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A NESTED FEEDBACK LOOP DECOMPOSITION FOR THE REDUCTION OF LINEAR MULTIVARIABLE SYSTEMS

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Abstract

Previous results on the reduction of single-input/single-output systems are extended to the case of square invertible multivariable systems. It is shown that such systems have a unique decomposition in the form of a forward and a feedback path each having special properties relating to the invariant zeros of the original system. Under certain generic conditions this decomposition can be extended to yield a representation of the system as a nested sequence of feedback loops. This provides a convenient method of deriving reducing - order models, which will give good matching of the asymptotic system root-locus and open-and closed-loop system dynamics, and hence will be a valid tool in closed-loop controller design.

1. Introduction

A recent paper[1] presented a method of obtaining reduced-order models (i.e. models of smaller state-dimension) of a single-input/single-output (SISO) linear time-invariant dynamical system described by a state-space model S(A,b,c) of the form

$$\dot{x}(t) = A\,x(t) + b\,u(t) \quad , \quad x(t) \in R^n$$
$$y(t) = c\,x(t) \tag{1}$$

It was shown by manipulation of the transfer function $g(s) \triangleq c(sI - A)^{-1}b$ that the system (1) has a unique representation as a sequence of nested feedback loops, with respect to which the state-space equations take a particular canonical form. This transfer function/state-space decomposition provided the basis of a reduction method having the following important features:

(a) the models generated preserve the asymptotic root-locus behaviour (i.e. the order, asymptotes and intercepts) of the original system;

(b) models can be generated to match a desired number of moments about $s = 0$ and $|s| = \infty$ of the original system;

(c) under well-defined conditions the reduced model has poles and zeros closely approximating a subset of the poles and zeros of the original system;

(d) simulation of the system in canonical form can give an a priori estimate of suitable reduced model order, thereby eliminating some of the trial and error methods commonly associated with model reduction procedures.

In this paper the concepts underlying the reduction technique[1] for SISO systems are extended to the case of square invertible multi-input/multi-output (MIMO) systems described either by a state-space model of the form

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad , \; x(t) \in R^n \; , \; u(t) \in R^m$$
$$y(t) = C\,x(t) \quad , \quad y(t) \in R^m \tag{2}$$

or equivalently by a strictly proper mxm invertible transfer function matrix, G(s).

The development is such that each result can be seen to be the direct generalisation to the square multivariable case of the equivalent result for SISO systems. In section 2 it is shown that a square invertible system has a unique decomposition into the form of an invertible forward path system having no zeros, and a feedback path with strictly proper transfer function matrix, characterising the invariant zeros of the original system. In terms of the state-space representation (2), this decomposition is achieved by factoring out the subspace ν^*, the maximal (A,B) - invariant subspace in the kernel of C, as described by Wonham[2], Bengtsson[3], Morse[4]. Under certain mild restrictions (corresponding to the generic condition of invertibility of certain subsystems) the decomposition can be extended naturally to give a nested sequence of feedback loops, each having no invariant zeros, and with respect to which the system state-space equations again take a particular canonical form. It is also possible to modify the theory to allow the retention in each loop of a specified set of invariant zeros. Section 3 describes how this nested feedback loop decomposition can be applied to the problem of model reduction, providing a method of obtaining reduced order models which again satisfies properties (a) - (d) above. Section 4 considers computational schemes for achieving the desired system decomposition, which are shown to be particularly simple in the generic case when each loop has uniform rank[5,6]. Section 5 presents a detailed example, illustrating the application of the proposed techniques to the reduction of a complex high-order two-input, two-output system.

2. Decomposition of a Square Invertible Linear System

We consider initially the decomposition of a square invertible MIMO system into an invertible forward path system, having no zeros, and a single strictly proper feedback system, and deomonstrate that such a decomposition can be equivalently derived from either of the two standard forms of system representation, i.e. a transfer function matrix, or a set of state-space equations.

2.1 Decomposition of Square Invertible Transfer Function Matrix

Consider an invertible linear MIMO system having a strictly proper mxm transfer function matrix G(s). The inverse transfer function matrix may be written as[6]

$$G^{-1}(s) = P_1(s) - H_1(s) \tag{3}$$

where $P_1(s)$ is an mxm polynomial matrix in s, and $H_1(s)$ is a strictly proper mxm transfer function matrix. Furthermore, $P_1(s)$ and $H_1(s)$ as defined in this way are unique. As G(s) is invertible, and likewise $P_1(s)$[6] equation (3) gives

$$G(s) = [P_1(s) - H_1(s)]^{-1} = [I - P_1^{-1}(s)\ H_1(s)]^{-1}\ P_1^{-1}(s) \tag{4}$$

and defining $G_1(s) \triangleq P_1^{-1}(s)$, this may be written as

$$G(s) = [I - G_1(s)\ H_1(s)]^{-1}\ G_1(s) \tag{5}$$

In this expression, $G_1(s)$ is a strictly proper[6] transfer function matrix with polynomial inverse, which is uniquely defined by (3), and has no invariant zeros[6]. Thus equation (5) gives a unique decomposition of G(s) into a forward path system $G_1(s)$ having no invariant zeros, and a strictly proper feedback system $H_1(s)$, as illustrated by Figure 1.

If $G_1(s)$ and $H_1(s)$ have minimal (i.e. controllable and observable) state-space realisations $S(A_1,B_1,C_1)$ and $S(A_2,B_2,C_2)$ of state dimension n_1 and n_2 respectively, then equation (5) implies that S(A,B,C), with

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix} \quad ; B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad ; C = [C_1 \quad 0] \tag{6}$$

is a state-space representation of G(s). Furthermore, S(A,B,C) is controllable and observable (c.f. Appendix 1).

Defining $F \triangleq [0\ C_2]$, and $\bar{A} = A - BF$, it is easily seen that $\nu_2 \triangleq \text{Span}\ \{e_{n_1+1}, e_{n_2+1}, \ldots, e_{n_1+n_2}\}$ (where e_i is the i^{th} unit vector in $R^{n_1+n_2}$, $n_1+1 \leq i \leq n_1+n_2$) is an (A,B) - invariant subspace in the kernel of C. In fact, as $S(A_1,B_1,C_1)$ has no invariant zeros, ν_2 is the largest such subspace, denoted by ν^*. Thus $S(A_1,B_1,C_1)$ is a factor system[2] obtained (by factoring out ν^*) from $S(\bar{A},B,C)$, and the poles of $S(A_2,B_2,C_2)$, or equivalently the roots of $|sI_{n_2} - A_2|$, are the invariant zeros of $S(A,B,C)$, or $G(s)$.

2.2 Decomposition of State-Space

Consider the mxm invertible system $S(A,B,C)$ of state dimension n, described by equation (2). Using the results of Morse[4], and defining ν^* to be the maximal (A,B) - invariant subspace in the kernel of C, and τ to be the smallest subspace satisfying $R(B) \subset \tau$, $A(N \cap \tau) \subset \tau$ (where R(B) denotes the range of B,N denotes the kernel of C), invertibility of $S(A,B,C)$ implies[7] that

$$\nu^* \cap R(B) = 0 \tag{7}$$

$$\nu^* \oplus \tau = R^n \tag{8}$$

(where $\oplus$ denotes a direct sum). Let $\dim \nu^* = n_z$ (= number of invariant zeros of $S(A,B,C)$), and set $n_1 \overset{\Delta}{=} n - n_z$.

Then the effect of transforming to the basis $\{\tau, \nu^*\}$ of R^n is characterised by the following result.

Theorem 1

For a square invertible system $S(A,B,C)$ of state dimension n, transformation to a basis $\{\tau , \nu^*\}$ yields an equivalent system $S(A(1), B(1), C(1))$ of the form

$$A(1) = \begin{bmatrix} \hat{A}_1 & \hat{B}_1 C_2 \\ B_2 \hat{C}_1 & A_2 \end{bmatrix} ; B(1) = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} , C(1) = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix} \tag{9}$$

for some matrices $\hat{A}_1, \hat{B}_1, \hat{C}_1, A_2, B_2, C_2$ of dimension $n_1 \times n_1$, $n_1 \times m$, $m \times n_1$, $n_z \times n_z$, $n_z \times m$, $m \times n_z$ respectively. The system $S(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ is controllable, observable and invertible, and has no invariant zeros, and the system $S(A_2,B_2,C_2)$ is controllable (resp. observable) if and only if $S(A,B,C)$ is controllable (resp. observable) (Proof: c.f. Appendix 2.)

It should be noted that $S(A_2,B_2,C_2)$ as defined above may not be invertible as, for instance, B_2 or C_2 need not have full rank.

Equations (9) are of the same form as equations (6), and by defining $G_1(s) \triangleq \hat{C}_1 (sI_{n_1} - \hat{A}_1)^{-1}\hat{B}_1$, and $H_1(s) \triangleq C_2(sI_{n_z} - A_2)^{-1}B_2$, and partitioning the state vector in the obvious way, it csn be seen that S(A(1), B(1), C(1)), and hence S(A,B,C), may again be represented in the forward/feedback loop configuration of Figure 1. Furthermore, as $S(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ has no invariant zeros, the results of Section 2.1 imply that $G_1(s)$ and $H_1(s)$ are uniquely defined. Thus it can be seen that the approaches of Sections 2.1 and 2.2 yield exactly equivalent results.

We now consider conditions under which the above decomposition can be applied repeatedly to give a nested sequence of feedback loops, the MIMO equivalent of the result obtained for SISO systems.

2.3 Nested Feedback Decomposition

If $S(A_2,B_2,C_2)$, or equivalently $H_1(s)$, as defined above is invertible it can be similarly decomposed into forward and feedback paths to give $G_2(s)$, $H_2(s)$. Proceeding in this way, if for some integer ℓ, the successively defined systems $S(A_i, B_i, C_i)$ $(3 \leq i \leq \ell)$ are all invertible then the system S(A,B,C), or equivalently G(s), admits the decomposition of Figure 2, in which $G_i(s) \triangleq \hat{C}_i(sI - \hat{A}_i)^{-1}\hat{B}_i$ $(1 \leq i \leq \ell)$ is an mxm strictly proper invertible transfer function matrix having no zeros, and $H_\ell(s) \triangleq C_{\ell+1}(sI - A_{\ell+1})^{-1} B_{\ell+1}$ is a strictly proper transfer function matrix. Thus G(s) may be expressed as

$$G(s) = [G_1^{-1}(s) - [G_2^{-1}(s) -[\ldots \quad -[G_\ell^{-1}(s) - H_\ell(s)]^{-1}]^{-1}\ldots]^{-1}. \tag{10}$$

The corresponding state-space equations have the form

$$\dot{x}(t) = A(\ell)\ x(t) + B(\ell)\ u(t)$$
$$y(t) = C(\ell)\ x(t) \tag{11}$$

with

$$A(\ell) = \left(\begin{array}{cccccccc|c} \hat{A}_1 & \hat{B}_1\hat{C}_2 & 0 & 0 & & & & & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 & \hat{B}_2\hat{C}_3 & 0 & & & & & 0 \\ 0 & \hat{B}_3\hat{C}_2 & \hat{A}_3 & \hat{B}_3\hat{C}_4 & \cdot & & & & 0 \\ 0 & 0 & \hat{B}_4\hat{C}_3 & \hat{A}_4 & & & & & 0 \\ & & & & \ddots & \hat{A}_{\ell-1} & \hat{B}_{\ell-1}\hat{C}_\ell & & 0 \\ & & & & & \hat{B}_\ell\hat{C}_{\ell-1} & \hat{A}_\ell & & \hat{B}_\ell C_{\ell+1} \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & B_{\ell+1}\hat{C}_\ell & & A_{\ell+1} \end{array}\right) \tag{12}$$

and $\hat{B}(\ell)=\hat{B}(1)$, $\hat{C}(\ell) = \hat{C}(1)$ are as defined by equations (9). $S(\hat{A}_i,\hat{B}_i,\hat{C}_i)$ has state dimension n_i $(1 \leqslant i \leqslant \ell)$, and $S(A_{\ell+1},B_{\ell+1},C_{\ell+1})$ has state dimension $n - \sum_{i=1}^{\ell} n_i$. Making the corresponding partition of the state vector, $x(t) = (x_1^T(t), x_2^T(t),\ldots x_{\ell+1}^T(t)^T$, the outputs $y_i(t)$ in Figure 2 are given by $y_i(t)=\hat{C}_i(t)x_i(t)$ $(1 < i < \ell)$, $y_{\ell+1}(t) = C_{\ell+1}(t)x_{\ell+1}(t)$. The uniqueness of each subsystem follows from Section 2.1 and 2.2, and this decomposition of the system into such a nested sequence of feedback loops with loop transfer matrices $G_i(s)$ having no zeros (so that $G_i^{-1}(s)$ is a <u>polynomial matrix</u> in s[6]), for $1 \leqslant i \leqslant \ell$, is unique. The existence of the decomposition depends only upon the invertibility of the successively defined subsystems $S(A_i,B_i,C_i)$. Although this condition is not always satisfied, it is true that if for some $j > 1$ the subsystems $S(A_i,B_i,C_i)$ $(1 \leqslant i \leqslant j)$ are all invertible, and $n - \sum_{i=1}^{j} n_i \geqslant m$, then the invertibility of $S(A_{j+1},B_{j+1},C_{j+1})$ holds generically.

