The dynamic behaviour of packed and tray-type binary distillation columns described by lumped-parameter models

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Summary

Parametric transfer-function matrix (T.F.M) models are derived from the differential equations describing the variation of liquid and vapour compositions within symmetrical distillation columns separating binary mixtures. Both packed and tray-type columns are considered, the models developed having minimal order 6 and 4 respectively, (implying perfectly-stirred mixtures and minimum number of trays in each column section) so removing the complications introduced by spatial variations encountered in earlier studies[1,2] of longer columns.

The T.F.M's obtained are again shown to be diagonal with the same choice of input and output vectors as were employed in the "long" column studies. Furthermore, the fundamental behavioural differences between packed and tray-type columns are again revealed using these low-order physical models as were demonstrated in the previous studies. In particular, the sign of the static gain of packed-columns, responding to total flow changes is again shown to be parameter-sensitive and the sign of the total-flow gain at high-frequency is shown to be the reverse of that for tray-type columns, so again causing nonminimum-phase behaviour in packed-columns in some circumstances.

Because of their behavioural and parametric similarities, it is proposed that the minimal-order models developed in this report be used as a future basis for control system design studies for spatially-distributed and multi-tray columns in situations where travelling-wave phenomena are unimportant.

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1. Introduction

The author's previous analyses of the dynamic behaviour of compositions within binary distillation columns have revealed that:

(a) Provided the processes are constructed and operated in a symmetrical manner then the transfer-function matrix (T.F.M) of the system, $\underline{G}(h,p)$, is diagonal, i.e.

$$\underline{G}(h,p) = \begin{bmatrix} g_1(h,p) & , & 0 \\ 0 & , & g_2(h,p) \end{bmatrix}$$

(where p denotes the Laplace variable w.r.t normalised time τ and h denotes the normalised distance measured from the two ends of the column) provided the output vector selected is:

$$\underline{q}(h,\tau) = \begin{bmatrix} y(h,\tau) - x'(h,\tau) \\ y(h,\tau) + x'(h,\tau) \end{bmatrix}$$

and the chosen input vector is:

$$\underline{u}(\tau) = \begin{bmatrix} v(\tau) + \ell(\tau) \\ v(\tau) - \ell(\tau) \end{bmatrix}$$

where y and x' denote the vapour and liquid composition perturbations in the rectifying and stripping sections respectively and v and ℓ denote the circulating vapour and liquid rate changes.

(b) Over a wide range of high frequencies g_1 and g_2 may be approximated by pure integrating processes but that the sign of g_1 for packed columns is the reverse of that for tray-type columns within this frequency range.

(c) The sign of the static gain $g_1(h,o)$ is dependent upon packed-column parameters with the result that long packed columns show nonminimum-phase behaviour, $g_1(h,o)$ having a sign opposite to that of the high-frequency gain.

(d) Long tray-type columns (having sufficient trays to permit the approximation of spatially discrete functions y(h,t) and x'(h,t) by continuous functions of h and t) do not exhibit nonminimum-phase behaviour, irrespective of parameter values, and therefore permit approximation by multivariable first-order lag T.F.M's.

Further conclusions concerning the effects of travelling waves on the system behaviour were also drawn but these were found to be relatively unimportant in long columns.

Because of the complexity in the analysis of spatially distributed systems and systems of very high order, and because of the complexity of the resulting transfer-function expressions it is interesting to speculate as to whether the above noted behavioural characteristics (a) to (d) can be reproduced by columns of minimal order and complexity. Figs. 1 and 2 illustrate two such conceptual columns, the first of the packed-type and the second of the tray-type. The crucial difference between the two is that the packed column requires consideration of both the liquid and the vapour capacitance in each section which do not run in equilibrium. The rate of cross-flow from liquid to vapour depends upon the departure of the liquid and vapour compositions from equilibrium and therefore the two capacitances are effectively separated by a cross-flow, or interphase, "resistance". The minimal packed-column model (ensuring liquids and vapours in each section to be thoroughly mixed) therefore requires a total of 6 discrete capacitances (vapour and liquid capacitance in rectifier and stripper plus the capacitances of the two terminating vessels) for its representation and hence a 6th-order dynamic model. The tray-type model, however, requires only 4 discrete capacitances for its representation since liquid and vapour are assumed here to be in continuous equilibrium so that interphase "resistance" is zero permitting either the neglecting of the vapour capacitance or lumping it with the liquid capacitance. A 4th order model therefore applies to the minimal tray-type column.

The following sections of the report consider each type of "lumped" system in turn, developing parametric T.F.M's in both cases, the implications of which for system behaviour are then considered.

2. Packed Column Modelling

2.1 The large-signal model

Taking mass balances on the four column capacitances H_v, H_ℓ, H_v' and H_ℓ' (shown in Fig. 1) in turn we get, if $Y(Y')$ and $X(X')$ denote compositions of vapour and liquid in the rectifier (stripper);

$$H_v \dot{Y}(o) = V_s Y'(o) - V_r y(o) + F_\ell z + k_r\{Y_e(o) - Y(o)\}$$

or

$$H_v \dot{Y}(o) = \alpha V_s X_e'(o) - V_r Y(o) + F_\ell z + k_r \{Y_e(o) - Y(o)\} \quad (1)$$

where

$$\left.\begin{aligned} Y_e &= X/\alpha + (\alpha-1)/\alpha \\ &\text{and} \\ X_e' &= Y'/\alpha \end{aligned}\right\} \quad (2)$$

$$H_\ell \dot{X}(o) = L_r X(1) - L_r X(o) - k_r \{Y_e(o) - Y(o)\}$$

or

$$\alpha H_\ell \dot{Y}_e(o) = \alpha L_r Y_e(1) - \alpha L_r Y_e(o) - k_r \{Y_e(o) - Y(o)\} \qquad (3)$$

$$H'_v \dot{Y}'(o) = V_s Y'(1) - V_s Y'(o) + k_s \{X'(o) - X'_e(o)\}$$

or

$$\alpha H'_v \dot{X}'_e(o) = V_s \alpha X'_e(1) - V_s \alpha X'_e(o) + k_s \{X'(o) - X'_e(o)\} \qquad (4)$$

and finally:

$$H'_\ell \dot{X}'(o) = L_r X(o) - L_s X'(o) + F_\ell Z - k_s \{X'(o) - X'_e(o)\}$$

or

$$H'_\ell \dot{X}'(o) = \alpha L_r Y_e(o) - L_s X'(o) + F_\ell Z - k_s \{X'(o) - X'_e(o)\} \qquad (5)$$

where F_ℓ and F_v are the feed rates of liquid and vapour respectively, the feed vapour composition being z and the feed liquid composition being Z. k_r and k_s are constant coefficients of crossflow from the liquid to vapour phase and suffix e denotes "equilibrium value". V_r (V_s) and L_r (L_s) denote vapour and liquid flow rates in the rectifier (stripper).

