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A NOTE ON THE ORDERS OF THE INFINITE ZEROS OF LINEAR MULTIVARIABLE SYSTEMS

by

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Abstract

A physical mechanism is suggested for the appearance of non-integer order infinite zeros. It is used to motivate a conjecture that attempts to characterize all possible orders of infinite zeros in terms of system structural invariants.

Multivariable root-locus theory is now well-developed (Macfarlane and Postlethwaite 1977, Owens 1978 a,b,c,d, Postlethwaite 1979, Postlethwaite and Macfarlane 1979) and a practical compensation theory is emerging (Owens 1979). If we consider the root-locus of the mxm system S(A,B,C) with transfer functionmatrix $Q(s) = C(sI_n - A)^{-1}B$ subjected to unity negative feedback with scalar gain $p \geqslant o$, then it is <u>anticipated</u> that, in almost all cases occuring in practice, the unbounded branches can be listed in the form

$$s_{j\ell}(p) = \eta_{j\ell}\, p^{1/\nu_j} + \varepsilon_{j\ell}(p)$$

$$\lim_{p\to\infty} p^{-\frac{1}{\nu_j}} \varepsilon_{j\ell}(p) = 0$$

$$1 \leq \ell \leq \nu_j \quad , \quad 1 \leq j \leq m \qquad (1)$$

where ν_j, $1 \leq j \leq m$, are strictly positive <u>integers</u>, p^{1/ν_j} is the positive-real ν_j^{th} root of p and $\eta_{j\ell}$, $1 \leqslant \ell \leqslant \nu_j$, are the distinct ν_j^{th} roots of a non-zero complex number. For simplicity $s_{j\ell}(p)$ is termed an infinite zero of <u>order</u> ν_j and asymptotic direction $\eta_{j\ell}$. It is known that the $\{\nu_j\}$ are generically integer (Owens 1978 a,d) and equal to the integer structural invariants $n_1, n_2, \ldots n_m$ of a transformation group (Owens 1978a). In this important case, the unbounded branches of the root-locus can be visualised as m group of ν_j pseudo-classical asymptotes emanating from common pivots α_j, $1 \leqslant j \leqslant m$. This is, in fact, the generic description but there are pathological cases (that can always be avoided by suitable choice of controller (Owens 1978d)) when the above description is not valid. In particular there are less than m groups of asymptotes and, if we continue to use the above terminology, the asymptotes appear to have fractional, non-integer order. The purpose of this note is to suggest a physical mechanism for this phenomenom and, on the basis of this interpretation, a conjecture is presented that claims to characterize the only possible fractional and integer order behaviours.

The vehicle for the study is the system

$$Q_o(s) = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix} \qquad (2)$$

which has one first order infinite zero and two second order infinite zeros. This is true for the system $Q(s) = Q_o(s)\, K$ for 'almost all' choices of constant nonsingular mxm 'controller' matrix K and, in fact, $n_1 = 1$ and $n_2 = 2$. Consider however the choice of

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (3)$$

to yield

$$Q(s) = \begin{pmatrix} 0 & \frac{1}{s} \\ \frac{1}{s^2} & 0 \end{pmatrix} \qquad (4)$$

The return difference determinant $|I_2 + p\, Q(s)| = (s^3 - p^2)/s^3$ suggests the presence of unbounded poles of the form $s(p) = p^{2/3}$. These poles appear, with the above definitions, to have the fractional order $^3/_2$ and it is certainly not clear how they can be split into 2(=m) pseudo-classical patterns. In fact, I suggest that we are being misled by the inadequacies of this terminology and that the correct viewpoint is that this system has 1(< m) third order pattern of asymptotes. It is hence a condition that is best avoided in controller design.

The physical interpretation and justification of this viewpoint is illustrated by noting that the closed-loop two-input/two-output system with K given by (3) takes the form of Fig. 1(a). It is easily verified that this configuration is equivalent to the single-loop system of Fig.1(b) consisting of one first order-block, one second order-block and two scalar blocks of gain p in a series configuration. This is, of course, equivalent to Fig. 1 (c), namely, a third order block operated upon by a scalar gain p^2, and confirms the viewpoint that the root-locus of the system is third order in character. The appearance of fractional orders is due totally to the squaring of the gain p.