Finally note that the matrix

$$\left(\begin{array}{cccccc|c} \hat{A}_2 & \hat{B}_2\hat{C}_3 & 0 & & & & 0 \\ \hat{B}_3\hat{C}_2 & \hat{A}_3 & \hat{B}_3\hat{C}_4 & & & & 0 \\ 0 & \hat{B}_4\hat{C}_3 & \hat{A}_4 & & & & 0 \\ & & & \ddots & & & \\ & & & & & \hat{A}_\ell & \hat{B}_\ell C_{\ell+1} \\ \hline 0 & 0 & 0 & & & B_{\ell+1}\hat{C}_\ell & A_{\ell+1} \end{array}\right) \tag{13}$$

is similarity equivalent to A_2, and hence has eigenvalues equal to the invariant zeros of S(A,B,C).

2.4 Retention of Zeros in the Forward Path

In the decomposition described above the forward path transfer function matrices have no invariant zeros. It is possible however to retain zeros in each loop by the following procedure.

Let S(A,B,C) be a controllable and observable mxm invertible system and suppose that ν^*, as defined in Section 2.2, can be expressed as

$$\nu^* = \nu_1^* \oplus \nu_2^* \tag{14}$$

with $A\nu_i^* \subset \nu_i^* + R(B)$ (i = 1,2). Equation (14) corresponds to a partitioning of the set of system zeros into two disjoint subsets, with ν_1^* representing those zeros which are to be retained in the forward loop system, and ν_2^* representing the remaining zeros which will constitute the poles of the feedback system. Considering the proof of Theorem 1 (in Appendix 2), the (A,B) - invariance of the subspaces ν_1^*, ν_2^* implies that

F may be chosen to satisfy $(\hat{A} + \hat{B}F)\nu_i^* \subset \nu_i^*$ (i = 1,2)

Then, as $C\,\nu_i^* = \{o\}$ (i = 1,2), it follows that

$$(\hat{A} + \hat{B}F + \hat{K}C)(\tau + \nu_1^*) = (\hat{A} + \hat{B}F + \hat{K}C)\tau + (\hat{A} + \hat{B}F + \hat{K}C)\nu_1^*$$

$$\subset \tau + (\hat{A} + \hat{B}F)\nu_1^* \subset \tau + \nu_1^*$$

and $\quad (\hat{A} + \hat{B}F + \hat{K}C)\,\nu_2^* = (\hat{A} + \hat{B}F)\nu_2^* \subset \nu_2^*$. Thus it can be seen that a transformation to a basis $\{\tau + \nu_1^*, \nu_2^*\}$ again yields a system $S(A^*,B^*,C^*)$ of the form

$$A^* = \begin{bmatrix} A_1^* & B_1^*C_2^* \\ B_2^*C_1^* & A_2^* \end{bmatrix} ; \; B^* = \begin{bmatrix} B_1^* \\ 0 \end{bmatrix} ; \; C^* = \begin{bmatrix} C_1^* & 0 \end{bmatrix} \tag{15}$$

where $S(A_1^*,B_1^*,C_1^*)$ has state dimension = n - dim ν_2^*, and $S(A_2^*,B_2^*,C_2^*)$ has state dimension = dim ν_2^*. The controllability and observability of $S(A_i^*,B_i^*,C_i^*)$ (i = 1,2) follow directly from the corresponding properties of S(A,B,C), and the invertibility of $S(A_1^*,B_1^*,C_1^*)$ is implied by the invertibility of S(A,B,C).

Defining $G_1(s) \triangleq C_1^*(sI - A_1^*)^{-1} B_1^*$, $H_1(s) \triangleq C_2^*(sI - A_2^*)^{-1}B_2^*$, we again have the forward and feedback path decomposition illustrated in Figure 1. Here, however, $G_1(s)$ has a set of zeros characterised by the maximal (A_1^*,B_1^*) invariant subspace in the kernel of C_1^*, which is precisely ν_1^*, and the poles of $H_1(s)$ are those zeros of S(A,B,C) corresponding to ν_2^*.

3. Application to Model Reduction

Suppose that for some $\ell \geq 1$ the invertible mxm system S(A,B,C), with transfer function matrix G(s), admits an equivalent representation of the form (12) corresponding to the state-space decomposition described in Section 2. (Sections 2.1 and 2.2 demonstrate that such a decomposition certainly holds for $\ell = 1$). An obvious method of obtaining a reduced-order model of S(A,B,C), and the natural generalisation of that proposed by the authors[(1)] for single-input/single-output systems, would be to retain loops 1,2,...j of the nested feedback decomposition, for some $1 \leq j \leq \ell$ and approximate $S(A_{j+1},B_{j+1},C_{j+1})$ or equivalently $H_j(s)$, by some lower order system $S(A_{j+1}^{(a)},B_{j+1}^{(a)},C_{j+1}^{(a)})$ (or $H_j^{(a)}(s)$). The reduced - order model, $S(A^{(a)},B^{(a)},C^{(a)})$, generated in this way would take the form

$$A^{(a)} = \left[\begin{array}{cccccccc|c} \hat{A}_1 & \hat{B}_1\hat{C}_2 & 0 & & & & & & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 & \hat{B}_2\hat{C}_2 & \cdot & & & & & 0 \\ 0 & \hat{B}_3\hat{C}_2 & \hat{A}_3 & & \cdot & & & & \vdots \\ & \cdot & & \cdot & & \cdot & & & \\ & & \cdot & & \cdot & & & & \\ & & & & & \hat{A}_{j-1} & \hat{B}_{j-1} & \hat{C}_j & 0 \\ & & & & & \hat{B}_j\hat{C}_{j-1} & \hat{A}_j & & \hat{B}_jC_{j+1}^{(a)} \\ \hline 0 & 0 & 0 & & & 0 & B_{j+1}^{(a)} & \hat{C}_j & A_{j+1}^{(a)} \end{array}\right]$$

$$B^{(a)} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \quad C^{(a)} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix} \tag{16}$$

with $\hat{A}_i, \hat{B}_i, \hat{C}_i$ $(1 \leq i \leq j)$ defined by equation (12), and transfer function matrix, $G^{(a)}(s)$, given by

$$G^{(a)}(s) = \left[G_1^{-1}(s) - \left[G_2^{-1}(s) - \ldots - \left[G_j^{-1}(s) - H_j^{(a)}(s)\right]^{-1} \ldots\right]^{-1} \tag{17}$$

If $S(A_{j+1}^{(a)}, B_{j+1}^{(a)}, C_{j+1}^{(a)})$ is chosen to have state dimension p then the order of the reduced model $S(A^{(a)}, B^{(a)}, C^{(a)})$ is $\sum_{i=1}^{j} n_i + p$, and $S(A^{(a)}, B^{(a)}, C^{(a)})$ is controllable (resp. observable) if and only if $S(A_{j+1}^{(a)}, B_{j+1}^{(a)}, C_{j+1}^{(a)})$ is controllable (resp. observable). (This follows by applying the proof of minimality given in Appendix 1 successively to the subsystems $S(A_j^{(a)}, B_j^{(a)}, C_j^{(a)})$, $S(A_{j-1}^{(a)}, B_{j-1}^{(a)}, C_{j-1}^{(a)})$,...which are derived from $S(A_{j+1}^{(a)}, B_{j+1}^{(a)}, C_{j+1}^{(a)})$ in the obvious way).

(Note: As indicated by the previous remark, the method of deriving reduced order models suggested above (and the computational procedure described in Section 4) can be applied to non-minimal system realisations S(A,B,C) and still generate controllable and observable models. This may be useful when the system to be reduced is described by some complex high-order transfer function matrix, and a non-minimal state-space realisation can be obtained by simple inspection by using, for instance, the method of Rosenbrock[8]).

The proposed model reduction technique has several important features paralleling those for the SISO case, which are discussed below.

3.1 Moment Matching about s = o and |s| = ∞

Consider the approximation $G^{(a)}(s) = \left[G_1(s) - H_1^{(a)}(s)\right]^{-1}$ to the system transfer function matrix

$G(s) = \left[G_1(s) - H_1(s)\right]^{-1}$. Evidently

$$G(s) - G^{(a)}(s) = G(s)\left[H_1(s) - H_1^{(a)}(s)\right]G^{(a)}(s) \tag{18}$$

Suppose that $H_1(s)$ has the series exansion about $|s| = \infty$ given by

$$H_1(s) = \sum_{i=1}^{\infty} s^{-1} M_i \tag{19}$$

where M_i , $i \geq 1$, are the system Markov parameters, and a series expansion about s = 0 given by

$$H_1(s) = \sum_{i=o}^{\infty} s^i N_i \tag{20}$$

If $H_1^{(a)}(s)$ matches the first m_o Markov parameters, M_i, $1 \leqslant i \leqslant m_o$, and the first n_o time moments N_i, $0 \leqslant i \leqslant n_o - 1$, of $H_1(s)$ then equation (18) implies that $G^{(a)}(s)$ matches the first $m_o + 2$ non-zero Markov parameters and the first n_o non-zero time moments of $G(s)$.

(Note: Under the above conditions on $H_1^{(a)}(s)$, $G^{(a)}(s)$ may well match more than the specified number of moments of $G(s)$ about $s = o$ or $|s| = \infty$. For instance, if $G(s)$ is of uniform rank[2] k_1, for some $k_1 > 1$, so that the first $k_1 - 1$ Markov parameters of $G(s)$ (and hence of $G^{(a)}(s)$!) are zero, then it can be shown from equation (18) that $G^{(a)}(s)$ matches the first $k + m_o + 1$ non-zero Markov parameters of $G(s)$. Similarly if the first k_2 time moments of $G(s)$ are all zero for some $k_2 > 0$, then $G^{(a)}(s)$ will match the first k_2+n_o non-zero time moments of $G(s)$).

Applying the above techniques inductively to $H_i(s)$, $1 \leqslant i \leqslant j$, it is possible to state the following generalisation of single-input/single-output results on the moment-matching properties of the reduced models.

Theorem 1

If $G^{(a)}(s)$ as defined by equation (17) is an approximation to $G(s)$, with $G(s) = [G_1^{-1}(s)-[G_2^{-1}(s) - \ldots -[G_j^{-1}(s) - H_j(s)]^{-1} \ldots]^{-1}$, such that $H_j^{(a)}(s)$ matches the first m_o Markov parameters and n_o time moments of $H_j(s)$, then $G^{(a)}(s)$ matches (at least) the first $2j + m_o$ non-zero Markov parameters and the first n_o non-zero time moments of $G(s)$.

3.2 Asymptotic Behaviour of the System Root-locus

It is known[6,9] that the orders, asymptotic directions and pivots of the unbounded infinite zeros of the root-locus of $G(s)$ $K(s)$, for any forward path controller K(s) satisfying the (practical) constraint that $\lim_{|s|\to\infty} K(s)$ is finite and non-singular, are identical to those of $G_1(s)\,K(s)$ with $G_1(s)$ defined as in Section 2.1. Moreover, this result is true for any reduced model of G(s) with forward path transfer function matrix $G_1(s)$ and strictly proper feedback system. That is, if system compensation is regarded as the systematic manipulation of asymptotic directions and pivots[10] then the reduced order models described above can be used with confidence as the basis of the design exercise.

The finite cluster points of the root-locus at high gains are identical to the system zeros. It is therefore important that the reduced model should give an adequate representation of the dominant zeros of the system. This is discussed in the next section.

3.3 Matching of the Open-Loop Poles and Zeros

Consider a square invertible system with a state-space representation of the form (12). If the elements of the off-diagonal block $\hat{B}_j\hat{C}_{j+1}$ (or $\hat{B}_{j+1}\hat{C}_j$) are 'small' in some sense for some $j \leqslant \ell$, then the states corresponding to the lower blocks $j+1, j+2, \ldots, \ell+1$ will be 'approximately' unobservable (resp. uncontrollable) and the eigenvalues of the matrix

$$\begin{bmatrix} \hat{A}_1 & \hat{B}_1\hat{C}_2 & & & \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 & & & \\ & & \ddots & & \\ & & & \hat{A}_{j-1} & \hat{B}_{j-1}\hat{C}_j \\ & & & \hat{B}_j\hat{C}_{j-1} & \hat{A}_j \end{bmatrix} \qquad (21)$$

will be a good approximation to a proper subset of the eigenvalues of $A(\ell)$, containing the dominant modes of the original system. Similarly the eigenvalues of

$$\begin{bmatrix} \hat{A}_2 & \hat{B}_2\hat{C}_3 & & & \\ \hat{B}_3\hat{C}_2 & \hat{A}_3 & & & \\ & & \ddots & & \\ & & & \hat{A}_{j-1} & \hat{B}_{j-1}\hat{C}_j \\ & & & \hat{B}_j\hat{C}_{j-1} & \hat{A}_j \end{bmatrix} \quad (22)$$

will be a good approximation to the lower r.h. submatrix of $\hat{A}(\ell)$ defined by equation (13), and hence to the dominant system zeros. Thus the reduced order model obtained simply by neglecting loops j+1, j+2,...ℓ+ 1, i.e. by putting $H_j^{(a)}(s) = 0$, can preserve the essential pole-zero structure of the original system. In practice a non-zero approximation $H_j^{(a)}(s)$ will probably be chosen to give better matching of the system steady-state behaviour, but, providing that this is an adequate representation of $H_j(s)$, the reduced order model $G^{(a)}(s)$ can still provide a good match to the dominant poles and zeros of G(s).

