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Owens, D.H. (1979) Computation and Characterization of the Zeros of Linear Multi-Variable Systems. Research Report. ACSE Research Report 94 . Department of Automatic Control and Systems Engineering

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COMPUTATION AND CHARACTERIZATION OF THE
ZEROS OF LINEAR MULTIVARIABLE SYSTEMS

by

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Research Report No. 94

July 1979

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Abstract

A natural extension of the results of Kouvaritakis and MacFarlane on the calculation of multivariable zeros is described.

Kouvaritakis and MacFarlane^{1,2} have suggested a conceptually simple numerical technique for the calculation of the invariant zeros of the ℓ -input/ m -output linear, time-invariant left-invertible system $S(A,B,C)$,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) & , & \quad x(t) \in \mathbb{R}^n \\ y(t) &= Cx(t) & , & \quad y(t) \in \mathbb{R}^m, \quad u(t) \in \mathbb{R}^\ell \end{aligned} \quad (1)$$

when, in particular, $m \geq \ell$, $\text{rank } B = \ell$, $\text{rank } C = m$. The purpose of this note is to describe a natural extension of their results that can make possible a reduction in dimension of the defining relationships.

Let k^* be the uniquely defined integer ≥ 1 such that

$$CA^{i-1}B = 0, \quad 1 \leq i \leq k^* - 1, \quad CA^{k^*-1}B \neq 0 \quad (2)$$

Lemma 1

The subspaces

$$V_0 \triangleq \{0\}, \quad V_k \triangleq R(B) \oplus AR(B) \oplus \dots \oplus A^{k-1}R(B) \quad (k \geq 1) \quad (3)$$

have dimension $k\ell$ for k in the range $0 \leq k \leq k^*$. By suitable choice of subspace W , the state space has the direct sum decomposition

$$\mathbb{R}^n = V_k^* \oplus V^* \oplus W \quad (4)$$

where V^* is the maximal $\{A,B\}$ -invariant subspace in the kernel of C . Sufficient conditions for $W = \{0\}$ are that $m = \ell$ and $|CA^{k^*-1}B| \neq 0$.

Proof

Follows directly from theorem 5 in ref. 3.

This fundamental lemma leads to the following construction

Lemma 2

Let $1 \leq k \leq k^*$ and, for each complex scalar λ , define

$$S_k(\lambda) \triangleq \{x \in \bigcap_{i=1}^k \ker CA^{i-1} : (\lambda I_n - A)x \in V_k\} \quad (5)$$

Then

$$S_k(\lambda) = \omega_1(\lambda) \oplus \{V_{k-1} \cap \{\bigcap_{i=1}^k \ker CA^{i-1}\}\} \quad (6)$$

where $\omega_1(\lambda) = \{(\lambda I_n - A)^{-1}R(B)\} \cap \ker C$.

Proof

It is known³ that $\omega_1(\lambda) \subset V^*$ and, using equation (2), $V^* \subset \bigcap_{i=1}^k \ker CA^{i-1}$. By definition $(\lambda I_n - A)\omega_1(\lambda) \subset V_1 \subset V_k$ from which we obtain $\omega_1(\lambda) \subset S_k(\lambda)$. It is also easily verified that $(\lambda I_n - A)V_{k-1} \subset V_k$ and hence that

$$\omega_1(\lambda) \oplus \{V_{k-1} \cap \{\bigcap_{i=1}^k \ker CA^{i-1}\}\} \subset S_k(\lambda) \quad (7)$$

(the sum being direct by lemma one). Now take $x \in S_k(\lambda)$ and write

$$(\lambda I_n - A)x = B\alpha_1 + AB\alpha_2 + \dots + A^{k-1}B\alpha_k \quad (8)$$

This expression can be written in the form

$$(\lambda I_n - A)(x + A^{k-2}B\alpha_k) = B\alpha_1 + AB\alpha_2 + \dots + A^{k-2}B(\alpha_{k-1} + \lambda\alpha_k)$$

and hence (by induction) in the form

$$(\lambda I_n - A)(x-b) \in R(B)$$

for some $b \in V_{k-1}$ ie $x-b \in \omega_1(\lambda)$. In particular, it is seen that $b \in S_k(\lambda) + \omega_1(\lambda) = S_k(\lambda)$ from which

$$b \in V_{k-1} \cap \left\{ \bigcap_{i=1}^k \ker CA^{i-1} \right\}$$

and hence

$$x \in \omega_1(\lambda) \oplus \left\{ V_{k-1} \cap \left\{ \bigcap_{i=1}^k \ker CA^{i-1} \right\} \right\}$$

which reverses the inclusion in equation (7). This proves the lemma.