The terminal boundary conditions are the mass balance equations pertaining to the accumulator and reboiler vessels (capacitances H_a and H_b respectively) which may be written

$$H_a \dot{X}(1) = V_r Y(o) - V_r X(1)$$

or

$$H_a \alpha \dot{Y}_e(1) = V_r Y(o) - V_r \alpha Y_e(1) + V_r(\alpha - 1) \qquad (6)$$

or

$$Y_e(1) = \{Y(o) + \alpha - 1\}/\alpha(1 + T_a D) \qquad (7)$$

where $D \equiv d/dt$ and $T_a = H_a/V_r$ (8)

and $H_b \dot{X}'(1) = L_s X'(o) - L_s Y'(1)$

or $H_b \dot{X}'_e(1) = L_s X'(o) - L_s \alpha X'_e(1)$ (9)

since $X_e(1) = X(1)$, assuming the reboiler to be in thermodynamic equilibrium.

Or, $X'_e(1) = X'(o)/\alpha\{1 + T_b D\}$ (10)

where $T_b = H_b/\alpha L_s$ (11)

2.1.1 Large signal steady-state solution

This solution is required to provide parameter values for the small signal model yet to be derived. We shall assume the following symmetry of operating conditions

$$\left.\begin{array}{l} z = \alpha/1+\alpha) \quad , \quad Z = 1/(1+\alpha) \\ F_\ell = F_v = F \\ V_r = L_s \quad , \quad V_r = \alpha L_r \quad , \quad \text{so that } L_s = \alpha V_s \\ \text{and} \quad F(= V_r - V_s) = V_r - L_r = V_r(\alpha - 1)/\alpha \end{array}\right\} \quad (12)$$

Putting the time-derivatives in equations (1), (3), (4), (5), (6) and (9) to zero and eliminating $Y_e(1)$ and $X'_e(1)$ we get

from (1) , $X'_e(o) - Y(o) + (\alpha-1)/(\alpha+1) = (k_r/V_r)\{Y(o) - Y_e(o)\}$ (13)

from (3) , $\{Y(o)-1\}/\alpha - \{Y_e(o)-1\} = (k_r/V_r)\{Y_e(o) - Y(o)\}$ (14)

from (4) , $X'(o)/\alpha - X'_e(o) = (k_s/V_r)\ \{X'_e(o) - X'(o)\}$ (15)

and from (5), $Y_e(o) - X'(o) - (\alpha-1)/(\alpha+1) = (k_s/V_r)\ \{X'(o) - X'_e(o)\}$ (16)

Dropping the argument (o), writing $V = V_r$, assuming for symmetry that

$$k_r = k_s = k \quad (17)$$

and writing $Q = 1 - Y$ and $Q_e = 1 - Y_e$ (18)

equations (13) to (16) become, respectively

$$X'_e + Q - 2/(\alpha+1) = (k/V)(Q_e - Q) \quad (19)$$

$$Q_e - Q/\alpha = (k/V)(Q - Q_e) \quad (20)$$

$$X'_e - X'/\alpha = (k/V)(X' - X'_e) \quad (21)$$

and $Q_e + X' - 2/(\alpha+1) = k/V)(X'_e - X')$ (22)

From the symmetry of equations (19) - (22) it is clear that

$$Q = X' \quad \text{and} \quad Q_e = X'_e \quad (23)$$

so that, from (19) and (22) we get:

$$Q_e + Q - 2/(\alpha+1) = (k/V)(Q_e - Q) \quad (24)$$

and, from (20) and (21)

$$\alpha Q_e - Q = (\alpha k/V)(Q - Q_e) \quad (25)$$

So that, eliminating Q_e between (25) and (26) gives, reinstating the argument "(o)"

$$Q(o) = X'(o) = \frac{2\alpha(1 + a)}{(1+\alpha)\{1+\alpha + (3\alpha-1)a\}} = 1 - Y(o) \quad (26)$$

where $a = k/V$ (27)

and substituting back for Q(o) gives

$$Q_e(o) = X'_e(o) = \frac{2(1+\alpha a)}{(1+\alpha)\{1+\alpha+(3\alpha-1)a\}} = 1 - Y_e(o) \tag{28}$$

and hence

$$Y_e(o) - Y(o) = X'(o) - X'_e(o) = \frac{2(\alpha-1)}{(1+\alpha)\{1+\alpha+(3\alpha-1)a\}} \tag{29}$$

2.2 Small-signal model

The large signal d.e's (1), (3), (4) and (5), upon eliminating $Y_e(1)$ and $X'_e(1)$ using boundary conditions (7) and (10) become

$$H_v\dot{Y}(o) = \alpha V_s X'_e(o) - V_r Y(o) + Fz + k\{Y_e(o) - Y(o)\} \tag{30}$$

$$\alpha H_\ell \dot{Y}e(o) = L_r\{Y(o)+(\alpha-1)\}/(1+T_aD) - \alpha L_r Y_e(o) - k\{Y_e(o)-Y(o)\} \tag{31}$$

$$\alpha H'_v \dot{X}'_e(o) = V_s X'(o)/(1+T_bD) - \alpha V_s X'_e(o) + k\{X'(o) - X'_e(o)\} \tag{32}$$

and

$$H'_\ell \dot{X}'(o) = \alpha L_r Y_e(o) - L_s X'(o) - L_r(\alpha-1) + FZ - k\{X'(o)-X'_e(o)\} \tag{33}$$

Dropping the common argument "(o)" and differentiating implicitly produces the following d.e's describing the variation of small changes y, y_e, x' and x'_e in Y, Y_e, X' and X'_e to small imposed changes v and ℓ in V and L respectively:

$$H_v\dot{y} = v(\alpha X_e - Y) + V_r x_e - V_r y + k(y_e - y) \tag{34}$$

$$\alpha H_\ell \dot{y}_e = \ell(\alpha Q_e - Q) + L_r\, y/(1+T_aD) - \alpha L_r y_e - k(y_e - y) \tag{35}$$

$$\alpha H'_v \dot{x}'_e = v(X' - \alpha X'e) - V_s x'/(1+T_bD) - \alpha V_s x'_e + k(x' - x'_e) \tag{36}$$

and

$$H'_\ell \dot{x}' = \ell(\alpha Y_e - X' - \alpha + 1) + V_r y_e - V_r x' - k(x' - x'_e) \tag{37}$$

in which the upper case symbols may be replaced by their steady-state values derived in Section 2.1.1. for the investigation of very small perturbations about the steady-state. From these solutions it is readily shown that coefficients $-(\alpha X'_e - Y) = \alpha Y_e - X' - \alpha + 1$

$$\triangleq R = \frac{(\alpha-1)\{(\alpha-1) + (\alpha+1)a\}}{(\alpha+1)\{\alpha+1 + (3\alpha-1)a\}} \tag{38}$$

and that $\alpha Q_e - Q = -(X - \alpha X_e)$

$$\triangleq S = \frac{2\alpha(\alpha-1)a}{(\alpha+1)\{\alpha+1 + (3\alpha-1)a\}} \tag{39}$$

Recalling conditions (12) for symmetrical plant operation and setting

$$H_v = H'_\ell = \alpha H_\ell = \alpha H'_v \triangleq H \tag{40}$$

for dynamic symmetry equations (34) to (37) become

$$(H/V_r)\dot{y} = -(R/V_r)v + x_e' - y + a(y_e - y) \quad (41)$$

$$(H/V_r)\dot{y}_e = (S/V_r)\ell + h_a' y/\alpha - y_e - a(y_e - y) \quad (42)$$

$$(H/V_r)\dot{x}_e' = -(S/V_r)v + h_b' x'/\alpha - x_e' + a(x'-x_e') \quad (43)$$

and

$$(H/V_r)\dot{x}' = (R/V_r)\ell + y_e - x' - a(x'-x_e') \quad (44)$$

where $h_a'(D)=1/(1+T_a D)$
and $h_b'(D) = 1/(1+T_b D)$ (45)