For certain systems the gain and phase margins, and hence the closed-loop stability and oscillation characteristics, will depend critically upon a particular subset of the system zeros, for example those situated in, or close to , the right-half complex plane. In such cases, these zeros can be retained explicitly in the forward path system, $G_1(s)$, by the techniques of Section 2.4, so that they will also be zeros of any reduced order model derived by the proposed methods.

3.4 Estimation of Suitable Reduced-Model Order

As in the SISO case,[1] the reduction technique proposed here can be used to provide a priori estimates of suitable reduced model order, although the multivariable nature of the problem does make it difficult to give rules concerning the reducibility of the system which will be applicable in all practical cases.

Consider a system represented by the state-space canonical form of equation (12). An examination of the magnitudes of the elements in the off-diagonal blocks of the state matrix, as suggested by the previous section, may indicate that an adequate model may be obtained from the first j loops only (for some $1 \leq j \leq \ell$). In general, though, the elements within each block will have an absolute magnitude dependent on the detailed choice of basis, and hence it is difficult to assess their overall significance in relation to the many other parameters within the state matrices. The significance of the respective feedback loops can be more effectively assessed from simulation data by considering the simulated response of the system in the form (12) to suitable inputs u(t) in each channel, and examining the output of each loop in the corresponding nested feedback decomposition (as described in Figure 2). More precisely, for $1 \leq j \leq m$, let $y_i^j(t) = (y_{1i}^j(t), y_{2i}^j(t), \ldots, y_{mi}^j(t))^T$ denote the output $y_i(t)$ of loop i $(1 \leq i \leq \ell+1)$ in the feedback configuration of Figure 2 to the system input $H(t).e_j$, i.e. a unit step in channel j (H(t) is the unit step function: $H(t) = 0, t \leq 0$, $H(t) = 1, t > 0$, and e_j is the j^{th} unit vector in R^m), and define $y_o^j(t) = (y_{1o}^j(t), \ldots y_{mo}^j(t))^T \triangleq H(t)\, e_j$. Then if for some $1 \leq k \leq \ell$ the loop outputs satisfy

$$|y_{i,k-1}^j(t)| \gg |y_{i,k+1}^j(t)| , \tag{23}$$

for all $1 \leq i, j \leq m$, and all times t of interest, it can be seen from Figure 2 that loops k+1, k+2,...ℓ+1 have little effect on the system input-output behaviour, and that an adequate model can intuitively be obtained by retaining loops 1,2,...k and taking some simple approximation to $H_k(s)$ (e.g. a steady-state approximation, as described in Section 3.5). Equation (23) represents m^2 inequalities, and in practice there may not exist a sufficiently small integer k for which they are all satisfied. However, experience with numerical examples indicates that it is frequently only necessary for equation (23) to be satisfied for some subset of the i's and j's, or that the loop outputs need simply satisfy a norm criterion

$$|| y^j_{k-1}(t) || >> || y^j_{k+1}(t) ||, \qquad (24)$$

for all $1 \leqslant j \leqslant m$ and times t of interest.

So far, only models obtained by taking strictly proper approximations $S(A^{(a)}_{j+1}, B^{(a)}_{j+1}, C^{(a)}_{j+1})$ to $S(A_{j+1}, B_{j+1}, C_{j+1})$ or $H_j(s)$ have to be considered. It may in some situations be useful to take approximations of the form $S(A^{(a)}_{j+1}, B^{(a)}_{j+1}, C^{(a)}_{j+1}, D^{(a)}_{j+1})$, where $D^{(a)}_{j+1}$ is an mxm constant matrix, described by the equations

$$\begin{aligned} \dot{x}(t) &= A^{(a)}_{j+1}\, x(t) + B^{(a)}_{j+1}\, u(t) \\ y(t) &= C^{(a)}_{j+1}\, x(t) + D^{(a)}_{j+1}\, u(t) \end{aligned} \qquad (25)$$

with transfer function matrix

$$H^{(a)}_j(s) = D^{(a)}_{j+1} + C^{(a)}_{j+1}\,(sI - A^{(a)}_{j+1})^{-1}\, B^{(a)}_{j+1} \qquad (26)$$

so that

$$\lim_{|s|\to\infty} H^{(a)}_j(s) = D^{(a)}_{j+1}.$$

Section 3.5 summarises briefly the extension of the previous results to this case.

3.5 Non-strictly Proper Approximations to $S(A_{j+1}, B_{j+1}, C_{j+1})$

Consider the reduced model of the square invertible mxm system S(A,B,C) in canonical state-space form (12), obtained by retaining loops 1,2,...j (for some $1 \leqslant j \leqslant \ell$) of the corresponding feedback decomposition and taking an approximation to $S(A_{j+1}, B_{j+1}, C_{j+1})$ of the form (26). Then it is not difficult to show that the resulting model has a state-space representation $S(A^{(a)}, B^{(a)}, C^{(a)})$ of the form

$$A^{(a)} = \left[\begin{array}{cccccc|c} \hat{A}_1 & \hat{B}_1\hat{C}_2 & 0 & & & & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 & \hat{B}_2\hat{C}_3 & & & & 0 \\ 0 & \hat{B}_3\hat{C}_2 & \hat{A}_3 & & & & 0 \\ & & & \cdot & & & \\ & & & & \hat{A}_j & \hat{B}_{j-1}\hat{C}_j & 0 \\ & & & & \hat{B}_j\hat{C}_{j-1} & \hat{A}_j+\hat{B}_j D^{(a)}_{j+1}\hat{C}_j & \hat{B}_j C^{(a)}_{j+1} \\ \hline 0 & 0 & 0 & & 0 & B^{(a)}_{j+1}\ \hat{C}_j & A^{(a)}_{j+1} \end{array}\right]$$

$$\hat{B}^{(a)} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \quad \hat{C}^{(a)} = [C_1 \quad 0] \tag{27}$$

The system $S(\hat{A}_j + \hat{B}_j D^{(a)}_{j+1} \hat{C}_j, \hat{B}_j, \hat{C}_j)$ can be obtained from $S(\hat{A}_j, \hat{B}_j, \hat{C}_j)$ by the application of constant output feedback, and therefore has no zeros, so that $S(A^{(a)}, B^{(a)}, C^{(a)})$ described by equation (27) can again be represented by the nested feedback loop decomposition of Figure 2, in which loops 1,2,...j have invertible transfer function matrices with no invariant zeros, and loop j+1 is strictly proper. Loop j has inverse transfer function matrix equal to $G_j^{-1}(s) - D^{(a)}_{j+1}$, and the controllability (resp observability) of $S(A^{(a)}, B^{(a)}, C^{(a)})$ and $S(A^{(a)}_{j+1}, B^{(a)}_{j+1}, C^{(a)}_{j+1}, D^{(a)}_{j+1})$ are again equivalent. Thus if $j > 1$, the reduced order model $S(A^{(a)}, B^{(a)}, C^{(a)})$ will still preserve the asymptotic properties of the system root-locus as described in Section 3.2, but can only be guaranteed to match the first $2j - 1$ Markov parameters of $G(s)$.

The most useful application of a non-strictly proper approximation to $H_j(s)$ occurs in the case where $\lim_{s\to o} H_j(s)$ is finite (i.e. A_{j+1} is non-singular) and $H_j^{(a)}$ is taken to be the steady-state approximation $H_j(o)$ (as illustrated in the example of Section 5). In this case, $D^{(a)}_{j+1} = - C_{j+1}A^{-1}_{j+1}B_{j+1}$, and the $j+1^{th}$ blocks in equation (27), and the last loop of the corresponding nested feedback configuration, disappear. $G^{(a)}(s)$ will now satisfy

$$G^{(a)}(o) = G(o) \tag{28}$$

i.e. $G^{(a)}(s)$ matches the first time moment of $G(s)$.

4. Computational Procedure for Decomposing the System State-Space

The implementation of the reduction techniques proposed above, for which the system must be transformed to the state-space canonical form[12], requires a procedure for calculating at each stage the subspaces ν^* and τ defined in Section 2.2. There is a well known recursive relation[11] (in fact the dual of equation (45)) which expresses ν^* as a limit of a sequence of subspaces, and this may be translated directly[3] into an algorithm to generate a basis matrix for ν^*. A basis for τ may be found either by noting that $\tau^\perp$ (the orthogonal complement of τ in R^n) is the maximal (A^T, C^T)-invariant subspace contained in the kernel of B^T, or by applying the algorithm presented in Appendix 3.

In the generic case when $S(A,B,C)$ is of uniform rank[5,6] k_1 so that $CB = CAB = \ldots = CA^{k_1-2}B = 0$, $|CA^{k_1-1}B| \neq 0$, (in fact, generically, $k_1 = 1$), it can be shown[12] that

$$\tau = R(B) \oplus AR(B) \oplus \ldots \oplus A^{k_1-1}R(B) \tag{29}$$

and

$$\nu^* = \bigcap_{i=1}^{k_1} \mathrm{Ker}\,(CA^{i-1}) \tag{30}$$

Equations (29) and (30) can then be used to generate simple algorithms for calculating basis for τ and ν^*, and furthermore, transformation to the basis

$$\left[B(CA^{k_1-1}B)^{-1}, AB(CA^{k_1-1}B)^{-1}, \ldots, A^{k_1-1}B(CA^{k_1-1}B)^{-1}, V\right] \tag{31}$$

where V is some basis matrix for ν^* yields $S(\hat{A}_1, \hat{B}_1, \hat{C}_1)$, as defined in equation (10), in the particularly simple 'observable block companion form'

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 & A_{11} \\ I_m & 0 & \ldots & 0 & A_{21} \\ 0 & I_m & \ldots & 0 & A_{31} \\ \vdots & & & & \\ 0 & 0 & & I_m & A_{k_1 1} \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} CA^{k_1-1}B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \hat{C}_1 = \begin{bmatrix} 0 \ldots 0 & I_m \end{bmatrix} \tag{32}$$

for some mxm matrices A_{j1} $(1 \leq j \leq k_1)$. The corresponding forward path transfer function matrix, $G_1(s)$, then satisfies

$$G_1^{-1}(s) = (CA^{k_1-1}B)^{-1}\left[s^{k_1}I_m - s^{k_1-1}A_{k_1 1} - \dots - sA_{21} - A_{11}\right] \quad (33)$$

Similarly, if $S(A_i,B_i,C_i)$ is of uniform rank k_i, for $2 \leq i \leq \ell$, then $S(\hat{A}_i,\hat{B}_i,\hat{C}_i)$ as defined in Section 2.3 take the form

$$\hat{A}_i = \begin{bmatrix} 0 & 0 & \dots & 0 & A_{1i} \\ I_m & 0 & & 0 & A_{2i} \\ 0 & I_m & & 0 & A_{3i} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & & I_m & A_{k_i i} \end{bmatrix}, \quad \hat{B}_i = \begin{bmatrix} C_iA_i^{k_i-1}B_i \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \hat{C}_i = [0 \dots 0 I_m] \quad (34)$$

for some matrices A_{ji} $(1 \leq j \leq k_i)$.

5. Illustrative Example

Consider the following 2 - input, 2 - output 16^{th} order linear system, $S_o = S(A,B,C)$, derived from linearisation of a flow-controlled counterflow heat exchanger with normalised variables.

The matrices A,B,C have a simple parametric form as follows:

$$A = \begin{bmatrix} X & Y \\ \frac{1}{2}Y^T & \frac{1}{2}X^T \end{bmatrix}$$, where the 8 x 8 matraices $X = (x_{ij})$, $Y = (y_{ij})$

are defined by

$$x_{ij} = \begin{cases} -59.0 & \text{if } i = j \\ 48.0 & \text{if } i = j+1 \\ 0.0 & \text{elsewhere} \end{cases}, \quad y_{ij} = \begin{cases} 11.0 \text{ if } j = i+1 \\ 0.0 \text{ elsewhere} \end{cases};$$

$B = (b_{ij})$ is defined by

$$b_{j1} = \begin{cases} 11.0 & 1 \leq j \leq 8 \\ 0.0 & 9 \leq j \leq 16 \end{cases}, \quad b_{j2} = \begin{cases} 0.0 & 1 \leq j \leq 8 \\ -11.0 & 9 \leq j \leq 16 \end{cases}$$

and $C = (c_{ij})$ is a matrix of zeros except for $c_{18} = c_{29} = 1$.

