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ON THE COMPUTATION OF OPTIMAL SYSTEM ASYMPTOTIC ROOT-LOCI

by

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Abstract

This note outlines a procedure for determining the asymptotic behaviour of the optimal closed-loop poles of a time-invariant linear regulator as the weight of the input in the performance criterion approaches zero. It is based on the systematic use of dynamic input/output transformations to the relevant return-difference. A sensitivity property of the pivots is noted.

I INTRODUCTION

It is well-known [1] that the stabilizable and detectable time-invariant linear system S(A,B,C)

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad , \qquad x(t) \in R^n$$
$$y(t) = C\,x(t) \quad , \quad y(t) \in R^m \quad , \quad u(t) \in R^{\ell} \tag{1}$$

with state feedback controller minimizing the performance criterion

$$J = \int_0^{\infty} \{y^T(t) Q y(t) + p^{-1} u^T(t)\; R\; u(t)\} dt \tag{2}$$

(where both Q and R are positive definite and $p>0$) has closed-loop poles equal to the left-half plane solutions of the equation,

$$|I_{\ell} + p\; G^T(-s)\; G(s)| = 0 \tag{3}$$

where

$$G(s) = Q^{\frac{1}{2}}\; C\; (sI_n - A)^{-1}\; B\; R^{-\frac{1}{2}} \tag{4}$$

A fairly complete theoretical analysis of the unbounded solutions of equation (3) as $p \to +\infty$ has been provided by Kwakernaak [1] but computational procedures were not suggested until quite recently [2,3]. However, in [2], proofs are provided for only the first few orders of infinite zero and the techniques of [3] are primarily suited for systems with small numbers of inputs. This note provides a complete analysis and computational method for the case of S(A,B,C) left-invertible and hence $m \geq \ell$ and $|G^T(-s)\; G(s)| \not\equiv 0$. The case of S(A,B,C) right-invertible can be deduced by replacing equation (3) by the equivalent equation $|I_m + G(s) G^T(-s)| = 0$. The approach used is that described in references [4-6].

II ASYMPTOTIC BEHAVIOUR OF THE OPTIMAL ROOT LOCUS

The following new lemma is fundamental

Lemma: Let $S(A,B,C)$ be left-invertible with $G(s)$ given in equation (4) and define $Q(s) = G^T(-s)G(s)$. Then there exists integers $q \geq 1$, $1 \leq k_1 < k_2 < .. < k_q$ and d_j, $1 \leq j \leq q$, a real nonsingular transformation T_1 and unimodular matrices of the form

$$L(s) = \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ O(s^{-1}) & I_{d_2} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ O(s^{-1}) & \cdots & \cdots & O(s^{-1}) & I_{d_q} \end{pmatrix}$$

$$M(s) = \begin{pmatrix} I_{d_1} & O(s^{-1}) & \cdots & O(s^{-1}) \\ 0 & I_{d_2} & & \vdots \\ & & \ddots & O(s^{-1}) \\ 0 & & 0 & I_{d_q} \end{pmatrix} \tag{5}$$

such that

$$L(s)T_1^{-1}Q(s)T_1M(s) = \text{block diag}\{Q_j(s)\}_{1 \leq j \leq q} + O(s^{-(2k_q+2)}) \tag{6}$$

where the $d_j x d_j$ transfer matrices $Q_j(s)$ have uniform rank $2k_j$, $1 \leq j \leq q$. Moreover it is always possible to choose T_1 to be orthogonal, $L(s) = M^T(-s)$ and

$$Q_j(s) = N_j^T(-s)N_j(s) \tag{7}$$

for some $m \times d_j$ left-invertible transfer matrix $N_j(s)$, $1 \leq j \leq q$.

Remark: The concept of a uniform rank system is fundamental to root-locus theory [4]-[6]. A square transfer function matrix $K(s)$ has uniform rank k if, and only if, $\lim_{|s|\to\infty} s^k K(s)$ is finite and nonsingular.