Now introducing vectors

$$\underline{q} = \begin{pmatrix} y - x' \\ y + x' \end{pmatrix} \quad (46)$$

$$\underline{r} = \begin{pmatrix} y_e - x_e' \\ y_e + x_e' \end{pmatrix} \quad (47)$$

and

$$\underline{u} = \frac{1}{V_r} \begin{pmatrix} v + \ell \\ v - \ell \end{pmatrix} \quad (48)$$

taking Laplace transforms in p w.r.t. normalised-time, τ, where

$$\tau = t\ V/H \quad (49)$$

by adding and subtracting analogous equations of set (42) to (44)

$$(p + 1 + a)\ \tilde{\underline{q}} - \begin{pmatrix} a-1 , & 0 \\ 0 , & a+1 \end{pmatrix} \tilde{\underline{r}} = - R\ \tilde{\underline{u}} \quad (50)$$

and

$$(p + 1 + a)\ \tilde{\underline{r}} - (h_e \alpha^{-1} + a)\tilde{\underline{q}} = S \begin{pmatrix} 1 , & 0 \\ 0 , & -1 \end{pmatrix} \tilde{\underline{u}} \quad (51)$$

provided the terminating vessels are similar such that

$$T_a = T_b \triangleq T' \quad (52)$$

so that

$$h_e(p) = 1/\ \{1 + (T'\ V/H)p\} \quad (53)$$

or

$$h_e(p) = 1/(1 + T\ p) \quad (54)$$

where,

$$T = T'\ V_r/H \quad (55)$$

, is the normalised time-constant of the two end-vessels.

$$\begin{bmatrix} (p+1+a)^2 + (1-a)(h_e\alpha^{-1}+a), & 0 \\ 0, & (p+1+a)^2 - (1+a)(h_e\alpha^{-1}+a) \end{bmatrix} \underline{q}$$

$$= \begin{bmatrix} (a-1)S - (p+1+a)R, & 0 \\ 0, & -(a+1)S - (p+1+a)R \end{bmatrix} \underline{u} \qquad (56)$$

so that, with this choice of input and output vectors the system T.F.M. is completely diagonal, as a result of the physical symmetry of the system. Thus, if $\underline{G}(p)$ is the system transfer-function matrix (T.F.M) then

$$\underline{q} = G(p)\ \underline{u} = \begin{bmatrix} g_1(p), & 0 \\ 0, & g_2(p) \end{bmatrix} \underline{u} \qquad (57)$$

where $$g_1(p) = \frac{(a-1)\ S - (p+1+a)\ R}{(p+1+a)^2 + (1-a)\ \{h_e(p)\alpha^{-1} + a\}} \qquad (58)$$

and $$-g_2(p) = \frac{(a+1)\ S + (p+1+a)\ R}{(p+1+a)^2 - (1+a)\ \{h_e(p)\alpha^{-1} + a\}} \qquad (59)$$

3. Interpretation of the packed-column model

3.1 Static Gains

For steady-state responses to step changes in $v + \ell$ and $v - \ell$ we set $p = o$ in equation (58) and (59) and substituting for parameter function R and S, using equation (38) and (39) yields the following expressions for the system's static gains:

$$g_1(o) = \frac{\alpha\ \varepsilon\{\varepsilon a^2 - 4\alpha\ a - \varepsilon\}}{\{1 + \alpha + a\ (3\alpha - 1)\}^2 (\alpha+1)} \qquad (60)$$

where $\varepsilon = \alpha - 1$ (61)

and $$g_2(o) = -\frac{\alpha\{\ \varepsilon + a(1+3\alpha)\}}{\{1 + \alpha + (3\alpha - 1)\}(\alpha+1)} \qquad (62)$$

from which it is clear that, whilst $g_2(o)$ is always negative, $g_1(o)$ may be positive or negative according to the system parameters, being positive for large k/V (=a) and negative for small k/V. These results are in complete accord with those obtained for the spatially-distributed packed-column model[1].

3.2 High-frequency gains

It is obvious from (58) and (59) that

$$\lim_{|p|\to\infty} \{p\ g_1(p)\} = -R \qquad (63)$$

and $$\lim_{|p|\to\infty} \{p\ g_2(p)\} = -R \qquad (64)$$

and inspection of the expression for R ,{equation (38)} shows this to be always positive so that, as in the case of distributed packed columns, the high-frequency gains are both negative.

It is interesting to note here in passing that results (63) and (64), i.e. the high-frequency performance of the system, is again obtainable immediately from the original transformed d.e. of the system (50), by merely ignoring all dependent-variable coefficients not involving the highest power of p. Equation (50) reduces in this case to simply

$$\lim_{|p|\to\infty} p\,\underline{\tilde{q}} = -R\,\tilde{u} \tag{65}$$

which yields results identical to (63) and (64).

It is also important to note that, as with distributed columns, time-constant T, and hence the end-capacitances, do not affect the high-frequency behaviour of the system since R does not involve T or any parameters related thereto.

Finally we note that if p is allowed to describe a clockwise infinite semi-circular contour around the positive half p-plane the $g_1^{-1}(p)$ and $g_2^{-1}(p)$ will describe clockwise infinite semi circular excursions around the negative half of their own complex planes as a result of the negative signs of the high-frequency gains. With $g_1(o)$ negative, and $g_2(o)$ which is always negative this therefore yields inverse Nyquist loci of the general form shown in Fig. 3. With $g_1(o)$ positive however the locus of $g_1^{-1}(p)$ must take the form shown in Fig. 4, (accepting the system to be open-loop stable), and requiring the presence of a positive zero in $g_1(p)$ because of the anticlockwise encirclement thus produced. It is therefore necessary to confirm the presence of a positive zero in $g_1(p)$ under the condition $g_1(o) > 0$ and its absence when $g_1(o) < 0$ to check that the system is indeed open-loop stable.

3.3 Location of system zeros

Returning to equation (58) and (59), since S and R are both positive {see equations (38) and (39)}, it is clear that $g_2(p)$ can only be made zero by negative values of p : as expected, whereas, for $g_1(p)$ to be zero, it requires that

$$p\,R = (a-1)\,S - (a+1)\,R \tag{66}$$

and, on substituting fur R and S , this relation becomes:

$$p = \frac{\varepsilon a^2 - 4\alpha a - \varepsilon}{(\alpha-1) + (a+1)a} \tag{67}$$

the solution clearly becoming positive only if $g_1(o)$ is positive - see equation (59). We therefore conclude that the system is indeed open-loop stable for either sign of $g_1(o)$ but that $g_1(p)$ is a non-minimum-phase system when $g_1(o) > 0$. This conclusion again coincides with the findings for spatially-distributed packed-columns.