Applying the state-space decomposition of Section 2, it is found that each successive loop has uniform rank 1, so that the system can be decomposed into eight loops, each having a state-space representation $S(\hat{A}_i,\hat{B}_i,\hat{C}_i)$ of state dimension 2. Choosing bases as described in

Section 4 so that $\hat{C}_i = I_2$ $(1 \leq i \leq 8)$, the matrices $\hat{A}_i, \hat{B}_i$ for the first 4 loops of the decomposed system are found to be

$$\hat{A}_1 = \begin{bmatrix} -11 & 0 \\ 0 & -5.5 \end{bmatrix} ; \quad \hat{B}_1 = \begin{bmatrix} 11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$\hat{A}_2 = \begin{bmatrix} -83 & -22 \\ -2.75 & -77.5 \end{bmatrix} , \quad \hat{B}_2 = \begin{bmatrix} 0 & 48 \\ -12 & 0 \end{bmatrix}$$

$$\hat{A}_3 = \begin{bmatrix} -11 & -11 \\ -5.5 & -5.5 \end{bmatrix} , \quad \hat{B}_3 = \begin{bmatrix} 5.5 & -22 \\ -2.75 & -22 \end{bmatrix}$$

$$\hat{A}_4 = \begin{bmatrix} -84.22 & -24.44 \\ -2.139 & -76.28 \end{bmatrix} . \quad \hat{B}_4 = \begin{bmatrix} -10.67 & 42.67 \\ -6.67 & 2.67 \end{bmatrix}$$

Table 1 shows the outputs $y_i^1(t) = \begin{bmatrix} y_{1i}^1(t) \\ y_{2i}^1(t) \end{bmatrix}$, $1 \leq i \leq 4$ (i.e. the outputs of the first 4 loops of the decomposed system for a unit step input to channel 1, as defined in section 3.4), and Table 2 displays the corresponding outputs $y_i^2(t)$, $1 \leq i \leq 4$, for a unit step into channel 2. It can be seen that: (i) $y_{11}^1(t) >> y_{13}^1(t)$ and $y_{21}^2(t) >> y_{23}^2(t)$ for all times $t > o$; and (ii) $y_{12}^1(t)$, $y_{22}^1(t)$, $y_{22}^2(t)$ are small for all $t > o$. Observation (i) suggests that loops j, for $j > 2$, may be having only a small effect on the system input-output behaviour, or at least on the direct input-output transmission in each channel. Retaining the first two loops and taking the constant steady-state approximation $H_2^{(a)}(s) = H_2(o)$ to $H_2(s)$, as described in Section 3.5, gives the 4^{th}-order reduced model $S_1 = S(A^{(a_1)}, B^{(a_1)}, C^{(a_1)})$ where

$$A^{(a_1)} = \begin{bmatrix} -11 & 0 & 11 & 0 \\ 0 & -5.5 & 0 & -11 \\ 0 & 48 & -116.8 & 57.66 \\ -12 & 0 & 7.207 & -58.41 \end{bmatrix} , \quad B^{(a_1)} = \begin{bmatrix} 11 & 0 \\ 0 & -11 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} ; \quad C^{(a_1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The open-loop step responses of the original system, S_o, and the reduced system, S_1, are compared in Figures 3 and 4 (for step inputs to channel 1 and channel 2 respectively) and it is seen that the reduced model is a good open-loop representation of the original system. Figures 5 and 6 compare the closed-loop step-responses of S_o and S_1, under the action of the forward path pre-compensator $K \triangleq \begin{bmatrix} \frac{1}{11} & 0 \\ 0 & -\frac{1}{11} \end{bmatrix}$ (chosen to retain the low level of interaction of high frequencies), diagonal loop gain matrix $p\, I_2 (p = 10)$, and unity negative feedback. It is clear that the reduced model is an adequate system representation for feedback design. The reduced system S_1 has zeros at -123.2, -52.0, and poles at -124.7, -47.4, -15.1, -4.5. The root-loci of S_o and S_1 are illustrated in Figures 7 and 8 respectively. Although of apparently different form, the tabular comparison of Table 3 verifies that the unbounded (dominant) poles are virtually indentical at high gains $p \to +\infty$ (see Section 3.2).

Finally, observation (ii), and the closeness of the zeros to two of the poles in S_1, suggest that there may still be some redundancy in this model, and that a simple 'first-order-like'[6] approximation to the original system may be adequate. Taking $H_1^{(a)}(s) = H_1(o)$ gives the second-order system, $S_2 = S(A^{(a_2)}, B^{(a_2)}, C^{(a_2)})$, where

$$A^{(a_2)} = \begin{bmatrix} -12.19 & 4.812 \\ 2.406 & -6.094 \end{bmatrix} \quad ; \; B^{(a_2)} = \begin{bmatrix} 11 & 0 \\ 0 & -11 \end{bmatrix} \quad ; \; C^{(a_2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has poles at -13.7, -4.6. The open- and closed-loop step responses of S_2 are almost identical to those of S_1, and are illustrated in Figures 9 - 12. The root locus of S_2 consists simply of two branches going to $-\infty$ along the negative real axis, and Table 3 illustrates that the unbounded closed-loop poles of S_2 and S_o (with forward path precompensator K as defined above) are again almost identical at high gains.

6. Conclusions

Using simple algebraic and geometric techniques, previously published results on the reduction of single-input/single-output systems have been generalised to the case of square invertible multi-input/multi-output linear time-invariant systems. It has been shown that such systems admit a natural decomposition into the form of a forward path system, having no invariant zeros, and a single feedback path, with strictly proper transfer function matrix, characterising the invariant zeros of the original system. Under certain generically satisfied conditions, repeated application of this decomposition allows the system to be represented as a nested sequence of feedback loops, to which there corresponds an equivalent state-space representation in block canonical form.

This multi-loop decomposition provides a convenient technique for obtaining reduced order models which has strong intuitive appeal, in that those loops which appear to be of least significance in terms of the system input/output behaviour can be simply neglected, or replaced by some simple (e.g. constant) approximation. The significance of each loop can be assessed by inspection of simulated time-responses of the system in decomposed canonical form, and moreover this will often suggest a choice of reduced model order. The most important feature of the proposed reduction technique is that by retention of the first loop of the decomposed system the reduced-order models obtained are guaranteed to preserve the asymptotic unbounded root-locus properties of the original system and can give a good representation of the dominant system zeros. They may therefore be used with confidence as the basis for high-gain closed-loop controller design. The method is given added flexibility by the freedom to retain dominant zeros explicitly in the reduced-order models, and the ability to match an arbitrary number of moments about $|s| = \infty$ and $s = o$ by suitable application and appropriate choice of approximation to the lower loops of the decomposition. As illustrated in the example of Section 5, the techniques presented in this paper offer a methodical approach to model reduction, applicable to 'almost all' square multivariable systems, and can be used to generate models which give a good representation of both the open-loop and closed-loop properties of the system.

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8. Appendices

8.1 Appendix 1

The state-space representation $S(A,B,C)$ of $G(s)$ defined by equation (6) is minimal.

Proof: (i) Observability

Suppose that

$$\begin{pmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; ; \; \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad (35)$$

Then

$$A_1x_1 + B_1C_2x_2 = \lambda x_1 \qquad (36)$$

$$A_2x_2 + B_2C_1x_2 = \lambda x_2 \qquad (37)$$

$$C_1x_1 = 0 \qquad (38)$$

(36) implies $A_1x_1 = \lambda x_1 - B_1(C_2x_1)$, and hence by (38) it follow that $x_1 = B_1(C_2x_2) = 0$, as $S(A_1,B_1,C_1)$ has no zeros. Therefore $C_2x_2 = 0$, as B_1 has full rank (by invertibility of $G_1(s)$). But (37) and (38) imply that $A_2x_2 = \lambda x_2$. Hence $x_2 = 0$ (by observability of $S(A_2,B_2,C_2)$). Thus $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, and $S(A,B,C)$ is observable.

(ii) Controllability

The observability of the dual system $S(A^T,C^T,B^T)$ can be proved as in (i) above, from which the controllability of $S(A,B,C)$ follows.

8.2 Appendix 2. Proof of Theorem 1

As $S(A,B,C)$ is invertible, it follows from Morse[4] that there exists a suitable choice of basis for $\{\tau,\nu^*\}$ and matrices F,K of dimension $m\times n$, $n\times m$ respectively such that the transformed system $S(\hat{A},\hat{B},\hat{C})$ takes the form

$$\hat{A} + \hat{B}F + K\hat{C} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} , \; \hat{B} = \begin{pmatrix} \hat{B}_1 \\ 0 \end{pmatrix} , \; \hat{C} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix} \qquad (39)$$

where A_1, A_2 are the restrictions of $\hat{A} + \hat{B}F + \hat{K}C$ to τ, ν^* respectively and $\hat{B}_1, \hat{C}_1$ are some matrices of dimension $n_1 xm$, mxn_1 respectively. Partitioning $F = [F_1, F_2]$, $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ gives

$$\hat{A} = \begin{pmatrix} A_1 - \hat{B}_1 F_1 - K_1 \hat{C}_1 & - \hat{B}_1 F_2 \\ - K_2 \hat{C}_1 & A_2 \end{pmatrix} \tag{40}$$

and setting $\hat{A}_1 \triangleq A_1 - \hat{B}_1 F_1 - K_1 \hat{C}_1, C_2 \triangleq - F_2, B_2 \triangleq - K_2$ gives the required form (9). Furthermore it is easily seen that transformations of the bases for τ and ν^* leave the form of equations (9) unchanged.

The geometric criterion for the invertibility of S(A,B,C) is that[7] $\nu^* \cap R(B) = \{o\}$ and dim R(B) = m. The nature of the state-space decomposition implies that the maximal $(\hat{A}_1, \hat{B}_1)$ - invariant subspace in the kernel of $\hat{C}_1$ is zero, so that $S(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ has no zeros and is invertible (as dim $R(\hat{B}_1)$ = dim R(B) = m). Again, as $S(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ has no zeros it has no uncontrollable or unobservable modes. Controllability (resp. observability) of S(A,B,C) obviously implies the corresponding property in $S(A_2, B_2, C_2)$, and the proof of the reverse implication follows the proof used (in Appendix 1) to establish the minimality of the state-space system (6).

8.3 Appendix 3. Algorithm to Generate a Basis Matrix for τ.

For an mxm invertible system S(A,B,C) the following algorithm may be used to generate a basis matrix for the subspace τ defined in Section 2.2.

(i) Define $B_1 = B$. Set i = 1

(ii) If $p_i \triangleq \text{rank } C B_i = m$, stop.

If $p_i < m$, choose an mxm non-singular matrix T_i such that

$$(C B_i) T_i = [E_i \quad 0] \tag{41}$$

where E_i has p_i columns (and hence full rank p_i).

(iii) Define

$$[\overline{B}_i\ \tilde{B}_i] \triangleq B_i T_i \tag{42}$$

where $\overline{B}_i$ has p_i columns and $\tilde{B}_i$ has $m - p_i$ columns

(iv) Define

$$B_{i+1} \triangleq [\overline{B}_i\ A\tilde{B}_i] \tag{43}$$

(v) Set $i = i+1$. Go to (ii)

Theorem 2:

For an mxm invertible system $S(A,B,C)$, of state dimension n, the above algorithm terminates for some integer $i = i_o \leqslant n$ (i.e. $p_{i_o} = m$ for some $i_o \leqslant n$), and then the matrix

$$[\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{i_o - 1}, B_{i_o}] \tag{44}$$

has full rank and its columns span τ, i.e. it is a basis matrix for τ (Note; This algorithm is effectively the dual of an algorithm presented by Silverman[13] for constructing an inverse of a linear multivariable system). To prove the theorem the following two results are required.

Lemma 1: (c.f. Morse[4])

The sequence of subspaces τ_i $(o \leqslant i \leqslant n)$ defined by

$$\tau_o \triangleq \{o\},\ \tau_i \triangleq A(N \cap \tau_{i-1}) + R(B)\ \ (1 \leqslant i \leqslant n) \tag{45}$$

satisfies $\tau_i \subset \tau_{i+1}$ $(o \leqslant i \leqslant n-1)$ and converges for some integer $i_o \leqslant n$, and then $\tau = \tau_{i_o} = \tau_n$ (where $N = \operatorname{Ker} C$)

Lemma 2: With the notation of Lemma 1, if $S(A,B,C)$ is invertible and Γ_i is a subspace satisfying

$$\tau_i = (\tau_{i-1} \cap N) \oplus \Gamma_i \tag{46}$$

for some $1 \leqslant i \leqslant n$, then

$$\tau_{i+1} = A(\Gamma_i \cap N) \oplus \tau_i \tag{47}$$

Proof From equation (45)

$$\tau_{i+1} = A(\tau_i \cap N) + R(B)$$

$$= A\left[(\Gamma_i \oplus (\tau_{i-1} \cap N)) \cap N\right] + R(B) \text{ (by assumption)}$$

$$= A\left[(\Gamma_i \cap N) \oplus (\tau_{i-1} \cap N)\right] + R(B)$$

$$= A(\Gamma_i \cap N) + A(\tau_{i-1} \cap N) + R(B)$$

$$= A(\Gamma_i \cap N) + \tau_i \qquad \text{(by (45))} \qquad (48)$$

To show that this sum is direct, suppose that $x \varepsilon A(\Gamma_i \cap N) \cap \tau_i$. Then (by (46)) $x = Ag = At + b$ for some $g \varepsilon \Gamma_i \cap N$, $t \varepsilon \tau_{i-1} \cap N$, $b \varepsilon R(B)$, so that $A(g-t) = b \varepsilon R(B)$, and $g-t \varepsilon N$. Hence, $g - t \varepsilon \nu^*$ Also, $g - t \varepsilon \Gamma_i + \tau_{i-1} \subset \tau$. But system invertibility implies that $\nu^* \cap \tau = \{o\}$. Hence $g - t = o$, so that $g = t \varepsilon \tau_{i-1} \cap N \cap \Gamma_i = \{o\}$ (by assumption), giving $x = Ag = o$. Thus the sum (48) is direct.