Proof of Lemma: The para-Hermitean structure of $Q(s)$ ensures the existence of an integer $k_1 \geq 1$ such that $\lim_{|s|\to\infty} s^{2k_1} Q(s)$ is finite and non zero and equal to a real, symmetric matrix P_{k_1}. If this matrix is non-singular, the result is proved with $T_1 \equiv L(s) \equiv M(s) \equiv I_\ell$. Suppose therefore that $d_1 \triangleq \operatorname{rank} P_{k_1} < \ell$ and let T_1 be a real orthogonal eigenvector matrix such that

$$T_1^T Q(s) T_1 = \begin{pmatrix} Q_1(s) & O(s^{-(2k_1+1)}) \\ O(s^{-(2k_1+1)}) & O(s^{-(2k_1+1)}) \end{pmatrix} \tag{8}$$

where $Q_1(s)$ is $d_1 \times d_1$ and of uniform rank $2k_1$. Noting that this matrix is para-Hermitean, it is easily verified that it is possible to construct a unimodular matrix of the form

$$M_1(s) = \begin{pmatrix} I_{d_1} & O(s^{-1}) \\ 0 & I_{\ell-d_1} \end{pmatrix} \tag{9}$$

such that

$$M_1^T(-s) T_1^T Q(s) T_1 M_1(s) = \begin{pmatrix} Q_1(s) & 0 \\ 0 & H_2(s) \end{pmatrix} \tag{10}$$

However, $H_2(s)$ has a decomposition of the form $H_2(s) = V^T(-s)V(s)$ where $V(s)$ is the $m \times (\ell-d_1)$ matrix generated by the last $\ell-d_1$ columns of $G(s)T_1M_1(s)$. In particular, the assumption of left-invertibility ensures that $|H_2(s)| \not\equiv 0$ and hence that $V(s)$ is left-invertible. Applying a similar procedure to $H_2(s)$, and continuing by induction, it is possible to find q, $d_j (1 \leq j \leq q)$ and unimodular matrices of the form

$$M_j(s) = \begin{bmatrix} I_{d_1} & 0 & & 0 \\ & & & \vdots \\ 0 & & & \\ \vdots & I_{d_{j-1}} & & 0 \\ \vdots & & I_{d_j} & O(s^{-1}) \\ 0 \; . \; . & 0 & & I_{\ell-d_1-\,..\,-d_j} \end{bmatrix}, \quad 1 \leq j \leq q-1 \tag{11}$$

and real orthogonal matrices

$$T_j = \begin{bmatrix} I_{d_1} & 0 \; . \; . \; . \; . & 0 \\ 0 & & \vdots \\ \vdots & I_{d_{j-1}} & 0 \\ 0 \; . \; . \; . \; . & 0 & \tilde{T}_j \end{bmatrix}, \quad 1 \leq j \leq q-1 \tag{12}$$

such that

$$M_{q-1}^T(-s)T_{q-1}^T M_{q-2}^T(-s)T_{q-2}^T \; \ldots \; M_1^T(-s)T_1^T Q(s)T_1 M_1(s) \; ..$$

$$\ldots \; T_{q-1}M_{q-1}(s) = \text{block diag } \{Q_j(s)\}_{1 \leq j \leq q} \tag{13}$$

where $Q_j(s)$, $1 \leq j \leq q$, are of uniform rank $2k_j$, $1 \leq j \leq q$. The existence of k_j at each stage of the decomposition is guaranteed by the left-invertibility assumption.

Noting that we can replace T_1 by $T_1T_2 \ldots T_{q-1}$ without changing the structure of the $M_j(s)$, we can set $T_j = I_\ell$, $2 \leq j \leq q-1$ when equation (6) holds with

$$M(s) = M_1(s)M_2(s) \ldots M_{q-1}(s)$$

$$L(s) = M_{q-1}^T(-s) \ldots M_1^T(-s) \equiv M^T(-s) \tag{14}$$

Finally the decomposition of equation (7) is valid with $M_j(s)$ equal to $m \times d_j$ matrix generated by the $d_1+d_2+..+d_{j-1}+1$, $d_1+d_2+..+d_{j-1}+2, \ldots$ $d_1+d_2+\ldots+d_j$ th columns of $G(s)T_1M(s)$. This completes the proof of the lemma.

A comparison of the lemma with previous work [4], [6] indicates immediately a main result of this paper ie that the transfer matrix $Q(s) = C^T(-s)G(s)$ satisfies the assumptions required for the application of the method of dynamic transformation to the calculation of the orders, asymptotic directions and pivots of the unbounded roots of equation (3). The technique is simple and numerically efficient and does not suffer from restrictions inherent in other techniques [2], [3].

