller parameters based on some arbitrary performance criteria which include a numeric evaluation of stability, but no proof is given. [9] uses a single value decomposition estimation algorithm to identify, on-line, a state space model of the controlled system; stability is shown solely through numerical simulations. A set membership identification scheme is used in [10], coupled with additional output constraints in the optimization problem. These extra constraints ensure boundedness of the system response and hence feasibility, but stability is considered as a standing assumption. Other authors have addressed the issue of stability by making suitable assumptions, such as in [1], [11], [12], where the response error produced by the uncertainty of the model is treated as a bounded disturbance, which allows for robust MPC implementations to be used. An additional assumption of a known bound on the initial estimates error is made in [11] and [1], where the approach is specifically tailored to non-linear parameter affine models. A novel algorithm is developed in [2], where two models of the plant are maintained by the controller; a nominal model is used to provide feasibility and stability guarantees, while the second, adaptive, model is used to improve performance. However, none of these approaches consider that closed-loop system identification and regulation are conflicting objectives: while the controller attempts to drive the plant to a steady state, the identification scheme requires a proper level of excitation to correctly estimate the system parameters [7].

In the MPC context, this has been addressed in different ways. In [13]–[16], an additional constraint over the input is explicitly added to guarantee enough information on the output. The receding horizon fashion of MPC (namely, the fact that only the first part of the optimised input sequence is applied) is considered only in [16], where the additional constraint is applied to the first element of the input sequence throughout the prediction. Alternatively, a two-step optimization is performed in [17], [18]; the first step solves a standard MPC problem while the second step adds an exciting behaviour to the optimised sequence, while limiting the cost increase (reduced optimality). The concept of zone-tracking MPC is used in [19] to drive the state of the plant to an invariant set, inside which a persistently exciting input sequence can be safely applied. In a recent implementation [20], the MPC cost function is augmented with a term depending on the covariance error of the estimated parameters, in an attempt to force the optimiser to choose an exciting input sequence.

In this paper, the dual-problem of regulation and system identification is addressed within the frame of robust MPC; the main feature of the present algorithm is the division of the input signal. The first part, called the persistently exciting (PE) part, aims to generate enough information for the identification process, while the second (regulator) part is designed following the main objective of regulating the plant to the desired steady state. From the control perspective, the PE part of the input may be treated as a bounded disturbance, hence a standard tube MPC formulation [21] represents a suitable selection for the regulation task. This allows to maintain the standard form of the MPC optimization problem, unlike [17], [18], [20], and also helps to establish guaranteed

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asymptotic stability (a property that algorithms in [13]–[16] do not possess). A standard RLS algorithm with forgetting factor [22] is used for the identification process. In order for this to be convergent, an additional constraint based on the persistence of excitation theory [7] is included. This ensures not only an accurate identification process but also that the PE input is automatically defined by the optimiser (contrary to the approach in [19], where it must be computed off-line). The PE constraint proposed in [16] is modified (tightened) so as to guarantee recursive feasibility while allowing for increased optimality.

The paper is organized as follows: Section II states defines the problem and required preliminaries. Section III describes the proposed persistently exciting tube MPC (PE-Tube MPC). Stability and feasibility are established in Section IV. Section Section V contains numerical simulation results.

Notation: The operator $\oplus$ denotes the Minkowski sum, defined as $A \oplus B := \{a + b \mid a \in A, b \in B\}$. The operator $\ominus$ denotes the Pontryagin difference, defined as $A \ominus B := \{a \mid a + b \in A, \forall b \in B\}$. The set $\mathbb{I}_{\geq 0}$ is the set of all the positive integers including 0. The zero vector and the identity matrix in $\mathbb{R}^n$ are represented respectively by $\mathbf{0}_n$ and I_n.