We obtain immediately the following geometric characterization of the invariant zeros of $S(A,B,C)$.

Theorem 1

Let $1 \leq k \leq k^*$. Then the complex number λ is an invariant zero of $S(A,B,C)$ if, and only if,

$$\dim S_k(\lambda) > \dim V_{k-1} \cap \left\{ \bigcap_{i=1}^k \ker CA^{i-1} \right\} \quad (9)$$

Proof

Follows directly from lemma 2 noting³ that λ is a zero if, and only if, $\omega_1(\lambda) \neq \{0\}$.

This geometric formulation can be converted into algebraic relations paralleling those of Kouvaritakis and MacFarlane by the following constructions. Define the matrices

$$B_k \triangleq [B, AB, \dots, A^{k-1}B] \quad , \quad C_k \triangleq \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \quad (10)$$

when, taking $1 \leq k \leq k^*$, lemma one indicates that we can define full rank matrices N_k, M_k of dimension $(n-kl) \times n$ and $n \times (n - \text{rank } C_k)$ respectively satisfying the relations

$$N_k B_k = 0 \quad , \quad C_k M_k = 0 \quad (11)$$

Lemma 3

$\text{rank } B_k = kl \leq \text{rank } C_k$, $1 \leq k \leq k^*$, equality holding if $m = \ell$.

Proof

$\text{rank } B_k = kl$, $1 \leq k \leq k^*$, follows directly from lemma one. If $m = \ell$ then a dual argument on the system $S(A^T, C^T, B^T)$ yields $\text{rank } C_k = km$ as required. Finally, taking $m > \ell$ and choosing an $\ell \times m$ matrix K such that the square system $S(A, B, KC)$ is invertible, we see that

$$kl = \text{rank} \begin{pmatrix} KC \\ KCA \\ \vdots \\ KCA^{k-1} \end{pmatrix} \leq \text{rank } C_k \quad , \quad 1 \leq k \leq k^*$$

as required.

Defining, for notational convenience, $N_0 = I_n$, we now prove the following main result of this paper.

Theorem 2

The complex number λ is an invariant zero of $S(A, B, C)$ if, and only if,

$$\text{rank } N_k(\lambda I_n - A)M_k < \text{rank } N_{k-1}M_k \quad (12)$$

for any $1 \leq k \leq k^*$.

Proof

The proof is obtained by evaluation of the expressions occurring in equation (9). Firstly, we see from the definitions and lemma three that

$$\begin{aligned} \dim S_k(\lambda) &= \dim \{x \in \bigcap_{i=1}^k \ker CA^{i-1} : N_k(\lambda I - A)x = 0\} \\ &= \dim \ker N_k(\lambda I_n - A)M_k \\ &= \text{rank } M_k - \text{rank } N_k(\lambda I_n - A)M_k \end{aligned} \quad (13)$$

Secondly, we can, without loss of generality, assume that $M_k = [X_k, Y_k]$ where the columns of X_k are a basis for $V_{k-1} \cap \{ \bigcap_{i=1}^k \ker CA^{i-1} \}$ and $\ker Y_k = \{0\}$. It follows directly that $\text{rank } N_{k-1}M_k = \text{rank } N_{k-1}Y_k$. Consider now the relation $N_{k-1}Y_k\beta = 0$. We see immediately that $Y_k\beta \in V_{k-1} \cap \{ \bigcap_{i=1}^k \ker CA^{i-1} \}$ ie $Y_k\beta = 0$ (from the definition of X_k and Y_k) and hence $\beta = 0$. It follows that

$$\text{rank } N_{k-1}M_k = \text{rank } M_k - \dim V_{k-1} \cap \{ \bigcap_{i=1}^k \ker CA^{i-1} \} \quad (14)$$

The theorem follows by combining equations (9), (13) and (14).

The results of Kouvaritakis and MacFarlane¹⁻³ are obtained by setting $k = 1$ when equation (12) reduces to

$$\text{rank } N_1(\lambda I_n - A)M_1 < \text{rank } M_1 = n - m \quad (15)$$

or, in the important case of $m = \ell$ when (Lemma 3) the matrices are square, we recover the well-known relation¹

$$|\lambda N_1 M_1 - N_1 A M_1| = 0 \quad (16)$$

For those systems with $k^* > 1$, the choice of $k > 1$ represents a reduction in dimension of the relationships defining the zeros.