Proof of Theorem 2.

Note that equations (41) and (42) imply that $\overline{B}_i$ has full rank p_i, and that $R(\overline{B}_i) \cap R(\tilde{B}_i) = 0$. (For if $\overline{B}_i x = \tilde{B}_i y$, then $C\overline{B}_i x = C\tilde{B}_i y = 0$, and hence $x = 0$ as $C\overline{B}_i = E_i$ which has full rank) We first show that if $p_j < m$, for $o \leqslant j < i$, then

$$\tau_i = R(\tilde{B}_1) \oplus R(\tilde{B}_2) \oplus \ldots \oplus R(\tilde{B}_{i-1}) \oplus R(B_i) \qquad (49)$$

For i=1, $R(B_i) = R(B) = \tau_1$ (by definition), so that (49) holds for i=1. Next suppose that (49) holds for i = r. Then if $p_r < m$, noting that $R(\tilde{B}_j) \subset N$, $N \cap R(\overline{B}_j) = 0$ $(1 \leqslant j \leqslant r)$, it follows (by assumption) that

$$\tau_j = R(\tilde{B}_1) \oplus \ldots \oplus R(\tilde{B}_{j-1}) \oplus R(B_j) \; (1 \leqslant j \leqslant r)$$

$$= R(\tilde{B}_1) \oplus \ldots \oplus R(\tilde{B}_{j-1}) \oplus (R(\tilde{B}_j) \oplus R(\overline{B}_j)) \quad (1 \leqslant j \leqslant r) \qquad (50)$$

and $$\tau_j \cap N = R(\tilde{B}_1) \oplus \ldots \oplus R(\tilde{B}_{j-1}) \oplus R(\tilde{B}_j) \qquad (1 \leqslant j \leqslant r) \qquad (51)$$

Hence $$\tau_r = R(\tilde{B}_1) \oplus \ldots \oplus R(\tilde{B}_r) \oplus R(\overline{B}_r)$$

$$= (\tau_{r-1} \cap N) \oplus (R(\tilde{B}_r) \oplus R(\overline{B}_r))$$

and so by Lemma 2,

$$\tau_{r+1} = \tau_r \oplus A((R(\tilde{B}_r) \oplus R(\overline{B}_r)) \cap N)$$

$$= \tau_r \oplus A\ R(\tilde{B}_r).$$

$$= R(\tilde{B}_1) \oplus \ldots \oplus R(\tilde{B}_r) \oplus R(\overline{B}_r) \oplus A\ R(\tilde{B}_r)$$

$$= R(\tilde{B}_1) \oplus \ldots \oplus R(\tilde{B}_r) \oplus R\ (B_{r+1}) \text{ (by (43))}$$

which by induction proves the given assertion.

We now consider conditions under which the given algorithm terminates. As $S(A,B,C)$ is invertible, $R^n = \nu^* + \tau$, and hence $N = N \cap R^n = \nu^* \oplus (\tau \cap N)$. Thus if X is any subspace satisfying $(\tau \cap N) \oplus X = \tau$, it follows that

$$R^n = \nu^* \oplus (\tau \cap N) \oplus X = N \oplus X$$

so that

$$\dim X = n\text{-}\dim N = n-(n-m) = m. \tag{52}$$

Again suppose that $p_j < m$ for $1 \leqslant j < i$. Then by equations (50) and (51) we have

$$\tau_j = (\tau_j \cap N) \oplus R(\overline{B}_j)$$

But $\dim R(\overline{B}_j) = p_j < m$, so that $\tau_j \neq \tau$, by (52).

If $\tau_i = \tau$, then equations (49) and (51) imply that

$$\tau = \tau_i = (\tau_{i-1} \cap N) \oplus R(B_i).$$

But $R(B_i) \leqslant m$, and hence by (52) it follows that $R(B_i) \cap N = \{o\}$, and $p = \dim R(B_i) = m$.

Conversely, if $p_j < m$, $1 \leqslant j < i$, and $p_i = m$, then $R(B_i) \cap N = \{o\}$ by assumption, and as $\tau_i = (\tau_{i-1} \cap N) \oplus R(B_i)$, Lemma 2 implies that

$$\tau_{i+1} = \tau_i \oplus A(R(B_i) \cap N) = \tau_i$$

Thus the given algorithm coverges precisely for $i = i_o$, where i_o is the smallest integer satisfying $\tau_{i_o} = \tau_{i_o+1} = \tau$.

It remains to show that the matrix (44) has full rank, as equations (49) implies that its columns span τ. But, for $1 \leq j < i_o$, equations (42) and (43) imply that

$$\begin{aligned} \text{rank}\ (B_{j+1}) &= \text{rank}\ [\overline{B}_j, A\,\tilde{B}_j] \\ &\leq \text{rank}\ (\overline{B}_j) + \text{rank}\ (A\,\tilde{B}_j) \\ &\leq \text{rank}\ (\overline{B}_j) + \text{rank}\ (\tilde{B}_j) \\ &= \text{rank}\ [\overline{B}_j\ \tilde{B}_j] \quad (\text{as } R(\overline{B}_j) \cap R(\tilde{B}_j) = \{o\}) \\ &= \text{rank}\ B_j. \end{aligned}$$

Thus $m = \text{rank}\ (B_1) \geq \text{rank}\ (B_2) \geq \ldots \geq \text{rank}\ (B_{i_o}) = m$, so that B_j has full rank m, $1 \leq j \leq i_o$, and in particular $\tilde{B}_j$, $1 \leq j < i_o$, and B_{i_o} have full rank. Hence by equation (49), the matrix (44) is full rank and is a basis matrix for τ as required.

Time	$y^1_{11}(t)$	$y^1_{21}(t)$	$y^1_{12}(t)$	$y^1_{22}(t)$	$y^1_{13}(t)$	$y^1_{23}(t)$	$y^1_{14}(t)$	$y^1_{24}(t)$
0.1	.683	.049	.035	-.110	.098	-.042	.518	.124
0.2	.904	.159	.007	-.185	.329	-.229	.409	.158
0.3	.981	.265	.030	-.214	.407	-.306	.100	.198
0.4	1.02	.337	.049	-.218	.371	-.349	-.054	.227
0.5	1.04	.377	.059	-.215	.329	-.373	-.107	.240
0.6	1.06	.399	.064	-.213	.304	-.385	-.128	.245
0.7	1.06	.410	.066	-.212	.292	-.392	-.138	.248
0.8	1.07	.416	.068	-.212	.285	-.395	-.144	.250
0.9	1.07	.419	.068	-.211	.282	-.397	-.147	.251
1.0	1.07	.420	.069	-.211	.280	-.398	-.148	.251

Table 1: Loop outputs of decomposed system for step input into channel 1.

Time	$y^2_{11}(t)$	$y^2_{21}(t)$	$y^2_{12}(t)$	$y^2_{22}(t)$	$y^2_{13}(t)$	$y^2_{23}(t)$	$y^2_{14}(t)$	$y^2_{24}(t)$
0.1	-.169	-.861	-.434	.047	-.104	.057	.029	-.054
0.2	-.449	-1.41	-.665	.106	-.105	.286	.707	-.269
0.3	-.632	-1.75	-.755	.103	.151	.484	1.32	-.400
0.4	-.733	-1.94	-.798	.087	.363	.592	1.53	-.452
0.5	-.786	-2.03	-.820	.078	.474	.648	1.61	-.476
0.6	-.813	-2.08	-.832	.074	.530	.677	1.66	-.488
0.7	-.828	-2.11	-.838	.072	.558	.693	1.68	-.495
0.8	-.836	-2.12	-.841	.071	.574	.701	1.70	-.499
0.9	-.840	-2.13	-.843	.070	.582	.705	1.70	-.501
1.0	-.842	-2.13	-.843	.070	.586	.708	1.71	-.502

Table 2: Loop-outputs of decomposed system for step input into channel 2.

Gain	S_o	S_1	S_2
100	-117.4 + 14.7j -117.4 - 14.7j	-102.8 + 5.5j -102.8 -5.5j	-113.7 -104.6
200	-204.7 -211.5	-204.9 -212.3	-204.6 -213.7
500	-505.4 -511.1	-505.4 -511.1	-504.6 -513.7
1000	-1011 -1005	-1011 -1005	-1014 -1005

Table 3: Closed-loop poles of S_o, S_1, S_2 at high gains, with precompensator K.