II. Problem statement and preliminaries

The problem is to control a linear time invariant (LTI) system, subject to input and state constraints, for which only a nominal discrete time state space model is known. Define,

$$(\bar{A}, \bar{B}) \quad \text{Nominal system.} \tag{1a}$$
$$(A, B) \quad \text{Real system.} \tag{1b}$$
$$\big(\tilde{A}(i), \tilde{B}(i)\big) \quad \text{Identified system at time } i. \tag{1c}$$

The nominal state space model takes the following form,

$$x(i+1) = \bar{A}x(i) + \bar{B}u(i) \tag{2a}$$
$$x(i) \in \mathbb{X} \subset \mathbb{R}^n, \ \forall i \in \mathbb{I}_{\geq 0} \tag{2b}$$
$$u(i) \in \mathbb{U} \subset \mathbb{R}^m, \ \forall i \in \mathbb{I}_{\geq 0} \tag{2c}$$

in which $x(i)$ and $u(i)$ are the state and input vectors at time i. The following general assumptions are supposed to hold.

Assumption 1 (Stabilizability)**.** The pairs (A, B), $(\bar{A}, \bar{B})$ and $\big(\tilde{A}(i), \tilde{B}(i)\big)$ are stabilizable $\forall i \in \mathbb{I}_{\geq 0}$.

Assumption 2 (Properties of constraint sets)**.** The set $\mathbb{X}$ is closed and the set $\mathbb{U}$ is compact. Both contain the origin.

A. Standard MPC formulation

The standard MPC optimization problem for the nominal system (2) with prediction horizon N at time i is $\mathbb{P}_N\big(x(i)\big)$:

$$\min_u \sum_{k=0}^{N-1} \Big(x^\top(k)Qx(k) + u^\top(k)Ru(k)\Big) + V_f\big(x(N)\big) \tag{3}$$

subject to:

$$x(0) = x(i) \tag{4a}$$
$$x(k+1) = \bar{A}x(k) + \bar{B}u(k) \tag{4b}$$
$$x(k) \in \mathbb{X}, \ k = 0, 1, \ldots, N-1 \tag{4c}$$
$$u(k) \in \mathbb{U}, \ k = 0, 1, \ldots, N-1 \tag{4d}$$
$$x(N) \in \mathbb{X}_f \subseteq \mathbb{X} \tag{4e}$$

where (Q, R) are the state and input weight matrices. Once the optimization is solved, the first part of the optimal input sequence is applied to the plant, a new state measurement is taken, and the process is repeated.

It is well known that, under Assumptions 1 and 2, an appropriate selection of the weight matrices, terminal cost $V_f(\cdot)$ and terminal constraint $\mathbb{X}_f$ provides closed-loop asymptotic stability of the origin [23] for the nominal system. In particular, we use the following standard assumption:

Assumption 3 (Stability assumption)**.** The function $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ is continuous and $V_f(\mathbf{0}_n) = 0$. Q is positive semidefinite and R is positive definite. The set $\mathbb{X}_f$ is a closed control invariant set for the system, containing the origin in its interior, for which,

$$\exists u \in \mathbb{U} \text{ s.t. } V_f\big(\bar{A}x + \bar{B}u\big) + \ell(x, u) \leq V_f(x), \ \ \forall x \in \mathbb{X}_f$$

Recursive feasibility is guaranteed by restricting the initial state to belong to the feasible space (region of attraction) defined by the constraints,

$$\mathcal{X}_N = \Big\{x(0) \in \mathbb{X} \mid \exists \ \{u(k) \in \mathbb{U}\}_{k=0}^{N-1} \text{ s.t. } x(k) \in \mathbb{X}, \ k = 1, 2, \ldots, N-1 \text{ and } x(N) \in \mathbb{X}_f \Big\}$$

B. Persistence of excitation

For many reasons, the nominal model $(\bar{A}, \bar{B})$ may not be an accurate representation of the real system (A, B). This could have a detrimental effect on the performance and stability of the MPC controlled system; therefore, to reduced model uncertainty, some form of closed-loop system identification can be implemented. Note that any state space model may be regarded as a set of $ARX(1,1)$ models, for which a predictor can be built for each component of the state vector,

$$\hat{x}_j(i) = \phi^\top(i-1)\tilde{\theta}_j(i), \quad j = 1, 2, \ldots, n \tag{5a}$$
$$\phi^\top(i) = [x^\top(i) \ \ u^\top(i)] \tag{5b}$$
$$\tilde{\theta}_j(i) = [\tilde{A}_j(i) \ \ \tilde{B}_j(i)]^\top, \quad j = 1, 2, \ldots, n \tag{5c}$$