Consider now the problem of interpretation of integer order infinite zeros for quite general systems. For almost all choices of forward path controller the infinite zeros will split into m-groups of $\nu_j = n_j$ asymptotes in a pseudo-classical manner. The fact that these asymptotes can be manipulated /compensated at will by dyadic loop controllers (Owens 1979) generating m parallel paths into the system suggests that each group of asymptotes corresponds to one of m parallel paths through the system with dynamics intuitively modelled by transfer-functions of the asymptotic form p/s^{nj}, $1 \leqslant j \leqslant m$ (any non-zero constants of proportionately being irrelevant to the arguement) and that these paths are, in some sense, non-interacting at high gains. All these ideas are easily illustrated with our simple numerical example taking, for simplicity, the case of $K = I_2$.

A solution to the problem of interpretation of 'apparent' fractional, possibly non-integer order infinite zeros is suggested by our example and the above discussions. That is, it is suggested that fractional order infinite zeros occur when the chosen forward path controller generates a series interconnection between two or more of these paths in a similar manner to that illustrated in Fig. 1(a). Suppose that the paths corresponding to the integers $n_{i_1}, n_{i_2}, \ldots, n_{i_\ell}$ are serially connected. The action of unity negative feedback in this case can be envisaged as reducing the ℓ parallel loops to the one loop illustrated is Fig. 1(d) consisting of the polynomial gain p^ℓ in series with a block of rank equal to $n_{i_1} + n_{i_2} + \ldots + n_{i_\ell}$. It is clear that this situation represents a reduction of ℓ groups of asymptotes of order n_{i_j}, $1 < j < \ell$, to one group of order $n_{i_1}+n_{i_2}+\ldots+n_{i_\ell}$. Alternatively with our original jargon, we appear to have an infinite zero of order $(n_{i_1} + n_{i_2} +\ldots+n_{i_\ell})/\ell$. In either terminology the consequent reduction in stability margins suggest that it is a situation that should be avoided in controller design.

A proof of these ideas in the general case has not yet been obtained. The above intepretation is so physically appealing however that I suggest that the following conjecture is true:

The mxm linear, time-invariant, invertible system S(A,B,C) possessing the integer structural invariants $N = \{n_1,n_2,\ldots n_m\}$ can only have infinite zeros of orders equal to arithmetic means of subsets of N.

For example for $m = 2$ and $N = \{1,2\}$, the conjecture suggest that the system can only have infinite zeros of order 1 , 2 and $(1+2)/2 = 3/2$ as in the above example. Alternatively, if $m=3$ and $N = \{1,3,4\}$, the conjecture suggests that the infinite zero can only have orders $1,3,4,(1+3)/2=2$, $(1+4)/2 = 5/2$, $(3+4)/2 = 7/2$ and $(1+3+4)/3 = 8/3$.

One important conclusion to be drawn from these ideas is that the orders of infinite zeros coincides with the number of the asymptotes only in the generic case of $\nu_j = n_j$. More generally, an infinite zero of order $(n_{i_1}+\ldots+n_{i_\ell})/\ell$ possesses $n_{i_1}+\ldots+n_{i_\ell}$ branches, even though the order may be integer. For example, if $m = 2$ and $N = \{1,3\}$, the infinite zeros of (integer!) order $(1+3)/2 = 2$ will possess $3+1 = 4$ asymptotes. One way out of this problem would be to attempt the formulation of a definition of concepts of order and multiplicity of asymptotes that can deal with both cases in a straightforward manner. Bearing in mind, however, that the series interconnection interpretation suggests that the pathological cases should be avoided in practical design, the present definitions are probably quite sufficient to describe situations of practical interest.