It is possible to obtain more detailed information on the general structure of the optimal root-locus,

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Theorem: The optimal root-locus of the left-invertible system $S(A,B,C)$ with control minimizing the performance criterion (2) has unbounded branches as $p \to +\infty$ of the form

$$s_{j\ell r}(p) = p^{\frac{1}{2k_j}} \eta_{j\ell r} + \alpha_{jr} + \varepsilon_{j\ell r}(p)$$

$$\lim_{p\to+\infty} \varepsilon_{j\ell r}(p) = 0 \qquad 1\le\ell\le k_j \;,\; 1\le r\le d_j \;,\; 1\le j\le q \tag{15}$$

where each α_{jr} is pure imaginary and the $\eta_{j\ell r}$, $1\le\ell\le k_j$, take the form $\eta_{j\ell r} = \lambda_{jr}\mu_{j\ell}$ where λ_{jr} is real and strictly positive and the $\mu_{j\ell}$, $1\le\ell\le k_j$, are the distinct left-half-plane $2k_j$th roots of $(-1)^{k_j+1}$.

Remark: In other terminology the optimal system root-locus has only even order infinite zeros with pivots on the imaginary axis of the complex plane.

Proof of theorem: Following the results of [4], [6] the lemma indicates that the unbounded solutions of equation (3) can be split into groups of even orders $2k_1, 2k_2, \ldots, 2k_q$. The more detailed structure of the $2k_j$th order roots is assessed by consideration of the unbounded roots of

$$|I_{d_j} + pQ_j(s)| = 0 \tag{16}$$

In particular, equation (7) and the uniform rank structure of Q_j indicates that

$$Q_j(s) = s^{-2k_j}P^{(j)}_{k_j} + s^{-(2k_j+1)}P^{(j)}_{k_j+1} + O(s^{-(2k_j+2)}) \tag{17}$$

where $P^{(j)}_{k_j}$ is real, symmetric and nonsingular and $P^{(j)}_{k_j+1}$ is real and skew-symmetric. Also, suitable modifications to T_1 enable us to assume that $P^{(j)}_{k_j}$ is diagonal with real, non-zero diagonal elements that are negative if k_j is odd and positive if k_j is even. The theorem is now easily proved by direct application of [4], [6], separating out only the left half plane branches and noting that the $\{\alpha_{jr}\}$ must be pure imaginary as they are the eigenvalues of diagonal blocks of $P^{(j)}_{k_j+1}$ multiplied by real numbers. The diagonal blocks are, of course, real and skew-symmetric.

We can obtain a little more information on the structure of the pivots $\{\alpha_{jr}\}$ as follows:

<u>Corollary</u>: The pivots are 'almost always' equal to zero.

<u>Proof</u>: If the eigenvalues of $P^{(j)}_{k_j}$ are distinct (the generic case!) then the pivots are equal [4], [6] to a real number multiplying the diagonal terms of $P^{(j)}_{k_j+1}$!

The corollary suggests that the optimal system root-loci may suffer from a sensitivity problem analogous to that noticed in more general studies [6]. This is easily illustrated by considering the case of

$$G(s) = \begin{pmatrix} \frac{(1+\varepsilon)}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{pmatrix} \tag{17}$$

when

$$G^T(-s)G(s) = \frac{1}{s^2}\begin{pmatrix} -(1+\varepsilon)^2 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{s^3}\begin{pmatrix} 0 & -(1+\varepsilon) \\ (1+\varepsilon) & 0 \end{pmatrix} + O(s^{-4}) \tag{18}$$

has uniform rank two. Application of the algorithms of [4] and [6] yields the left-half plane infinite zeros of the asymptotic forms for $\varepsilon \neq 0$

$$s \simeq - p^{\frac{1}{2}} + \varepsilon_1(p) \quad , \quad s \simeq -(1 + \varepsilon)p^{\frac{1}{2}} + \varepsilon_2(p)$$

$$\lim_{p\to\infty} \varepsilon_j(p) = 0 \quad , \quad j = 1,2 \tag{19}$$

and, for $\varepsilon = 0$,

$$s \simeq - p^{\frac{1}{2}} + \frac{j}{2} + \varepsilon_1(p) \quad , \quad s \simeq - p^{\frac{1}{2}} - \frac{j}{2} + \varepsilon_2(p)$$

$$\lim_{p\to\infty} \varepsilon_j(p) = 0 \quad , \quad j = 1,2 \tag{20}$$

Note the discontinuous behaviour of the pivots in the vicinity of $\varepsilon = 0$.

III SUMMARY

It has been shown that a recently derived computational method [4], [6] can always be applied to the calculation of the asymptotic behaviour of the root-locus of optimal linear regulators. The analysis has also demonstrated that the optimal root-locus has only even order infinite zeros with pivots on the imaginary axis of the complex plane. In particular the pivots are almost always (but not always) equal to zero suggesting that the optimal root-locus has sensitivity characteristics similar to those noted in multivariable root-locus studies [6].

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