In (5), $\hat{x}_j(i)$ represents the prediction of the state component j, at time i, $\phi(i)$ is the regressor vector and $(\tilde{A}_j(i), \tilde{B}_j(i))$ are the j^{th} rows of the currently estimated matrices $(\tilde{A}(i), \tilde{B}(i))$. A standard RLS algorithm with constant forgetting factor λ [22] is employed to identify a new model every time step. The recursion at time i is computed as follows,

$$\tilde{\theta}_j(i) = \tilde{\theta}_j(i-1) + R_{\text{ID}}^{-1}(i)\phi(i)\Big[x_j(i) - \phi^\top(i-1)\tilde{\theta}_j(i-1)\Big], \quad j = 1, 2, \ldots, n \tag{6a}$$

$$R_{\text{ID}}(i) = \lambda R_{\text{ID}}(i-1) + \phi(i)\phi^\top(i), \quad j = 1,2,\ldots,n \quad (6b)$$

A sufficient condition to guarantee convergence of the estimated parameters, under an RLS identification algorithm, is that the regressor is strongly persistently exciting [7].

Definition 1 (Strongly persistently exciting sequence)**.** The sequence $\{\phi(i)\} = \phi(0), \phi(1), \cdots, \phi(i)$, is said to be *strongly persistently exciting* of order N_p at time i, if there exists an integer l_p and real numbers $\rho_0, \rho_1 > 0$ such that,

$$\rho_1 I_{(n+m)N_p} > \sum_{j=0}^{l_p-1} \left(\boldsymbol{\phi}_{i-j}\boldsymbol{\phi}_{i-j}^\top\right) > \rho_0 I_{(n+m)N_p}$$

$$\boldsymbol{\phi}_{i-j} = \begin{bmatrix} \phi(i-j) \\ \phi(i-j-1) \\ \vdots \\ \phi(i-j-Np+1) \end{bmatrix}$$

The variable N_p defines the length of a time window that is going to be observed and the variable l_p defines the number of time instants into the past that this window will be observed. Definition 1 is identical to definition 3.4.A given in [7] but after a time shift. The objective of the time shift is to set the current time i as the upper time limit (i.e. the window is placed at time i and it moves backwards). In this way, coupling with the receding horizon fashion of MPC is achieved in a straightforward way.

Persistence of excitation of the regressor vector is not a suitable condition to use as a constraint in the MPC context, mainly because the state vector is not an explicit decision variable of the optimization (3)–(4). Within MPC framework, it is more convenient to focus on the input, which is the decision variable, and how the persistence of excitation of propagates from the input to the regressor. To do this, the concept of state reachability is employed.

Definition 2 (State reachability)**.** System (2) is said to be *state reachable* if, for any $x \in \mathbb{X}$, there exists an input sequence $\{u(j) \in \mathbb{U}\}_{j=0:s<\infty}$ such that at time s, $x(s) = x$.

Theorem 1 (Persistence of excitation of reachable systems)**.** The sequence $\{\phi(i)\} = \phi(0), \phi(1), \cdots, \phi(i)$, with $\phi(\cdot)$ defined as in (5b), is said to be *strongly persistently exciting* of order N_p at time i if, the system (5a) is state reachable and there exists an integer l_p and real numbers $\rho_0, \rho_1 > 0$ such that,

$$\rho_1 I_{mN_p} > \sum_{j=0}^{l-1} \left(\boldsymbol{u}_{i-j}\boldsymbol{u}_{i-j}^\top\right) > \rho_0 I_{mN_p} \quad (7a)$$

$$\boldsymbol{u}_{i-j} = \begin{bmatrix} u(i-j) \\ u(i-j-1) \\ \vdots \\ u(i-j-N_p+1) \end{bmatrix} \quad (7b)$$

Proof. This proof can be found in [24] (Theorem 2.1). ■

In [16] it is shown that the lower bound of inequality (7a) characterizes the outside of an ellipsoid, hence the PE constraint is non-convex. Also note that $\boldsymbol{u}_{i-j} = \left[\boldsymbol{0}_{m(N_p-1)}\right]$ (or any other steady value) violates (7a), which means that regulation to a steady state and persistence of excitation cannot be simultaneously attained.