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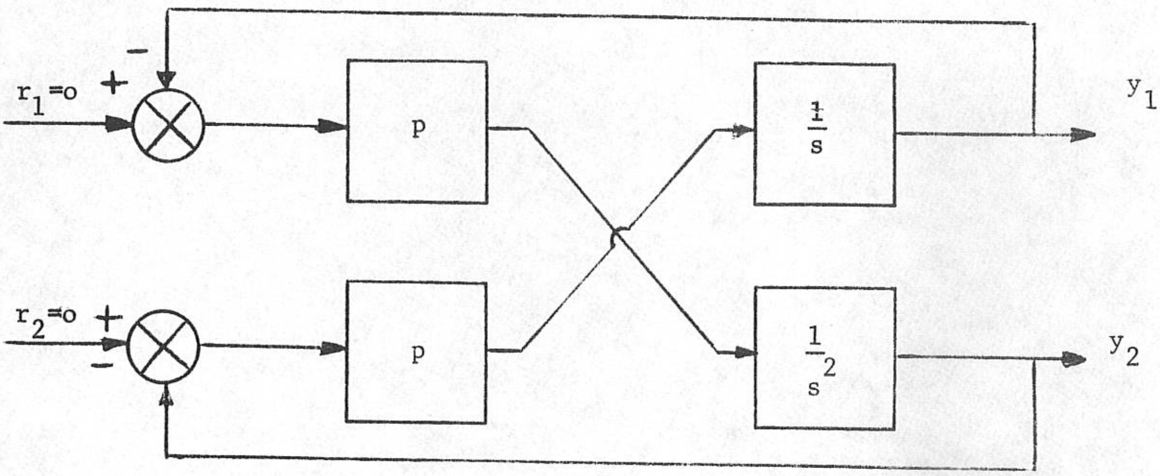
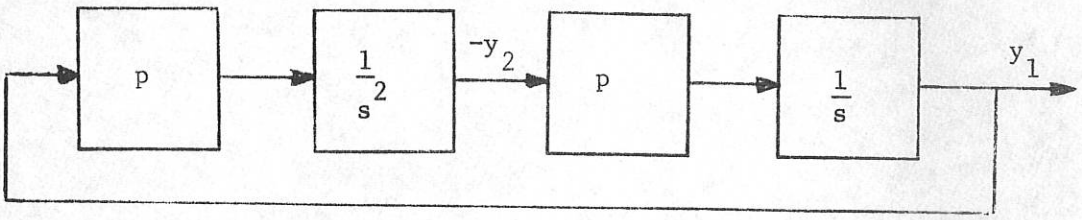


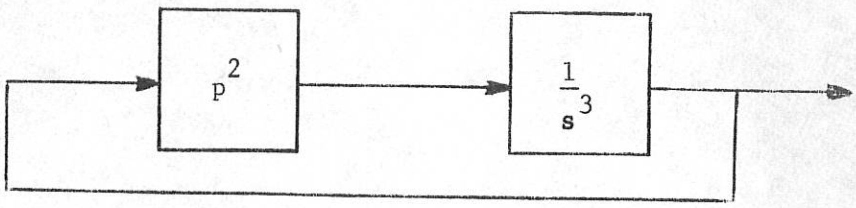
r_1=o
+
−
p
$\frac{1}{s}$
y_1
r_2=o
+
−
p
$\frac{1}{s^2}$
y_2

(a)



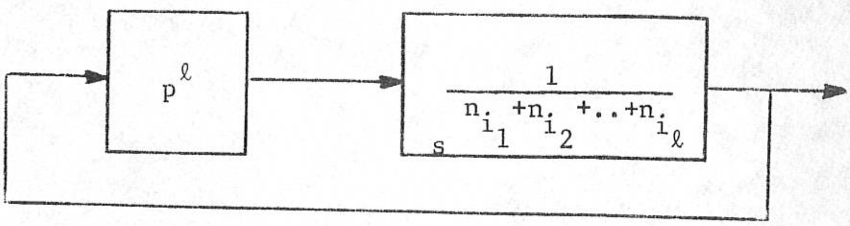
p
$\frac{1}{s^2}$
$-y_2$
p
$\frac{1}{s}$
y_1

(b)



p^2
$\frac{1}{s^3}$

(c)



p^ℓ
$\frac{1}{s^{n_{i_1}+n_{i_2}+\ldots+n_{i_\ell}}}$

(d)

Fig. 1