4. Modelling the tray-column

4.1 Taking mass-balances for the more-volatile component of the binary mixture for each of the two trays shown in Fig. 2 yields the differential equations:

$$H_r \dot{X}(o) = L_r\{X(1)-X(o)\} + V_s Y'(o) - V_r Y(o) + F_v z \qquad (68)$$

and

$$H_s \dot{X}'(o) = V_s\{Y'(1)-Y'(o)\} + L_r X(o) - L_s X'(o) + F_\ell Z \qquad (69)$$

where H_r and H_s are the molar liquid capacitances† of the trays in the rectifier and stripper respectively. Treating the accumulator and reboiler vessels similarly yields:

$$H_a \dot{X}(1) = V_r\{Y(o) - X(1)\} \qquad (70)$$

or

$$X(1) = Y(o)/(1 + T_a D) \qquad (71)$$

where

$$T_a = H_a/V_r \qquad (72)$$

and

$$H_b \dot{X}(1)' = L_s\{X'(o) - Y'(1)\} \qquad (73)$$

and since the reboiler is in equilibrium:

$$Y'(1) = \alpha X'(1) \qquad (74)$$

so that

$$Y(1)' = X'(o)/(1 + T_b D) \qquad (75)$$

where

$$T_b = H_b/\alpha L_s \qquad (76)$$

where H_a and H_b are the accumulator and reboiler molar capacitances.

Now tray-column models assume each tray's liquid to be in continuous equilibrium with the vapour above it so that, in this case,

$$Y = X/\alpha + (\alpha-1)/\alpha \qquad (77)$$

and

$$Y' = \alpha X' \qquad (78)$$

† Vapour capacitances are generally neglected in tray column models but could be lumped with the liquid capacitances slightly modifying equations (68) and (69) with insignificant consequences - see introduction.

assuming the usual piecewise linear approximation to hold for the vapour/liquid equilibrium line. The use of these two relationships together with boundary conditions (71) and (75) therefore allows all dependent variables other than, say, Y(o) and X'(o) to be eliminated from the tray equations (68) and (69) giving

$$H_r \alpha \dot{Y}(o) = L_r\{(\alpha-1)(1-Y(o)\} + (h_a-1)Y(o)L_r + V_s \alpha X'(o) - V_r Y(o) + F_v z$$

and

$$H_s \dot{X}'(o) = -V_s\{(\alpha-1)X'(o)\} + (h_b-1)X'(o)V_s + L_r\{\alpha Y(o)-\alpha+1\} - L_s X'(o) + F_\ell Z \qquad (79)$$

where again

$$H_a(D) = 1/(1 + T_a D)$$

and

$$H_b(D) = 1/1 + T_b D) \qquad (80)$$

As with the packed column we now assume symmetrical feed conditions, viz:

$$z = 1 - Z \quad , \quad z = \alpha Z$$

so that

$$z = \alpha/(1+\alpha) \quad \text{and } Z = 1/(1+\alpha) \qquad (81)$$

which, on substitution into the differential equations above and noting that

$$F_v = V_r - V_s \quad \text{and} \quad F_\ell = L_s - L_r \qquad (82)$$

gives

$$\alpha H_r \dot{Y}(o) = \{1 - Y(o)\}\{L_r(\alpha-1) + V_r\} + (h_a-1)Y(o) L_r + V_s \alpha X'(o) - V_r/(1+\alpha) - V_s \alpha/(1+\alpha) \qquad (83)$$

and

$$-H_s/\alpha \dot{X}'(o) = X'(o)\{V_s(\alpha-1) + L_s\} - (h_b-1)X'(o) V_s + L_r \alpha\{1-Y(o)\} - L_s/(1+\alpha) - L_r\alpha/(1+\alpha) \qquad (84)$$

4.1.1 Large-signal steady-state solution

Assuming the same conditions (12) to apply for symmetrical column operation as were assumed for the packed-column analysis, setting

$$Q(o) \triangleq 1 - Y(o) \qquad (85)$$

then, on putting the time derivatives in d.e.'s (83) and (84) to zero, we obtain

$$Q(o)(2\alpha-1)/\alpha + X'(o) - 2/(1+\alpha) = 0 \qquad (86)$$

and

$$X'(o)(2\alpha-1)/\alpha + Q(o) - 2/(1+\alpha) = 0 \qquad (87)$$

from which we deduce that

$$Q(o) = 1-Y(o) = X'(o) = 2\alpha/(3\alpha-1)(\alpha+1) \qquad (88)$$

It is interesting and reassuring to note at this point that the steady state-solution (26) obtained for the packed-column approaches the tray column solution (88) as

$$a\ (=k/V_r)\ \rightarrow\ \infty \tag{89}$$

i.e. as the packed-column sections approach near-equilibrium conditions

4.2 Small-perturbation model

Differentiating implicitly the large-signal d.e's (80) produces the following relationships between the small composition changes $y(o), x'(o)$ and the imposed flow changes v, ℓ, about the steady state:

$$\alpha H_r \dot{y}(o) = -y(o)(\alpha L_r + V_r) + L_r h_a(D)\, y(o) + V_s\, \alpha\, x'(o)$$
$$+ (\ell\, \varepsilon + v)\{1 - Y(o)\} + \alpha X'(o) v - v \tag{90}$$

and

$$-(H_s/\alpha)\dot{x}'(o) = x'(o)\{\alpha V_s + L_s\} - V_s h_b(D) x'(o) - L_r\, \alpha\, y(o)$$
$$+ (\varepsilon v + \ell) X'(o) + \ell\, \alpha\{1 - Y(o)\} - \ell \tag{91}$$

where again, $\varepsilon = \alpha - 1$

Substitution of the symmetrical steady-state working condition (12) and the resulting solution (88) for $Y(o)$ and $X'(o)$ into (90) and (91) produces the matrix equation:

$$\begin{bmatrix} h_e'(D)/\alpha - (2+T_t D) & , & 1 \\ -1 & , & -\{h_e(D)/\alpha - (2+T_t D)\} \end{bmatrix} \begin{bmatrix} y(o) \\ x'(o) \end{bmatrix} = \frac{\varepsilon}{V_r(3\alpha-1)} \begin{bmatrix} 1 & , & -2\alpha/(\alpha+1) \\ -2\alpha/(\alpha+1) & , & 1 \end{bmatrix} \begin{bmatrix} v \\ \ell \end{bmatrix} \tag{92}$$

$$\left.\begin{aligned} &\text{where} \quad T_t = \alpha H_{r/V_r} \quad \text{also set} = H_{s/\alpha V_s} \\ &\text{and} \quad h_e'(D) = h_a'(D) = h_b'(D) \end{aligned}\right\} \tag{93}$$

Introducing the now familiar vectors of [composition "tilt", composition "total"] and [flow "total" and flow "tilt"], viz

$$\underline{q}(o) = \begin{bmatrix} y(o) - x'(o) \\ y(o) + x'(o) \end{bmatrix} \quad , \quad \underline{u} = \frac{1}{V_r} \begin{bmatrix} v + \ell \\ v - \ell \end{bmatrix} \tag{94}$$

and taking Laplace transforms in p w.r.t normalised time $\tau(=t/T_t)$ thus allows (92) to be expressed

$$\underline{\tilde{q}}(o) = \underline{G}(p)\ \underline{\tilde{u}} \tag{95}$$

where T.F.M. $\underline{G}(p)$ is again diagonal:

$$\underline{G}(p) = \begin{pmatrix} g_1(p) & , & 0 \\ 0 & , & g_2(p) \end{pmatrix} \tag{96}$$

where

$$g_1(p) = \varepsilon^2/\{3 + p - h_e(p)\alpha^{-1}\}(3\alpha-1)(\alpha+1) \tag{97}$$

$$g_2(p) = -(3\alpha+1)\,\varepsilon/\{1 + p - h_e(p)\alpha^{-1}\}(3\alpha-1)(\alpha+1) \tag{98}$$

where

$h_e(p)$ is of course given by

$$h_e(p) = 1/(1 + Tp/T_t) \tag{99}$$

which is readily shown to be identical to the related expression for the packed column {equation (54)}.