III. Tube MPC with Persistence of Excitation

The main contribution of this paper is presented in this section. The underlying idea is to include a persistence of excitation constraint in a standard MPC formulation, to guarantee enough information for an accurate identification of the system parameters. This is done from a robust control perspective, where the excitation is treated as a bounded disturbance. The proposed approach uses tube MPC, which is a robust control technique with guaranteed stability under bounded additive uncertainties, but complexity similar to conventional MPC [21], [23].

A. Tube MPC for uncertain systems with partitioned input

Tube MPC solves the regulation problem for an undisturbed nominal model, while securing that the state of the uncertain system will always be in a robust positive invariant (RPI) set [5], centered around the nominal system trajectory. This robust control technique is inherently capable of dealing with model uncertainties as long as these can be *quantified*, i.e., treated as a bounded additive disturbance. This requires a certain insight on how different the real system (1a) and prediction model (1c) may be. To account for this, the following assumption is supposed to hold,

Assumption 4 (Size of parametric uncertainty)**.** A set $\mathbb{W}_S := \left\{w_S = (A - \tilde{A}(i))x + (B - \tilde{B}(i))u \mid (x,u) \in \mathbb{X} \times \mathbb{U}, \quad \forall i \in \mathbb{I}_{\geq 0}\right\}$ is known.

Consider the model structure of (2). Henceforth, the input will be divided into a regulator part, $\hat{u}$, and a persistently exciting part, w. The nominal model (2) is rewritten as,

$$x(i+1) = \bar{A}x(i) + \bar{B}\big(\hat{u}(i) + w(i)\big) \quad (8a)$$

$$x(i) \in \mathbb{X}, \;\; \forall i \in \mathbb{I}_{\geq 0} \quad (8b)$$

$$\hat{u}(i) \in \hat{\mathbb{U}}, \;\; \forall i \in \mathbb{I}_{\geq 0} \quad (8c)$$

$$w(i) \in \mathbb{W}, \;\; \forall i \in \mathbb{I}_{\geq 0} \quad (8d)$$

$$\bar{B}w(i) = \hat{w}(i) \in \hat{\mathbb{W}} = \bar{B}\mathbb{W}, \;\; \forall i \in \mathbb{I}_{\geq 0} \quad (8e)$$

Assumption 5 (Properties of the divided input constraint sets)**.** The sets $\mathbb{W}$ and $\hat{\mathbb{U}}$ are compact and contain the origin. Also $\hat{\mathbb{U}} \oplus \mathbb{W} \subseteq \mathbb{U}$.

Assumption 5 implies $\hat{\mathbb{U}} \subseteq \mathbb{U} \ominus \mathbb{W}$ which must be non-empty. Note that $\hat{\mathbb{W}}$ is a linear mapping of $\mathbb{W}$ therefore it maintains compactness [25].

Within the tube MPC implementation, $\hat{w}$ is treated as a bounded additive disturbance. The undisturbed model takes the form,

$$z(i+1) = \bar{A}z(i) + \bar{B}v(i) \quad (9a)$$

$$z(i) \in \mathbb{Z} = \mathbb{X} \ominus \mathbb{S}, \;\; \forall i \in \mathbb{I}_{\geq 0} \quad (9b)$$

$$v(i) \in \mathbb{V} = \hat{\mathbb{U}} \ominus K_t\mathbb{S}, \;\; \forall i \in \mathbb{I}_{\geq 0} \quad (9c)$$

in which K_t is any stabilizing gain for $(\bar{A}, \bar{B})$, guaranteed to exist in view of Assumption 1. The set $\mathbb{S}$ is an RPI set for the dynamic model of the error between the trajectories of the nominal and uncertain models,

$$e(i) = x(i) - z(i) \tag{10a}$$
$$e(i+1) = (\bar{A} + \bar{B}K_t)e(i) + \hat{w}(i) + w_S(i) \tag{10b}$$
$$\hat{w}(i) \in \hat{\mathbb{W}}, \ \forall i \in \mathbb{I}_{\geq 0} \tag{10c}$$
$$w_S(i) \in \mathbb{W}_S, \ \forall i \in \mathbb{I}_{\geq 0} \tag{10d}$$

For a nominal solution to exists, the constraint space defined by constraints (9b) and (9c) must be non-empty,

Assumption 6 (Allowable disturbance size for constraint satisfaction)**.** The set $\mathbb{S}$ is such that $\mathbb{S} \subset \mathbb{X}$ and $K_t\mathbb{S} \subset \hat{\mathbb{U}}$.