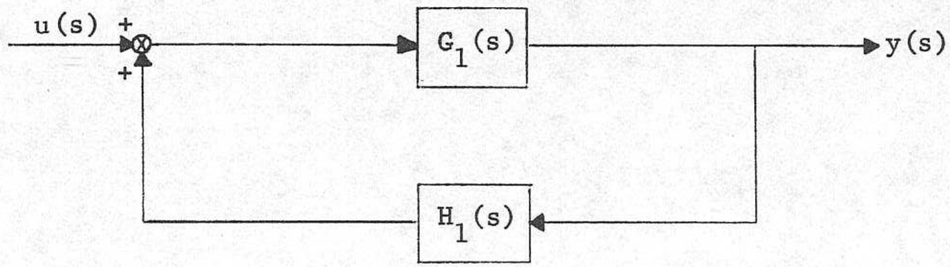

Figure 1

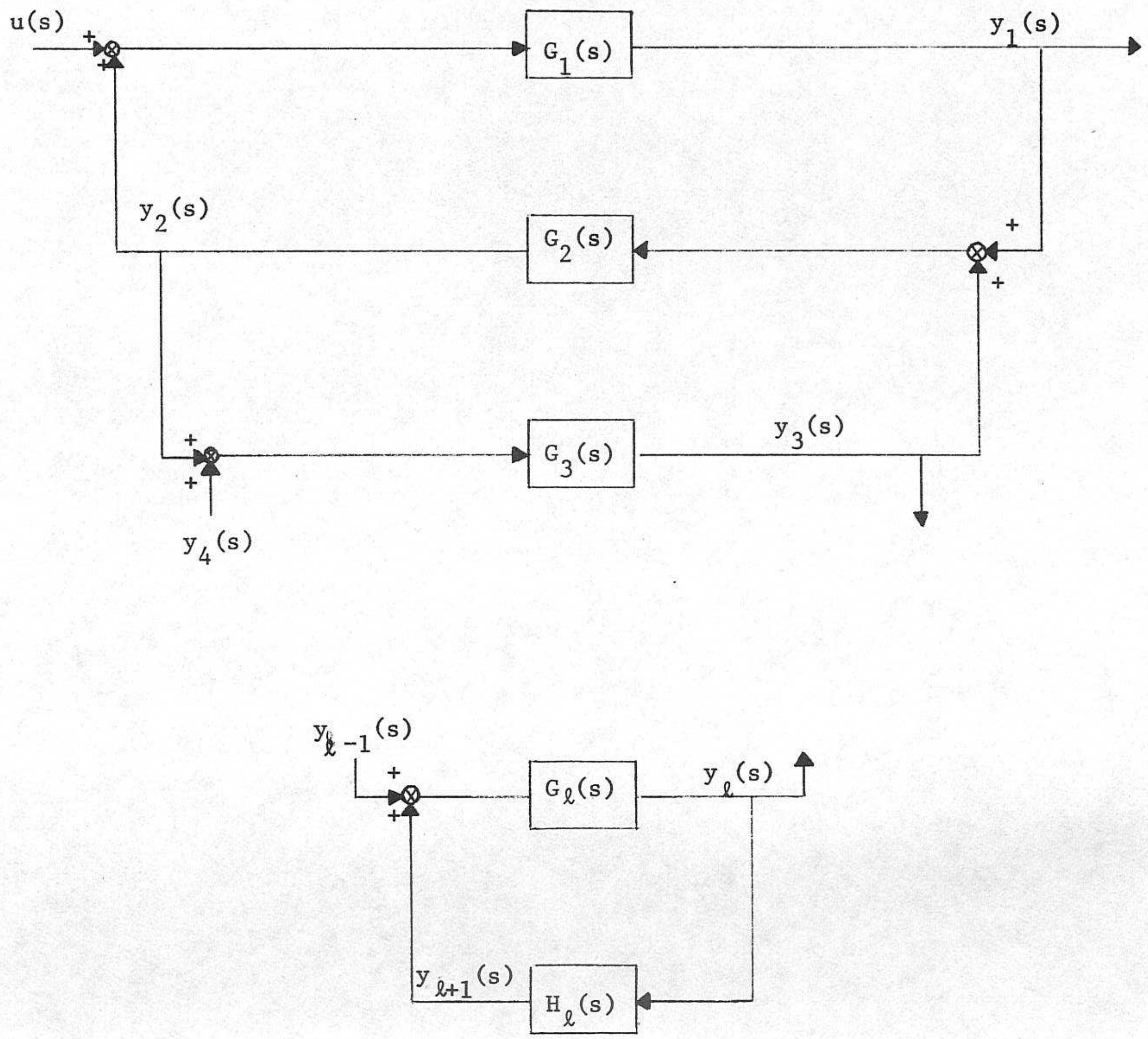

Figure 2

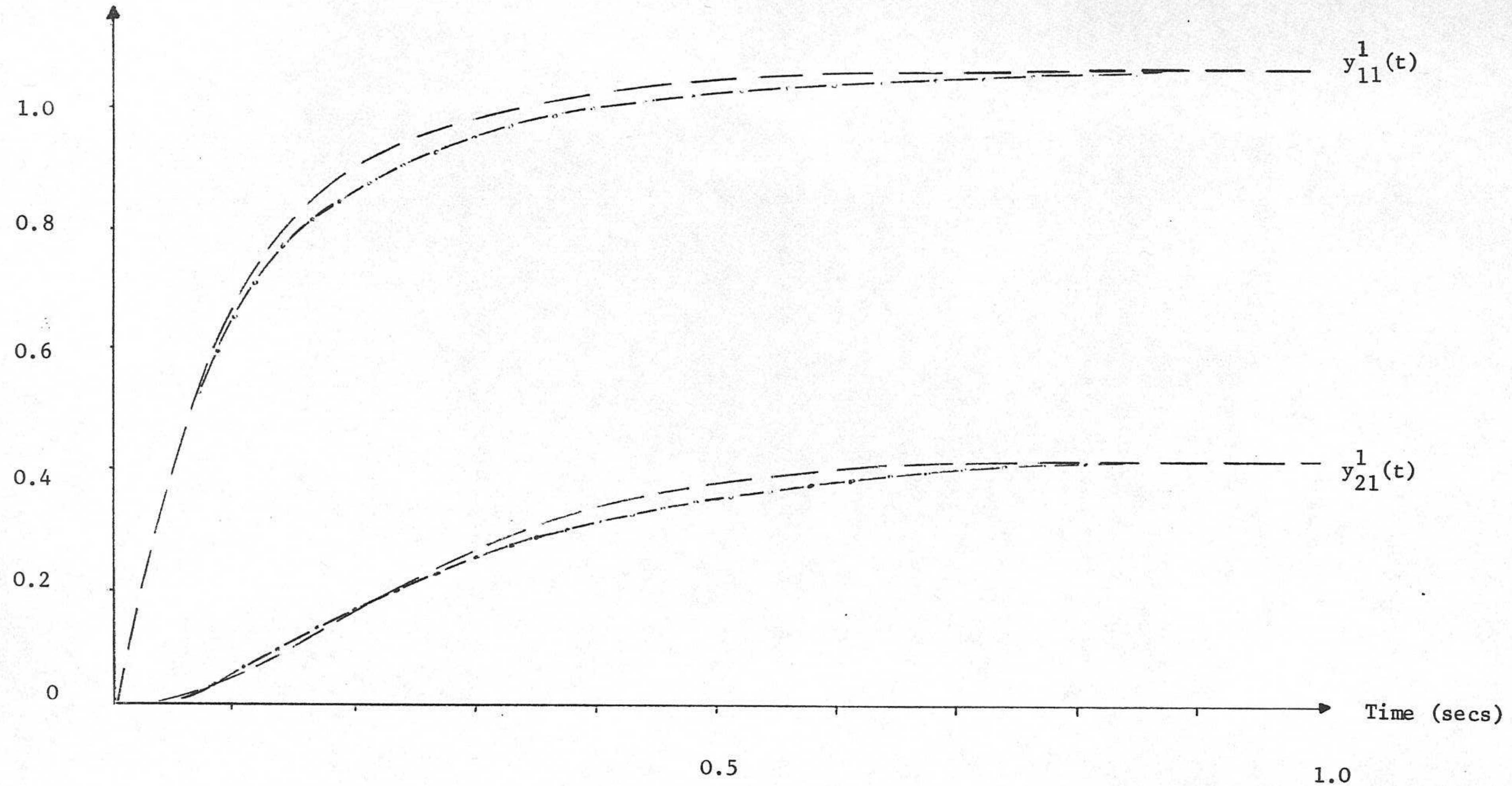

Figure 3 Open loop response of S_o and S_1 to a step change in channel 1

- - - - Response of S_o

-·-·-· Response of S_1

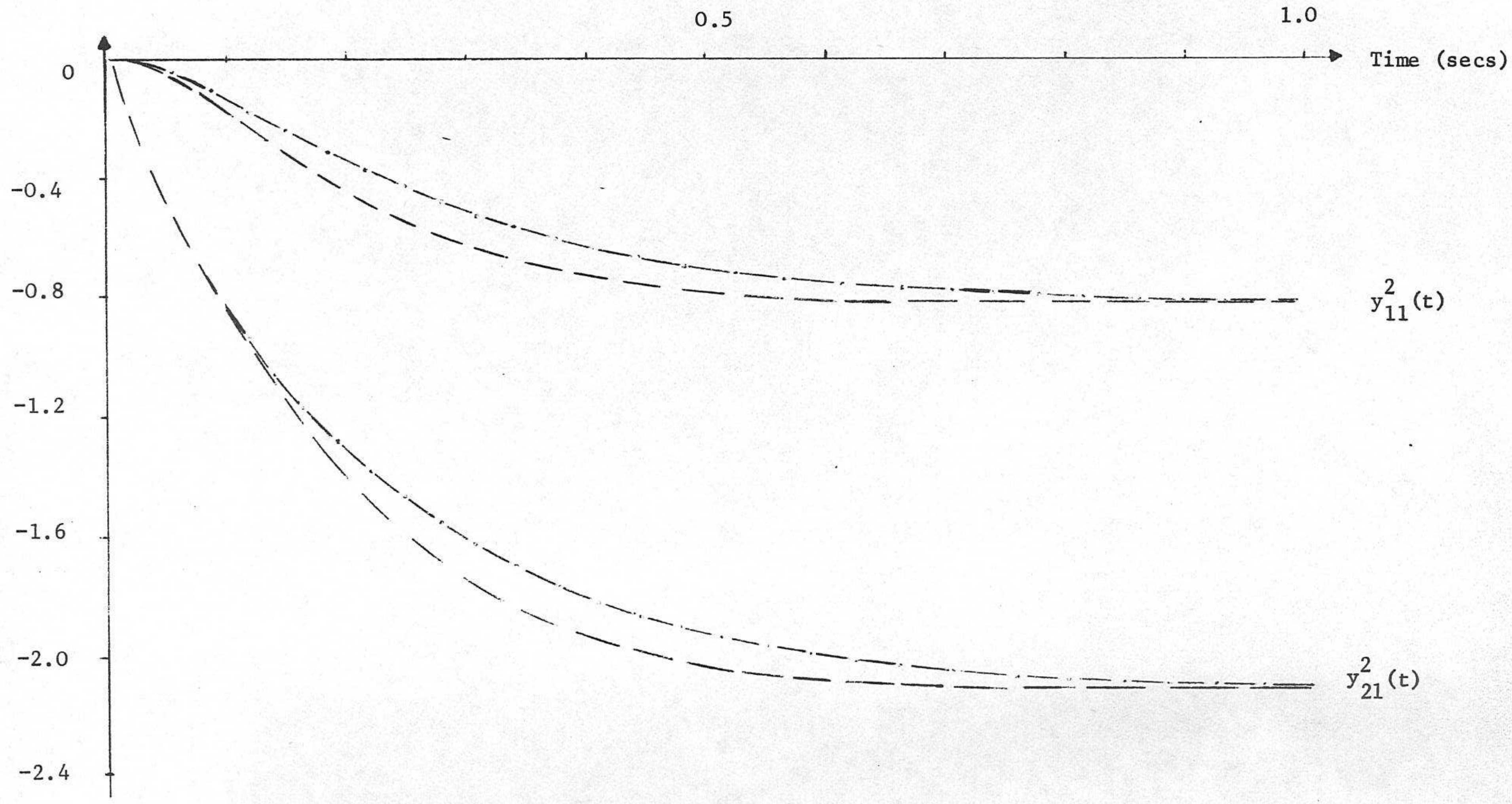

Figure 4 Open-loop response of S_o and S_1 to a step change in channel 2

------Response of S_o -·-·-·- Response of S_1

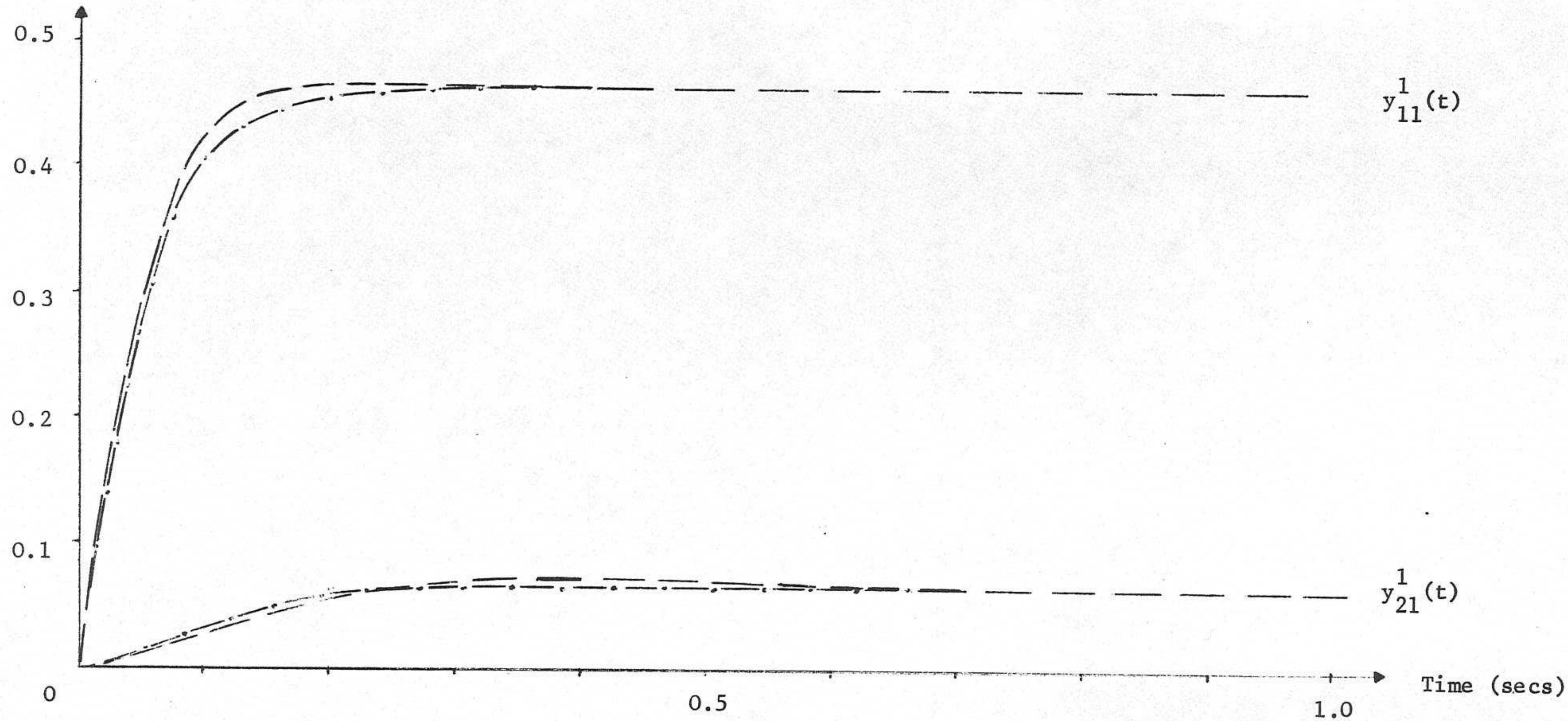

Figure 5 Closed-loop response of S_o and S_1 to a step change in channel 1