5. Comparison of packed-and tray-column models: Conclusions

5.1 Low-frequency behaviour

Setting p to zero in (97) and (98) produces the following expressions for the static gain of the tray-type column:

$$g_1(o) = \alpha\,\varepsilon^2/(3\alpha-1)^2(\alpha+1) \tag{100}$$

and

$$g_2(o) = -\,\alpha(3\alpha+1)/(3\alpha-1)(\alpha+1) \tag{101}$$

the expressions agreeing precisely with the limiting values of the equivalent static gains for the packed column as a, $(=k/V_r)$, tends to infinity: as would be expected. Unlike the case of the packed-column however, $g_1(o)$ for the tray-column can only be positive. Both these findings are in complete accord with the analytical results obtained for spatially-distributed and multi-tray column.

5.2 High-frequency behaviour

Now by substituting for terminal transfer-function $h_e(p)$ in (97) and (98) and denoting the normalised time constant of the end vessels by T_e,

i.e. $$T_e = T/T_t \tag{102}$$

$g_1(p)$ and $g_2(p)$ may be expressed in the following forms for the tray-type column

$$g_1(p) = \frac{(1 + T_e p)}{1 + \{\alpha(3T_e+1)/(3\alpha-1)\}p + \{T_e\alpha/(3\alpha-1)\}p^2} \tag{103}$$

and

$$g_2(p) = \frac{g_2(o)\ (1 + T_e p)}{1 + (\alpha/\varepsilon)(T_e+1)p + (\alpha T_e/\varepsilon)p^2} \tag{104}$$

from which it is readily deduced that

$$\lim_{|p| \to \infty} p\, g_1(p) = g_1(o) \{(3\alpha - 1)/\alpha\} \qquad (105)$$

and

$$\lim_{|p| \to \infty} p\, g_2(p) = g_2(o) (\varepsilon/\alpha) \qquad (106)$$

so that the signs of the high-and low-frequency gains are identical in the case of both elements of the T.F.M. for the tray-column. This contrasts with the case of element - 1 of the packed-column's T.F.M. which always exhibits a negative high-frequency gain and a low-frequency gain whose sign is parameter-dependent. Again identical conclusions were drawn from the earlier studies of "long" columns of the two types[1,2]. It is also obvious, from (103) and (104) that again end-capacitances have no effect on the high-frequency behaviour of the "short" tray-type column, as in all the cases studied previously. Finally it may be concluded that, because of the non-minimum-phase nature of the "short" tray-type column's T.F.M., a multivariable first-order-lag approximation should be particularly applicable†, as was found to be the case with "long" tray-types. The inverse Nyquist loci for the short tray type column will therefore take the general form illustrated in Fig. 5.

† This conclusion is further supported by the fact that the phase-lag of $g_1(p)$ and $-g_2(p)$ can never exceed 90^{o} as is quickly proven from (103) and (104).

6. References

(1) Edwards J.B. 'The analytical modelling and dynamic behaviour of a spatially-continuous binary distillation column'. University of Sheffield, Department of Control Engineering, Research Report No. 86, April 1979, 45 pp.

(2) Edwards J.B. 'The analytical modelling and dynamic behaviour of tray-type binary distillation columns', University of Sheffield, Department of Control Engineering, Research Report No. 90 , June 1979, 37 pp.

7. List of Symbols

a	$= k/V_r$
α	= initial slope of equilibrium curve approximation
ε	$= \alpha - 1$
F_ℓ	= liquid molar feed rate
F_v	= vapour molar feed rate
$\underline{G}$	= transfer-function matrix (T.F.M.)
g_1, g_2	= diagonal elements of $\underline{G}$
h'_a, h'_b	= transfer function of accumulator and reboiler
h_e	= h'_a and h'_b normalised
H_ℓ, H'_ℓ	= liquid capacitances of rectifier and stripper (packed-column)
H_v, H'_v	= vapour capacitances of rectifier and stripper (packed-column)
H_r, H_s	= liquid tray-capacitance of rectifier and stripper (tray-column)
H_a, H_b	= accumulator and reboiler capacitance
k_r, k_s	= evaporation rates for rectifier and stripper =k when equal
L_r, L_s, ℓ	= molar flows of liquid in rectifier and stripper and small changes therein.
p	= Laplace variable for transforsm w.r.t τ.
$\underline{q}$	= vector of difference and total composition changes
R	= function of packed-column parameters
$\underline{r}$	= vector of difference and total equilibrium composition changes (packed-column)
S	= function of packed-column parameters
t	= time
τ	= normalised time
T_a, T_b	= time-constants of accumulator and reboiler
T'	= values of T_a and T_b when equal
T	= normalised value of T' (packed-column) = end-vessel time-constant (tray-column)
T_t	= tray time-constant
T_e	= normalised value of T (tray-column)

$\underline{u}$	= vector of total and difference of vapour and liquid flow changes
V_r, V_s, v	= molar flows of vapour in rectifier and stripper and small changes therein
$X(o), X'(o)$	= liquid molfractions in rectifier and stripper
$x(o), x'(o)$	= small changes in $X(o)$ and $X'(o)$
$Y(o), Y'(o)$	= vapour molfractions in rectifier and stripper
$y(o), y'(o)$	= small changes in $Y(o)$ and $Y'(o)$
$Y_e(o), X_e'(o)$	= equilibrium values of vapour and liquid composition associated with $X(o)$ and $Y'(o)$
$y_e(o), x_e'(o)$	= small changes in $Y_e(o)$ and $X_e'(o)$
$X(1), X'(1)$ $x(1), x'(1)$ $Y(1), Y'(1)$ $y(1), y'(1)$ $Y_e(1), X_e'(1)$ $y_e(1), x_e'(1)$	as above but pertaining to accumulator and reboiler respectively
z	= vapour feed composition
Z	= liquid feed composition