Assumption 6 it is not uncommon in robust control implementations, it simply states the fact that it will not be possible to satisfy the constraints if the disturbances are *too* large. After a standard MPC problem (section II-A) is solved for the nominal system (9a) under tightened constraints (9b) and (9c), the input to the uncertain system is computed from the following control policy,

$$\hat{u}(i) = v(i) + K_t(x(i) - z(i)) \tag{11}$$

B. Additional PE constraint

Since w is bounded (8d), the upper bound in Theorem 1 is trivially fulfilled [15], therefore the focus is placed on achieving the lower bound. At time i define,

$$\boldsymbol{M}\big(w(i)\big) = \boldsymbol{M}_i = \sum_{j=0}^{l-1} \Big(\boldsymbol{w}_{i-j}\boldsymbol{w}_{i-j}^{\top}\Big) - \rho_0 I_{mN_p} \tag{12a}$$

$$\boldsymbol{w}_{i-j} = \begin{bmatrix} w(i-j) \\ w(i-j-1) \\ \vdots \\ w(i-j-N_p+1) \end{bmatrix} \tag{12b}$$

That $\boldsymbol{M}_i$ depends only on the past and current exciting input makes the following a suitable *persistence of excitation* constraint within the receding horizon fashion of MPC

$$\boldsymbol{M}_i > 0 \tag{13}$$

Remark 1. The realisation of constraint (13), i.e. a persistently exciting behaviour of w, does not necessarily imply persistence of excitation in the absolute input signal u. This is because of the control policy (11), which tries to reject disturbances. However, numerical simulations have shown that a proper selection of the linear gain K_t secures that the PE condition is transmitted to u. The investigation of conditions for gain selection is beyond the scope of this paper, hence we require the following assumption.

Assumption 7 (Persistence of excitation transmission)**.** For the linear gain K_t, the persistence of excitation of w is transmitted to the absolute input sequence u.

C. Tube MPC with additional PE constraint

The optimization problem of the proposed model predictive controller, at time i is, $\mathbb{P}_N\big(z(i)\big)$:

$$\min_{v,w} \sum_{k=0}^{N-1} \Big(z^{\top}(k)Qz(k) + d^{\top}(k)\mathcal{R}d(k)\Big) + V_f\big(z(N)\big) \tag{14}$$

subject to:

$$z(0) = z(i) \in \mathcal{Z}_N \tag{15a}$$
$$z(k+1) = \bar{A}z(k) + \bar{B}v(k) \tag{15b}$$
$$z(k) \in \mathbb{Z}, \ k = 0, 1, \ldots, N-1 \tag{15c}$$
$$v(k) \in \mathbb{V}, \ k = 0, 1, \ldots, N-1 \tag{15d}$$
$$z(N) \in \mathbb{Z}_f \subseteq \mathbb{Z} \tag{15e}$$
$$w(k) \in \mathbb{W}, \ k = 0 \tag{15f}$$
$$w(k) = 0_m, \ k = 1, \ldots, N-1 \tag{15g}$$
$$\boldsymbol{M}_i > 0 \tag{15h}$$

where $\mathcal{Z}_N$ and $\mathbb{Z}_f$ are the nominal equivalents of $\mathcal{X}_N$ and $\mathbb{X}_f$ respectively. The cost term $d^{\top}(k)\mathcal{R}d(k)$ is a straight forward augmentation of the usual input cost with,

$$d^{\top}(k) = [v^{\top}(k) \ w^{\top}(k)], \ k = 0, 1, \ldots, N-1 \tag{16a}$$
$$\mathcal{R} = \begin{bmatrix} R & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & R \end{bmatrix} \tag{16b}$$
$$\bar{A} = \tilde{A}(i) \quad \bar{B} = \tilde{B}(i) \tag{16c}$$

IV. Stability and feasibility

In this section stability and recursive feasibility proofs are derived for the PE-Tube MPC.