------- Response of S_o

-.-.-.- Response of S_1

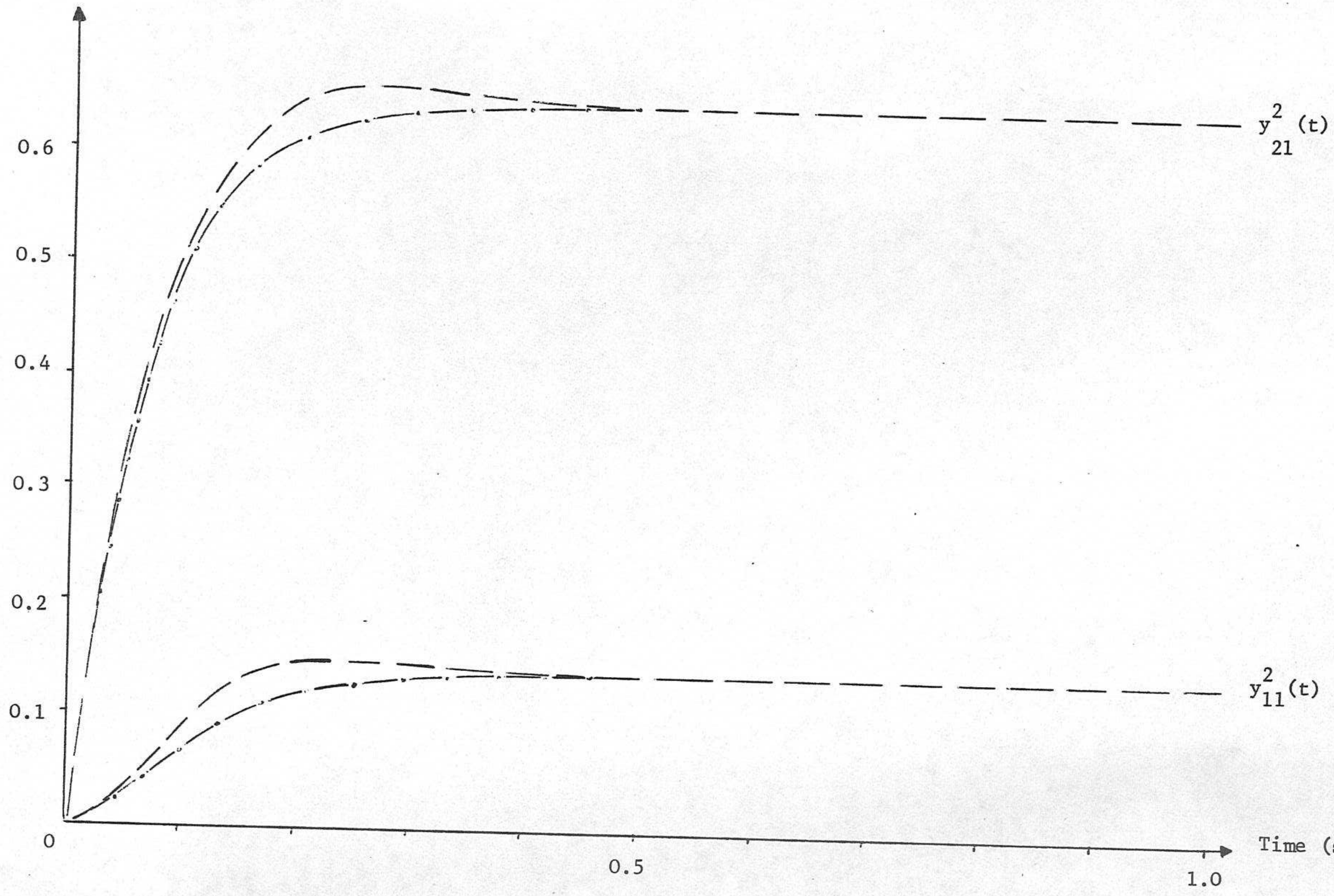

Figure 6 Closed-loop response of S_o and S_1 to a step change in channel 2

------ Response of S_o -·-·-·- Response of S_1

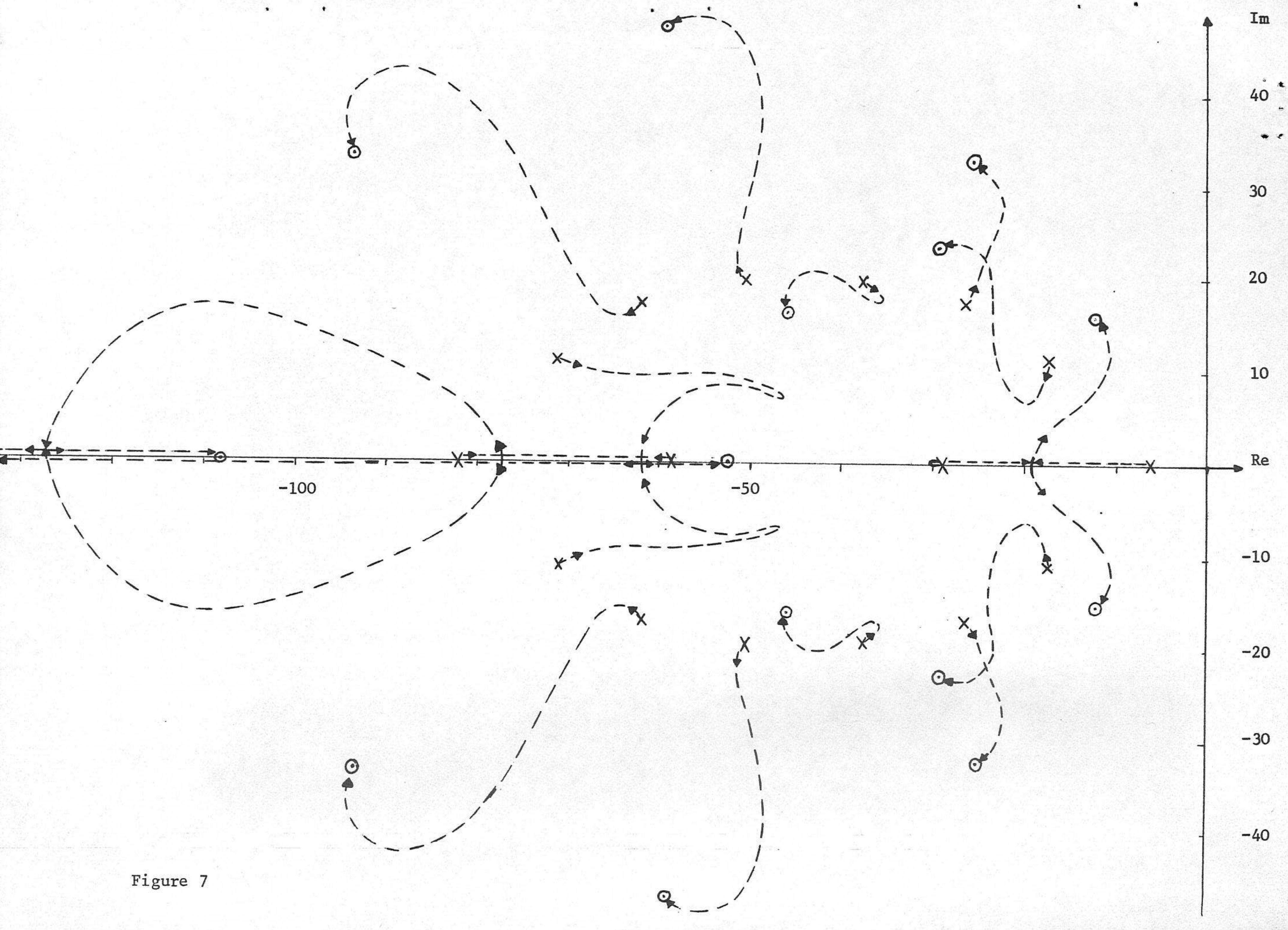

Figure 7

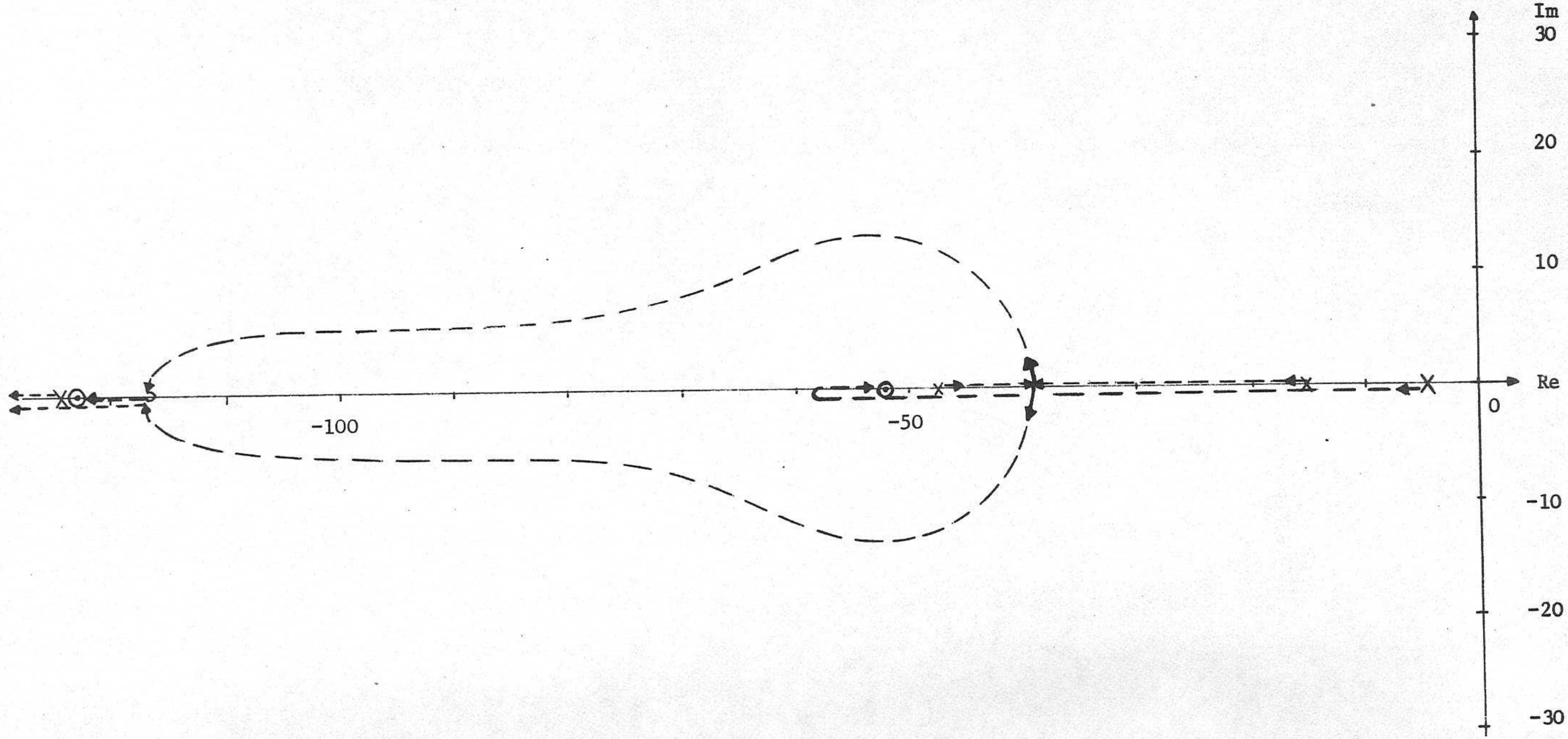

Figure 8

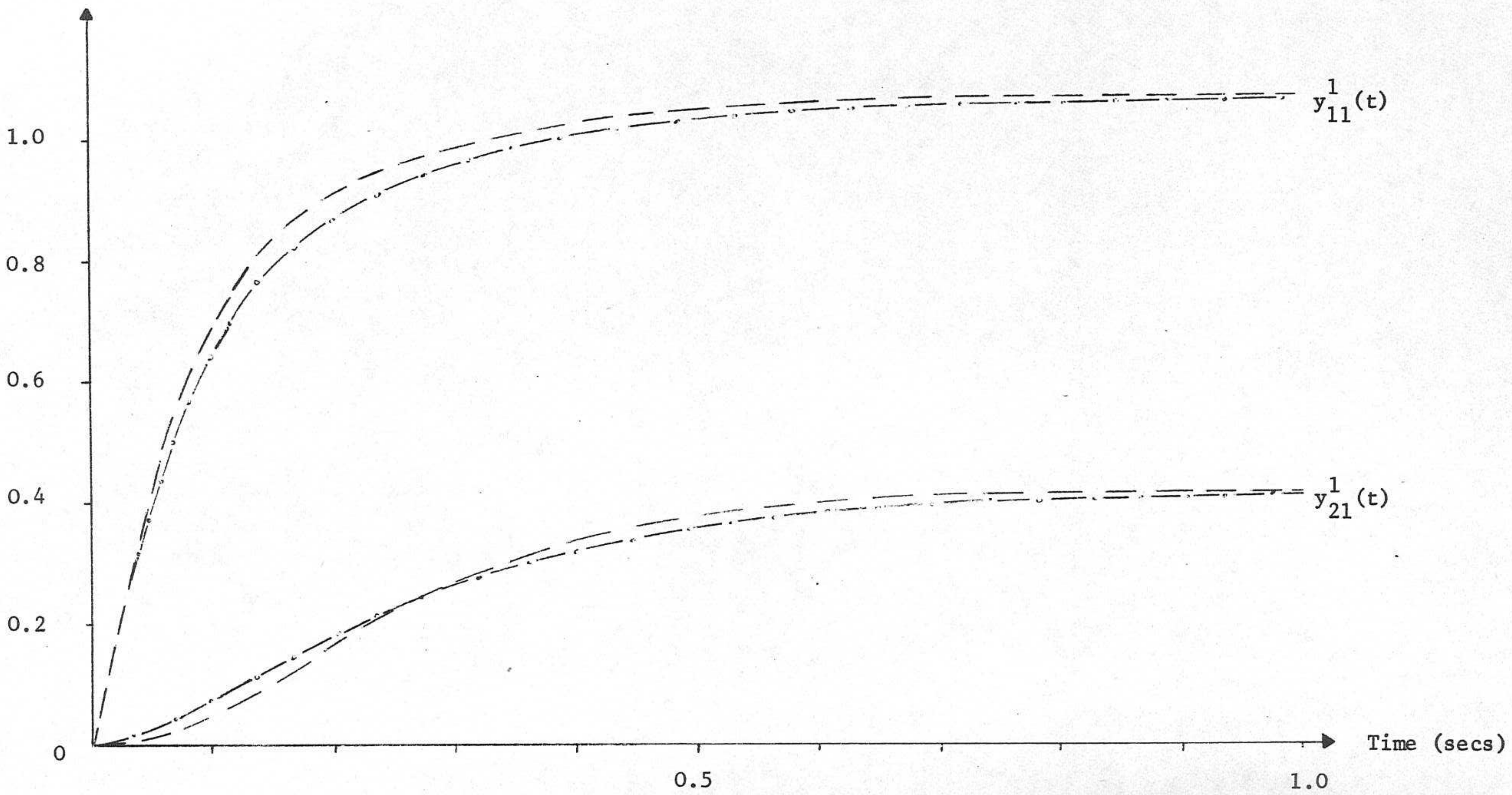

Figure 9 Open-loop response of S_o and S_2 to a step change in channel 1

------ Response of S_o

-·-·-·- Response of S_2