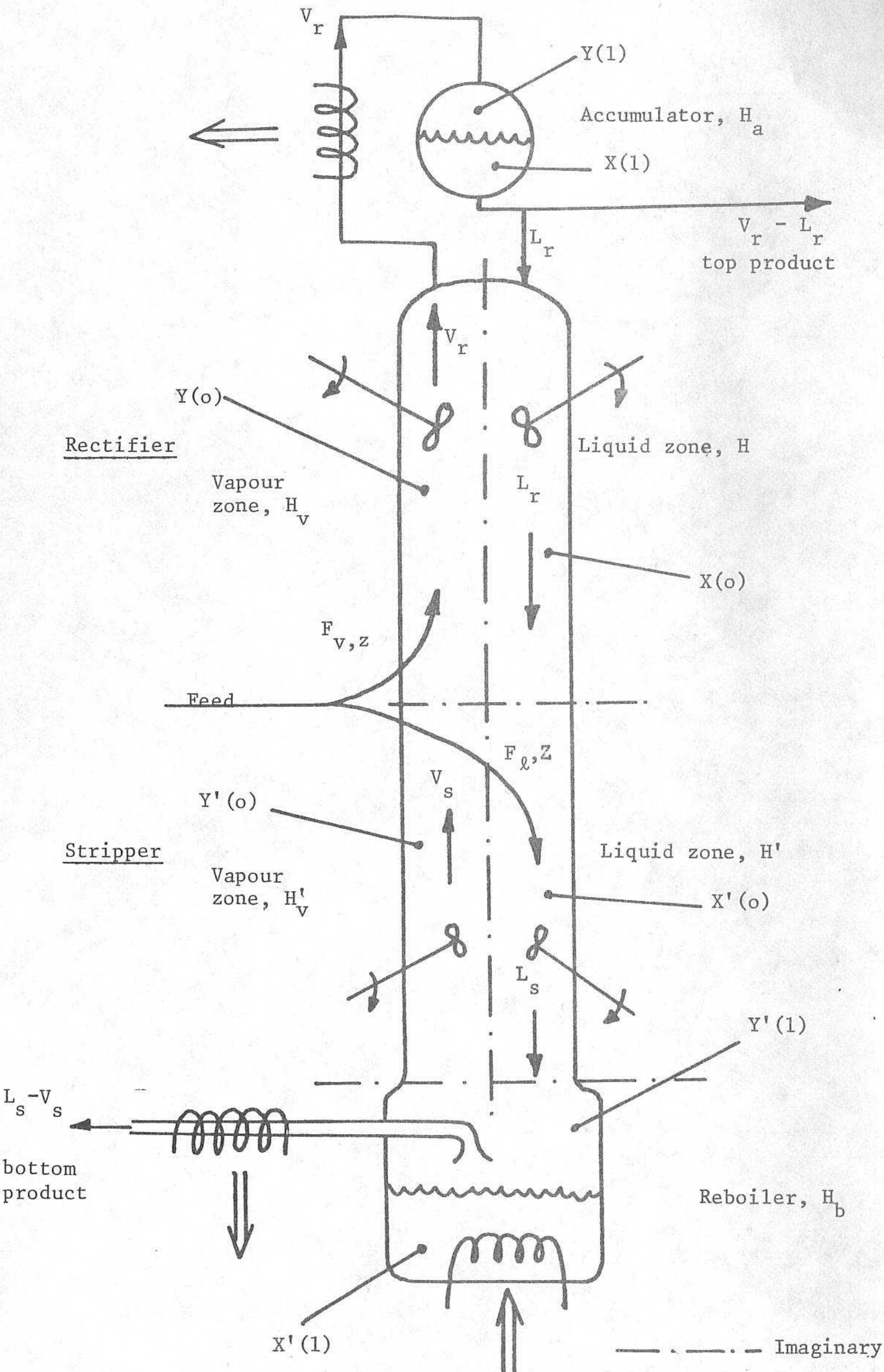

Fig. 1 Zoning and main variables in packed-column

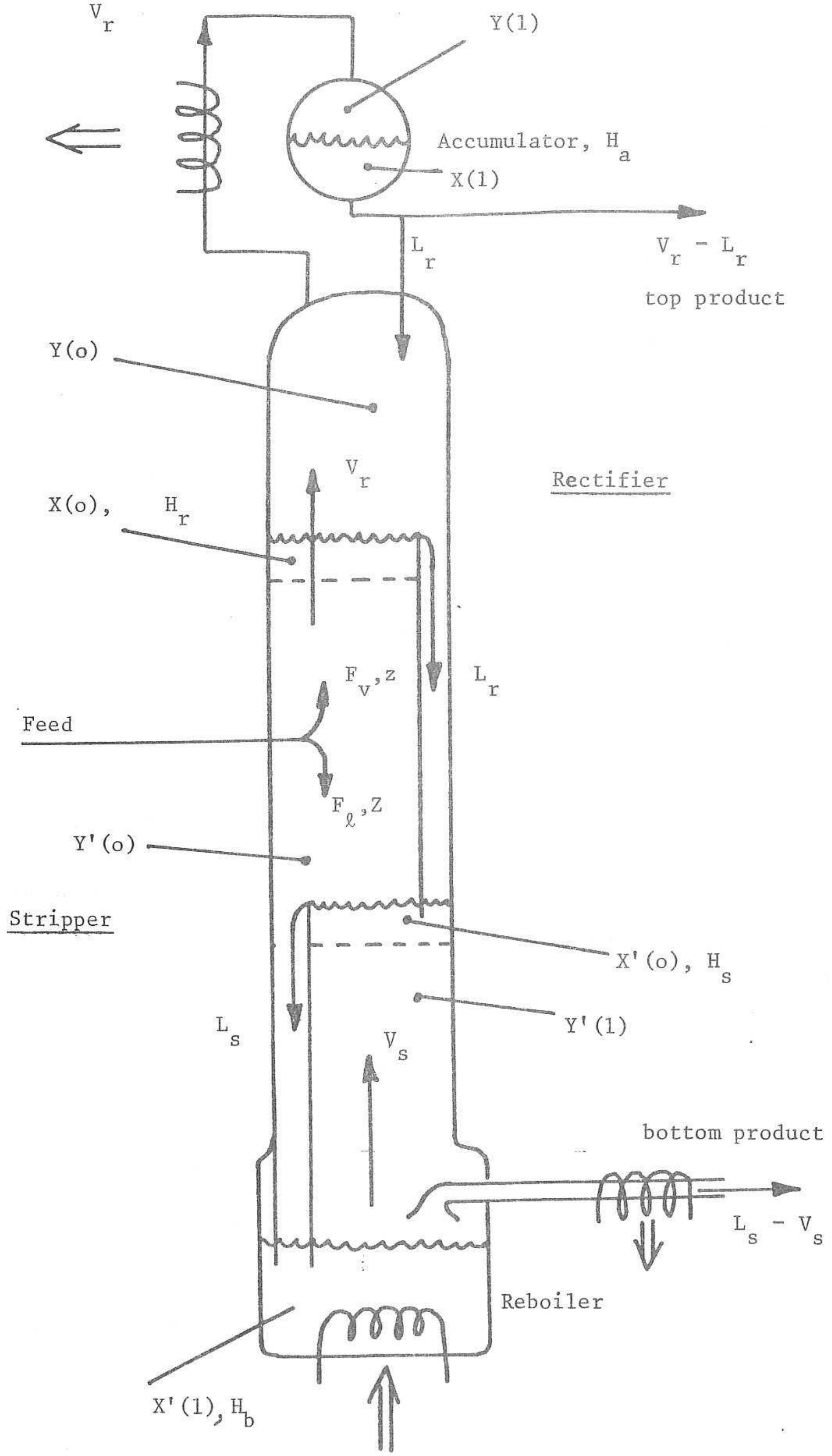

Fig. 2 Tray column, showing main