A. Stability

The objective of performing closed-loop system identification is to reduce model uncertainty and thereby improve the performance of the MPC. Therefore it is safe to assume that the initially known model (1a) will differ from the actual plant being controlled (1b) and from any transitional model (1c) given by the recursive identification (6). This represents a considerable drawback, as one of the main requirements of tube MPC is to know an RPI set, which is model dependent. Different approaches can be used to compute such a set; in [19] for example, Assumption 4 is bypassed by showing that, under a type of parametric affine model uncertainty, an RPI set computed for a certain model is also RPI for a family of models. The definition of a specific set of rules for computing $\mathbb{S}$ is out of the scope of this paper; for the examples shown in Section V a suitable RPI set is computed on the basis of Assumption 4 and the error dynamics (10).

Theorem 2 (Stability of the PE-Tube MPC)**.** If assumptions 1–6 hold, then the set $\mathcal{A} := \mathbb{S} \times \{\mathbf{0}_n\}$ is asymptotically stable with a region of attraction $\big(\mathcal{Z}_N \oplus \mathbb{S}\big) \times \mathcal{Z}_N$ for the constrained composite system,

$$x(i+1) = \bar{A}x(i) + \bar{B}\hat{u}(i) + \hat{w}(i) + w_S(i)$$
$$z(i+1) = \bar{A}z(i) + \bar{B}v(i)$$

under the closed-loop control laws defined by (10) and (14)–(15) respectively.

Proof. Follows directly from the stability proofs in [23]. ■

B. Recursive feasibility

Recursive feasibility of a standard tube MPC formulation is provided by constraint (15a). In [16] an additional assumption is used to provide a proof of recursive feasibility under the effects of the non-convex PE constraint (15h),

Assumption 8. A feasible solution is available at time $i-1$, i.e., $\boldsymbol{M}_{i-1} > 0$

Theorem 3 (Recursive feasibility: Trivial solution)**.** If Assumption 8 holds, then there exists a feasible solution at time i for the persistently exciting tube MPC (14)–(15).

Proof. This theorem is proved in [16]. It is repeated here for clarifying purposes. From (12),

$$\boldsymbol{M}_i = \boldsymbol{M}_{i-1} + \boldsymbol{w}_i \boldsymbol{w}_i^\top - \boldsymbol{w}_{i-l_p} \boldsymbol{w}_{i-l_p}^\top$$
$$\boldsymbol{w}^i = \boldsymbol{w}_i \boldsymbol{w}_i^\top - \boldsymbol{w}_{i-l_p} \boldsymbol{w}_{i-l_p}^\top$$

therefore, a sufficient condition for $\boldsymbol{M}_i > 0$ is that $\boldsymbol{w}^i \geq 0$. The proof is completed by noticing that $w(i) = w(i-l_p) \implies \boldsymbol{w}^i = 0$. ■

Theorem 3 provides recursive feasibility under the trivial periodic repetition of a previous solution, but it does not analyse the effect of choosing a different one. In fact, $w(i)$ is a decision variable in the proposed optimization problem (14), hence the optimiser is *free* to choose $w(i) \neq w(i-l_p)$ as long as constraints (15f) and (15h) are not violated. Numerical simulations (conducted on the same system used as an example in [16]) show that recursive feasibility may be lost if periodicity is broken, namely, if the optimization algorithm lands in a solution such that $w(i) \neq w(i-l_p)$. The observed behaviour can be summarised in,

$$\exists w(i) \in \mathbb{W} \text{ s.t., } \ w(i) \neq w(i-l_p)$$
$$\wedge \ \ \boldsymbol{M}_i > 0, \text{ but } \boldsymbol{M}_{i+1} \leq 0 \ \ \forall w(i+1) \in \mathbb{W}$$

According to (12), the non-trivial optimised $w(i)$ remains in $\boldsymbol{w}_i$ for $N_p - 1$ time steps. To take this into account the following constraint is proposed to replace (15h),

$$\boldsymbol{M}_{i+l} > 0, \ \ l = 0, 1, \ldots, N_p - 1 \tag{17a}$$
$$w(i+l) = w(i+l-l_p), \ \ l = 1, \ldots, N_p - 1 \tag{17b}$$

Theorem 4 (Recursive feasibility: Non-trivial solution)**.** If Assumption 8 holds, and the constraint (15h) is replaced by (17), then there exists a feasible, not necessarily trivial, solution at time i for the PE-tube MPC (14)–(15).