The rank conditions in equation (12) can be simplified when $m = \ell$ and $|CA^{k^*-1}B| \neq 0$. In this case, applying lemma three $\text{rank } M_k = \text{rank } N_k = n - km$, $1 \leq k \leq k^*$. Noting that (using equation (2) and lemma one)

$$V_{k-1} \cap \left\{ \bigcap_{i=1}^k \ker CA^{i-1} \right\} = V_{\min\{(k-1), (k^*-k)\}} \quad (17)$$

and that $\dim V_k = km$, $1 \leq k \leq k^*$, then a combination of equations (9) and (13) indicates that λ is a zero of $S(A, B, C)$ if, and only if,

$$\text{rank } N_k (\lambda I_n - A) M_k < n - km - \min\{(k-1), (k^*-k)\}m \quad (18)$$

In particular, if $k = k^*$ this reduces to the $(n - k^* m) \times (n - k^* m)$ determinental relationship

$$z_0(\lambda) \stackrel{\Delta}{=} |\lambda N_k^* M_k^* - N_k^* A M_k^*| = 0 \quad (19)$$

As $|CA^{k^*-1}B| \neq 0$, then it is easily verified that $N_k^* M_k^*$ is non-singular and hence that, if N_k^* is replaced by $\bar{N}_k^* = (N_k^* M_k^*)^{-1} N_k^*$ in equation (19), then the invariant zeros are the eigenvalues of $\bar{N}_k^* A M_k^*$. This result is a direct generalization of the result¹ that the invariant zeros are the eigenvalues of $N_1 A M_1$ if

$|CB| \neq 0$ and we choose N_1 such that $N_1 M_1 = I_{n-m}$.

Finally, note that equation (19) will also give the correct algebraic multiplicity of each zero. To prove this, note that the above discussion indicates that $z_0(\lambda)$ has degree equal to $n-k$. The decomposition of lemma one makes possible the consideration of arbitrarily small perturbations δ to A such that $\delta v \subset v^*$ and $\delta[V_k^*] = \{0\}$. It is easily verified that the number of zeros including multiplicities (ie³ $\dim v^* = n-k$) is unaffected by any such perturbation, that the invariant zeros vary continuously with δ and that both N_k^* and M_k^* are unchanged. The invariant zeros are hence defined by the relation

$$\begin{aligned}
 z_\delta(\lambda) &\stackrel{\Delta}{=} \left| \lambda N_k^* M_k^* - N_k^* (A+\delta) M_k^* \right| \\
 &\equiv \left| N_k^* M_k^* \right| \cdot \left| \lambda I_{n-k} - \bar{N}_k^* (A+\delta) M_k^* \right| \\
 &= 0
 \end{aligned}
 \tag{20}$$

Note that the degree of the polynomial $n-k$ is independent of δ and that the $n-k$ zeros are distinct for 'almost all' choices of δ . A simple continuity argument now proves that $z_0(s)$ provides the correct algebraic multiplicity.

To illustrate the results, consider the system defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 3 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 1 & 2 & 1 & -1 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & -4 \end{pmatrix}$$

$$\begin{aligned}
 B^T &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
 C &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (21)
 \end{aligned}$$

The system has one zero at the point $\lambda = -4$ and $k^* = 3$.

Consider now the application of equation (12) to the calculation of the zero. Take initially $k = 2$, then $C_1 = C$,

$$\begin{aligned}
 C_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 B_2^T &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad (22)
 \end{aligned}$$

yielding

$$\begin{aligned}
 N_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
 N_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
 M_2^T &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (23)
 \end{aligned}$$

Substituting into equation (12) indicates that λ is a zero if, and only if,

$$\text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda+4 \end{pmatrix} < \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (24)$$

ie $\lambda = -4$ is the only zero. Considering now the case of $k = k^* = 3$ we obtain

$$N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$M_3^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (25)$$

Substituting into equation (12) indicates that λ is a zero if, and only if,

$$\text{rank} \begin{pmatrix} 0 & 0 \\ 0 & \lambda+4 \end{pmatrix} < \text{rank} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (26)$$

again verifying that $\lambda = -4$ is the only zero of the system.

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