flow and composition variables

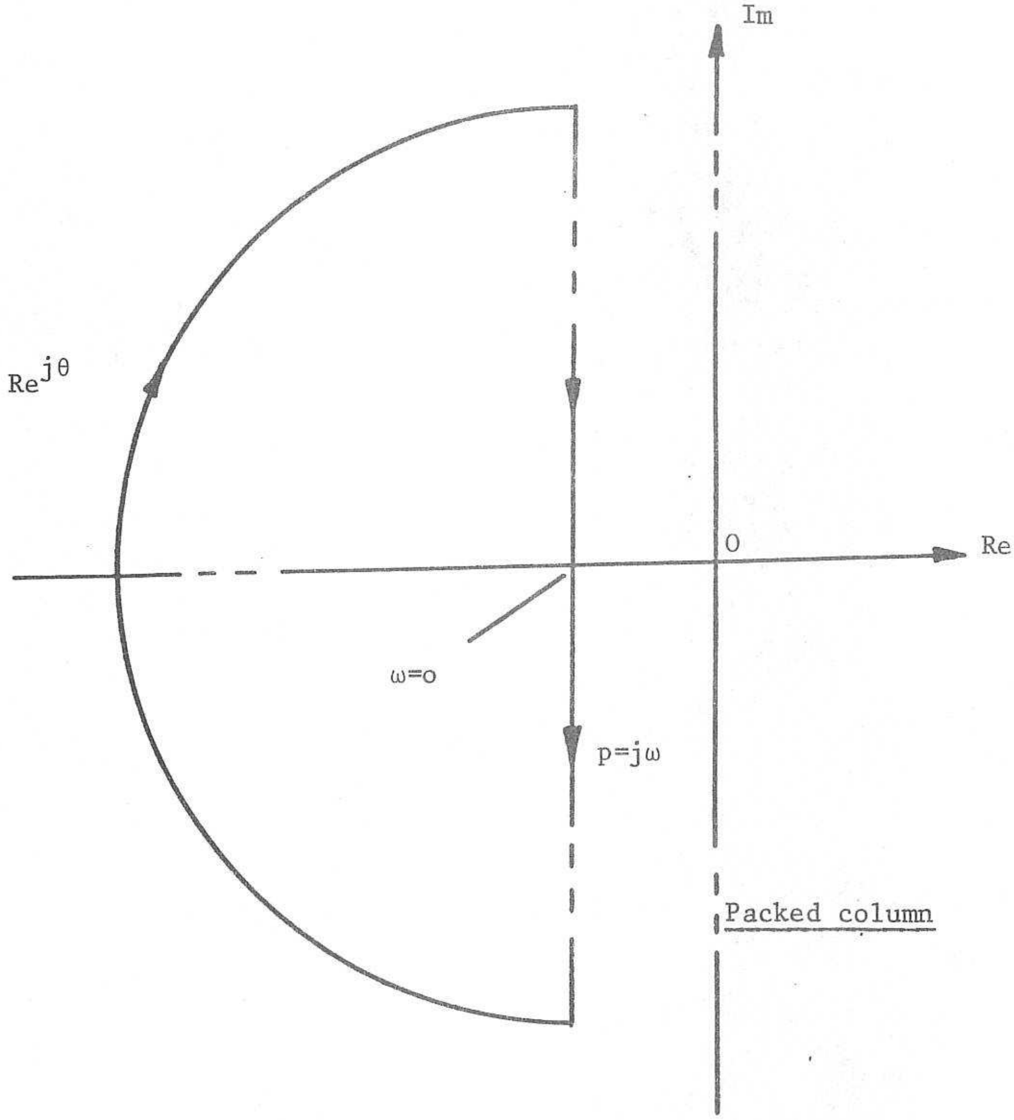

Fig. 3 Basic shape of inverse Nyquist loci, $g_1^{-1}(p)$ and $g_2^{-1}(p)$, $g_1(o) < 0$