Proof. This result is established by extending the proof for Theorem 3 to cover $N_p - 1$ time steps. ■

V. Simulation results

This section shows the behaviour of the proposed algorithm through two numerical examples. The task is to perform closed-loop identification while regulating the states of the following multi-variable system (taken from the examples used in [19]):

$$A(\delta) = \bar{A} + \delta \hat{A} = \begin{bmatrix} 0.42 & -0.28 \\ 0.02 & 0.60 \end{bmatrix} + \delta \begin{bmatrix} -0.6 & -0.4 \\ -0.6 & -0.85 \end{bmatrix} \tag{18a}$$
$$B(\delta) = \bar{B} + \delta \hat{B} = \begin{bmatrix} 0.30 \\ -0.40 \end{bmatrix} + \delta \begin{bmatrix} -0.2 \\ 0.4 \end{bmatrix} \tag{18b}$$

subject to the following constraints,

$$\mathbb{X} = \{x \in \mathbb{R}^2 \text{ s.t, } |x_j| \leq 17, \ \ j = 1, 2\} \tag{19a}$$
$$\mathbb{U} = \{u \in \mathbb{R} \text{ s.t, } |u| \leq 4\} \tag{19b}$$
$$\mathbb{W} = \{w \in \mathbb{R} \text{ s.t, } |w| \leq 0.2\} \tag{19c}$$
$$|\delta| \leq 0.15 \tag{19d}$$

Since no particular performance requirements are being considered, the controller parameters are loosely set to $N = 3$, $Q = I_{2\times 2}$ and $R = 1$. The terminal cost V_f, and the terminal constraint set $\mathbb{Z}_f$ are computed according to Assumption 3. $A(\delta)$ is inherently stable for any δ following (19d), this provides flexibility in choosing the linear gain $K_t = [-0.112 \ \ 0.354]$ which is stabilizing and complies with Assumption 7. A set $\mathbb{W}_S$ following Assumption 6 is defined for (18)–(19) and the corresponding RPI set $\mathbb{S}$ is computed.

Following the directions given in [7], the PE constraint parameters are set to $N_p = 6$ and $l_p = 11$; given the size of $\mathbb{W}$ a value of $\rho_0 = 0.05$ is employed. To guarantee recursive feasibility at initialization, a feasible PE sequence of length $N_p + l_p - 1$ has to be compute available (Assumption 8). However, this sequence is not explicitly used, it only acts as a buffer for feasibility purposes (which means that some of its elements may indeed be used).

For the RLS algorithm a forgetting factor $\lambda = 0.97$ is employed. The estimates vector $\tilde{\theta}_j$ is initialized at the initially known values $(\bar{A}, \bar{B})$. The information matrix R_{ID} is initialized as the null matrix, therefore a *pseudo*-inverse is computed for (5a) until R_{ID} becomes invertible. Albeit a recursion is computed at every time instant, the prediction model update (16c) is performed only every 3 time steps.

A. Closed-loop identification capabilities

An initial state $x(0) = [0 \ \ 0]^\top \in \mathcal{Z}_N$ is considered to assess the closed-loop identification capabilities of the proposed algorithm. This is done to avoid the additional information that would be generated in the process of regulation, in that way effect of the PE constraint can be observed independently. Fig. 1 shows the optimised input signal generated by the PE-Tube MPC for both, nominal (v) and disturbed (u) systems. As expected, given the initial state, the nominal input remains at the origin while the input for the uncertain system is, indeed, disturbed by the PE part w. Fig. 2 shows the evolution of the states x_1 and z_1. During the initial time steps, the algorithm optimises a PE sequence on

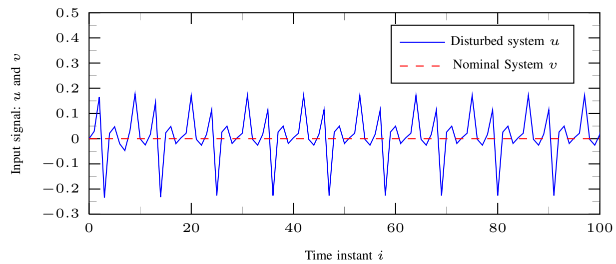

Fig. 1. Evolution of the input signal for the nominal and disturbed systems.