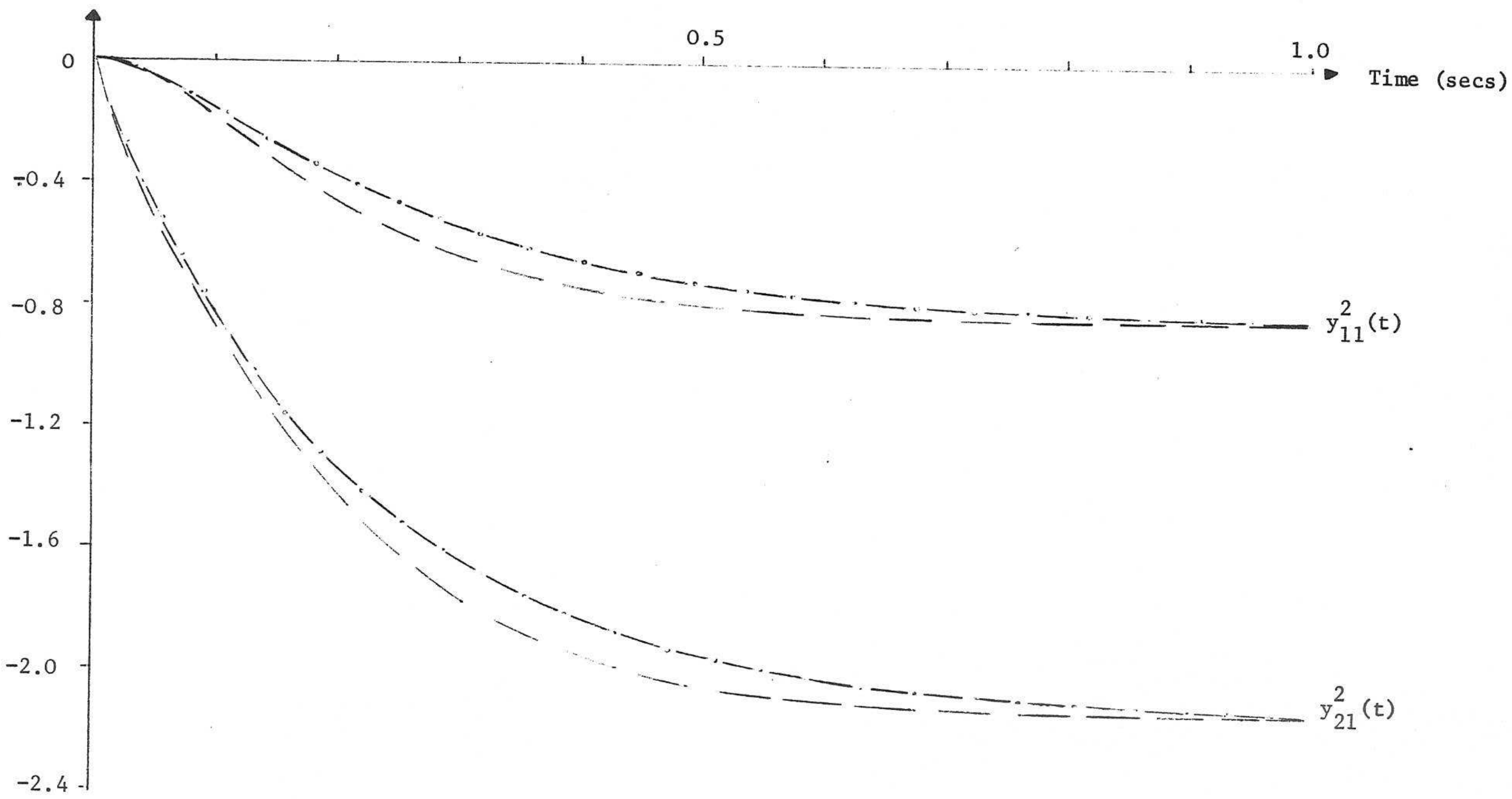

Figure 10 Open-loop response of S_Φ and S_2 to a step change in channel 2

------ Response of S_o -·-·- Response of S_2

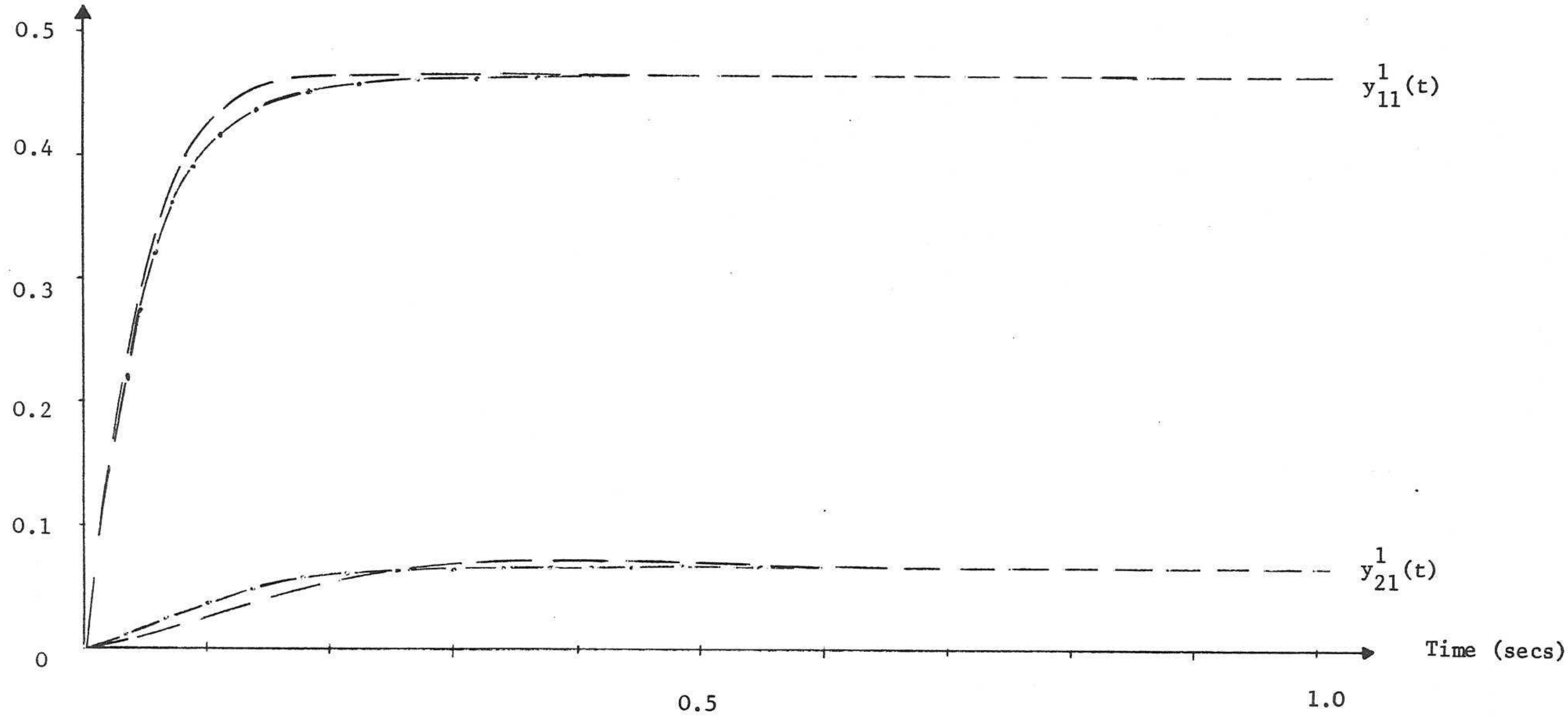

Figure 11 Closed-loop response of S_o and S_2 to a step change in channel 1

--------- Response of S_o

-·-·-·-·- Response of S_2

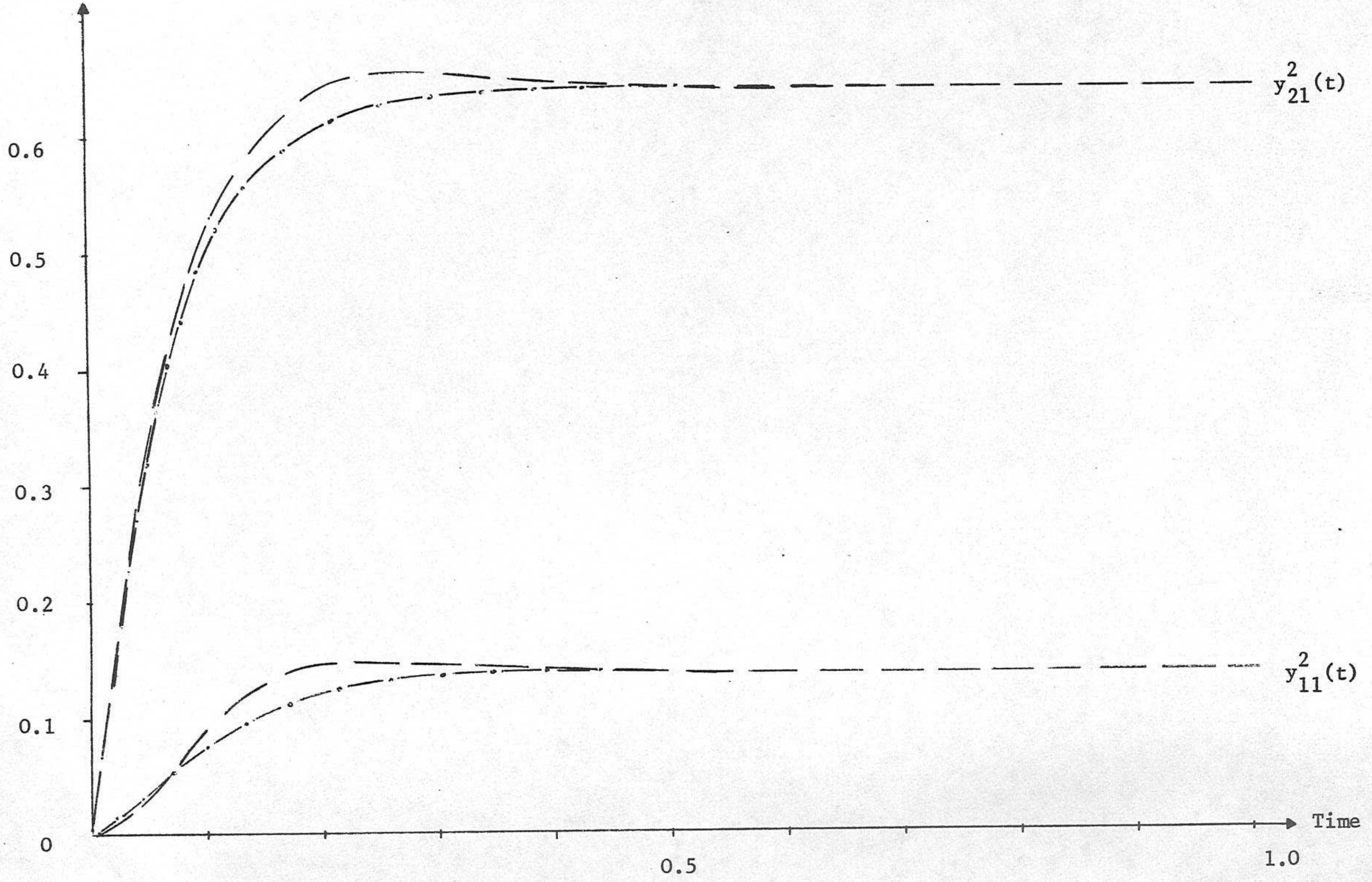

Figure 12 Closed loop response of S_o and S_2 to a step change in channel 2

------- Response of S_o -·-·-·- Response of S_2