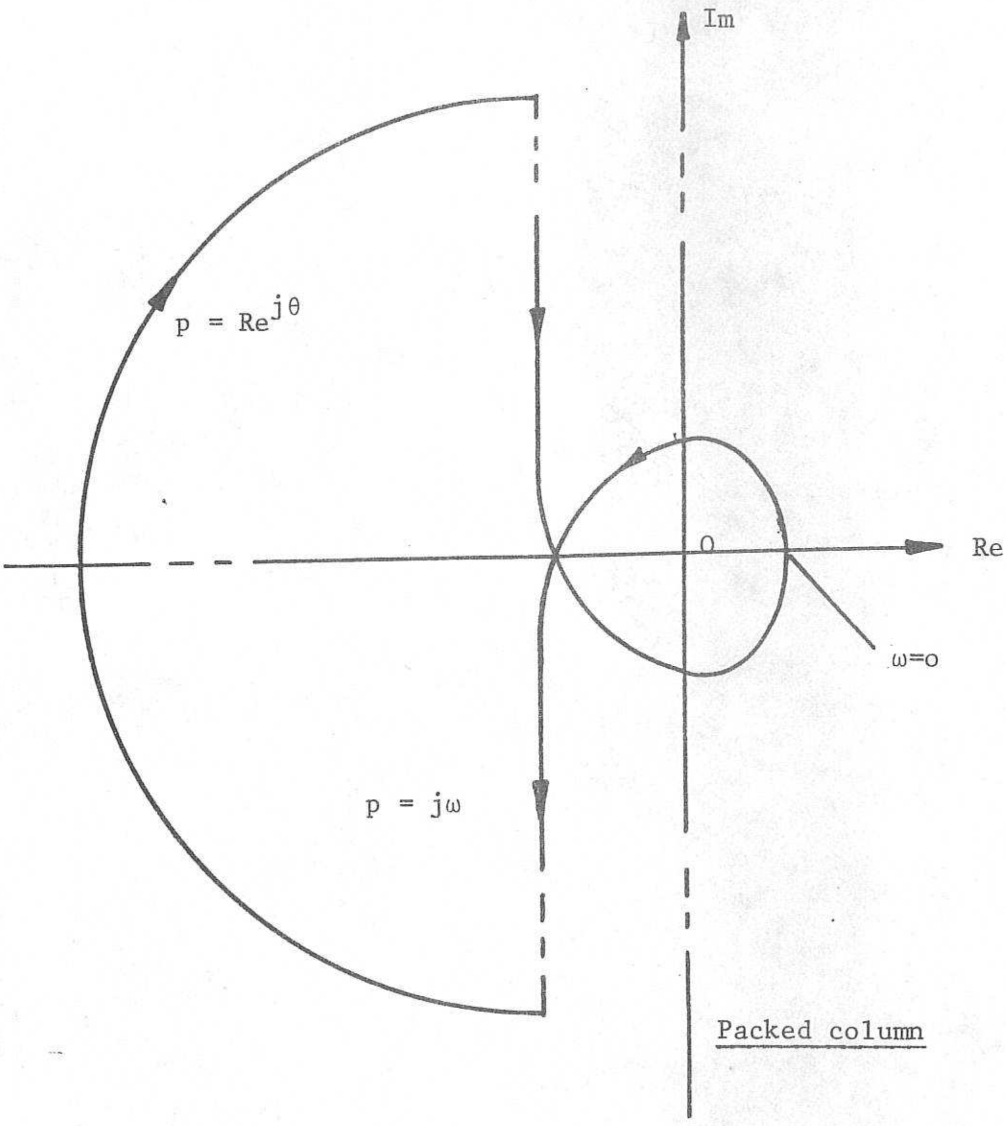

Fig. 4 Basic shape of inverse Nyquist locus

$g_1^{-1}(p)$, $g_1(o) > 0$

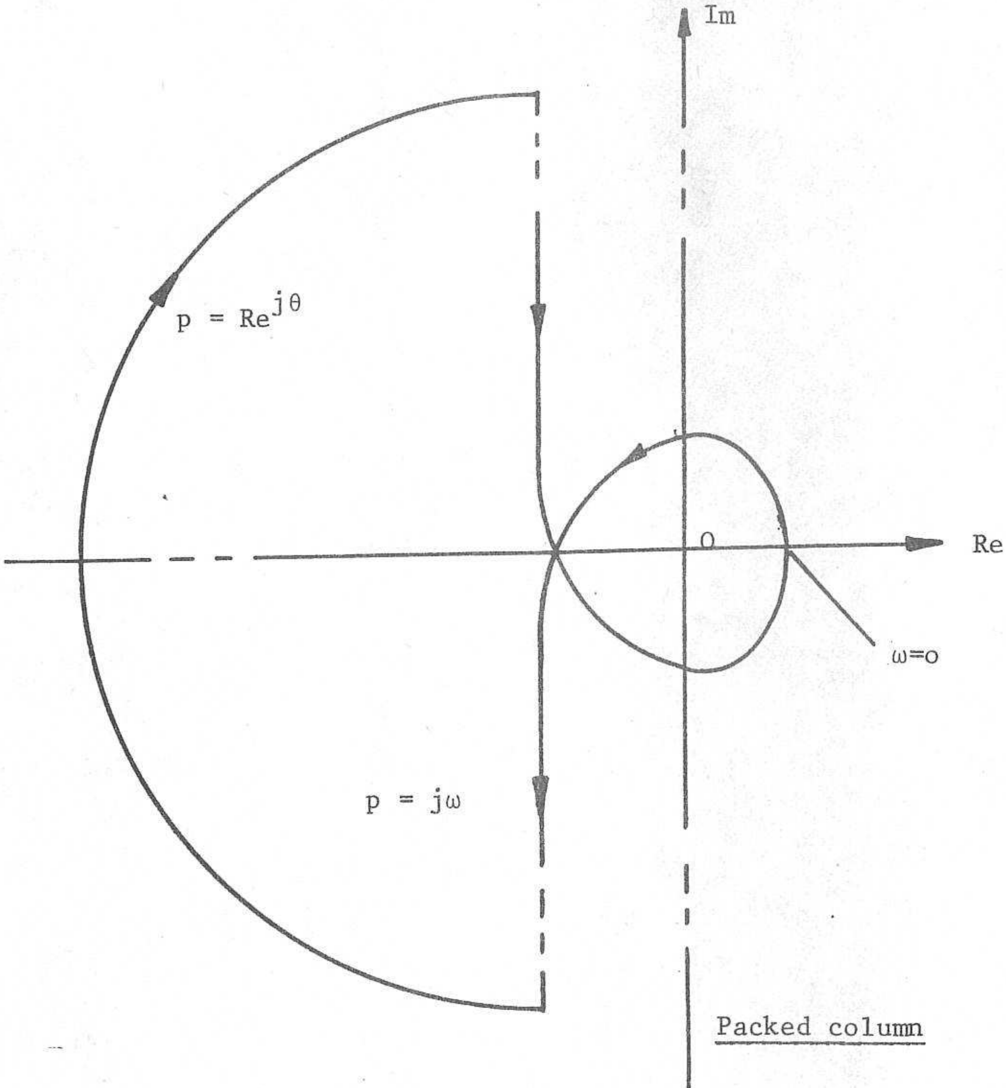

Fig. 4 Basic shape of inverse Nyquist locus

$g_1^{-1}(p)$, $g_1(o) > 0$

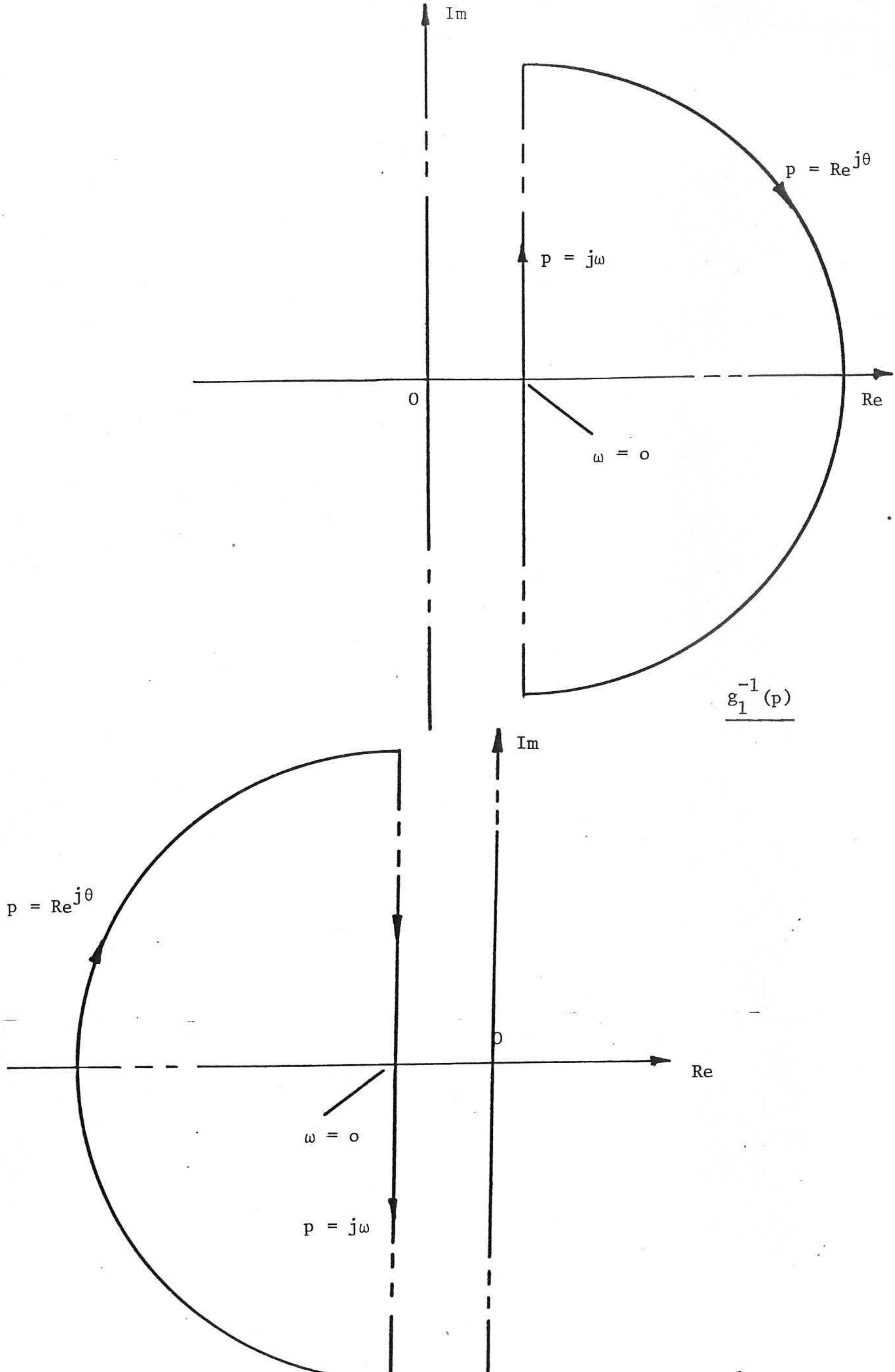

Fig. 5 Basic shapes of inverse Nyquist loci for tray-column