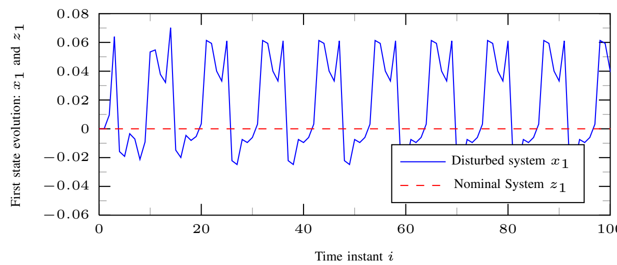

Fig. 2. Evolution of the first state for the nominal and disturbed systems.

TABLE I

EVOLUTION OF ESTIMATED PARAMETERS ERROR [%].

Parameter	$i = 0$	$i = 3$	$i = 6$	$i = 9$
A_{11}	-17.6	-1.76×10^{1}	-2.83×10^{-11}	-2.24×10^{-11}
A_{12}	-17.6	-1.76×10^{1}	1.25×10^{-11}	9.48×10^{-12}
A_{21}	-81.8	-8.18×10^{1}	-5.09×10^{-10}	-4.04×10^{-10}
A_{22}	-17.5	-1.75×10^{1}	-2.27×10^{-11}	-1.71×10^{-11}
B_{11}	-9.09	1.68×10^{-14}	-2.35×10^{-13}	4.37×10^{-13}
B_{21}	-13.0	0	6.87×10^{-13}	-1.19×10^{-12}

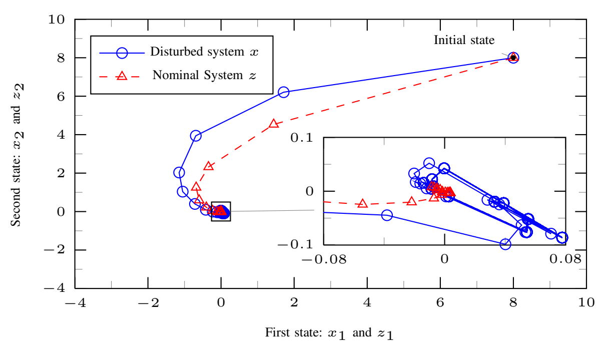

Fig. 3. Evolution of the state for the nominal and disturbed systems.

the basis of the feasible sequence initially supplied, hence the transient behaviour observed on the uncertain system (x). Feasibility is maintained during this period thanks to the tighter constraint (17). An optimised periodic solution is attained fairly fast.

Table I shows the error of the identified value (w.r.t. real value) for all the system parameters, at several time instants. As expected, by cause of the persistence of excitation constraint, the RLS algorithm gets enough information and the true values, within an acceptable tolerance, are reached.

B. *Regulation capabilities*

The regulation capabilities of the PE-Tube MPC are evaluated by initializing the scheme at $x(0) = [8\ 8]^\top \in \mathcal{Z}_N$; Fig. 3 shows the state evolution for both, nominal and real system. As expected, given the stabilizing characteristics of the proposed algorithm, the nominal state shows an asymptotic behaviour towards the origin. Due to the periodic PE disturbance (w), the evolution of the disturbed state is ultimately bounded to lie inside the set $\{\mathbf{0}_n\} \oplus \mathbb{S}$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, a new way to approach the dual-problem of system identification and regulation via a model predictive controller has been presented. At each time instant, the input used to control the system is divided into a persistently exciting (PE) part and a regulator part. The PE part is treated as a bounded disturbance and a tube MPC, enhanced with a PE constraint, is used to regulate the plant. At the same time, thanks to the PE constraint, enough information is generated for the identification process. Under the proper assumptions, the PE-Tube MPC has proved robust stability and recursive feasibility. Future work will be focused on the analysis of the transmission of persistence of excitation and the implementation of on-line variation of the allowable size of perturbations with the objective of feasibility enlargement.

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