ON THE ABSOLUTE STABILITY PROBLEM FOR NONLINEAR FEEDBACK SYSTEMS

by

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Research Report No. 103

November 1979

Abstract

The paper considers the use of a pole/zero canonical form in the investigation of the absolute stability problem for nonlinear feedback systems. The canonical form is easily computed and has an interpretation in terms of a nested system feedback structure.

1. Introduction

The derivation of stability criterion for the nonlinear multivariable feedback system illustrated schematically in Fig. 1 is an important theoretical and practical problem. A particularly important problem is that of absolute stability (Narendra and Taylor 1973) and the derivation of sector-type conditions on the nonlinear characteristic N_o that guarantee the global asymptotic stability of the system. Several absolute stability criteria are available and can be derived using functional analytic methods (Cook 1979b), Lyapunov methods and frequency domain conditions based on positive realness concepts (Narendra and Taylor 1973). It can however be difficult to check such conditions, particularly in the case of large scale systems. In such cases it can be very useful to have easily checkable sufficient conditions.

It is anticipated that stability conditions will be most easily expressed in terms of a suitable system structural model and, as the absolute stability problem is intuitively related to the root-locus of the linear part of the system, it is anticipated that the structure should reflect the system pole-zero configuration. Cook (1979a) has examined stability using modal/pole type concepts. It is expected however that the zero structure must play a fundamental role and hence that the geometric definition and construction of system zeros (Owens, 1977) will be an important theoretical tool.

This paper presents the results of an investigation into pole-zero effects on absolute stability for scalar and multivariable systems possessing a certain symmetric nested feedback structure. In such cases, stability conditions can be easily deduced by operations equivalent to sequential inversion operations on the linear system transfer function matrix. The approach taken is based on the use of quadratic Lyapunov functions and a careful choice of basis in the system state space. The elementary general stability criteria are described in section 2. The simplifying effect of the nested feedback structure is described in section 3.

2. General Stability Criteria

The foundation of the results presented are obtained by considering the differential system

$$\dot{x}(t) = A_1 x(t) + f(x(t),t) \quad , \quad x(t) \in R^n \qquad \ldots(1)$$

where A_1 is a constant, real nxn matrix and f is a real vector function that is sufficiently well-behaved for unique solutions of the equation to exist for all initial conditions. The following simple lemma provides sufficient conditions for the global asymptotic stability of this system:

Lemma 1: The system of equation (1) is globally asymptotically stable if

(a) $x^T f(x,t) \leq 0 \qquad \forall \quad x \; \forall \, t$ ...(2)

(b) $\lambda_n < 0$ ...(3)

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the (ordered) eigenvalues of $(A_1 + A_1^T)/2$.

Proof: Choosing the Lyapunov function $V(x,t) = \frac{1}{2} x^T x$ and multiplying (1) by $x^T(t)$ yields

$$\dot{V} = x^T(t) \frac{(A_1 + A_1^T)}{2} x(t) + x^T(t) f(x(t),t)$$

$$\leq x^T(t) \frac{(A_1 + A_1^T)}{2} x(t) \qquad \text{(by (a))}$$

$$\leq 2\lambda_n V \qquad \ldots(4)$$

and hence that $V(x(t),t) \leq V(x(o),0)e^{-2|\lambda_n|t}$ by (b). In particular

$$\|x(t)\|_2 \leq \|x(o)\|_2 \, e^{-|\lambda_n|t} \qquad \ldots(5)$$

and the result follows trivially.

Considering now the configuration of Fig. 1 with linear part L_o defined by the linear, time-invariant invertible system $S(A_1,B_1,C_1)$

$$\dot{x}(t) = A_1x(t) + B_1e(t) \quad , \quad e(t)\in R^m$$
$$y(t) = C_1x(t) \quad , \quad y(t)\in R^m \qquad \ldots(6)$$

and nonlinear part defined by the map $n : R^m \rightarrow R^m$

$$e(t) = -n(y(t)) \qquad \ldots(7)$$

It is trivially verified that the closed-loop system takes the form of equation (1) with

$$f(x(t),t) = -B_1n(Cx(t)) \qquad \ldots(8)$$

The problem arises as to the conditions on $n(y)$ that ensure that equation (2) is satisfied. The following lemma characterizes these conditions in the most general case of practical interest:

Lemma 2: Suppose that, for each $y_o \in R^m$, there is a scalar η and a vector $y \in R^m$ such that $y_o = \eta n(y)$. Then a necessary condition for $x^TB_1n(Cx)$ to be positive for all $x \in R^n$ is that $C_1 = LB_1^T$ for some choice of mxm matrix L.

(Note: the condition on $n(y)$ intuitively reflects the practical situation when its range 'fills' the whole of the output space.)

Proof: Invertibility guarantees that rank $C_1 = m$ and, without loss of generality, we can use the orthogonal transformation of state variables $x(t) \to U^T x(t)$ defined by $U = [C_1^T (C_1 C_1^T)^{-\frac{1}{2}}, M]$ where M is an orthonormal basis matrix for the kernel of C_1. In this basis $C_1 = [(C_1 C_1^T)^{\frac{1}{2}} \; 0]$ and $B_1 = \begin{pmatrix} (C_1 C_1^T)^{-\frac{1}{2}} C_1 B_1 \\ \hat{B}_2 \end{pmatrix}$ for suitable choice of $\hat{B}_2$.

Writing $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 \in R^m$ yields

$$x^T B_1 n(C_1 x) = y^T C_1 B_1 n(y) + x_2^T \hat{B}_2 n(y) \qquad \ldots (9)$$

where $y = (C_1 C_1^T)^{\frac{1}{2}} x_1$ is independent of the choice of x_2. The result will follow if $\hat{B}_2 = 0$. Suppose that $\hat{B}_2 \neq 0$ when, by assumption, we can choose y (and hence x_1) such that $\hat{B}_2 n(y) \neq 0$. A contradiction is easily obtained by considering $x_2 = \mu \hat{B}_2 n(y)$ and suitable choice of scalar $\mu < - y^T C_1 B_1 n(y) / \|\hat{B}_2 n(y)\|_2^2$.

(Note: A comparison with the work of Anderson (Anderson 1967, Narendra and Taylor 1973) indicates that the condition is connected with the idea of the positive real property of $G(s) = C_1 (sI_n - A_1)^{-1} B_1$. The two conditions are not however identical.)

In fact we are lead to the following stability theorem underlying the remainder of this paper:

Theorem 1

The system defined by equations (6) and (7) is globally asymptotically stable if there exists a nonsingular mxm matrix L such that

(a) $C_1 = L\,B_1^T$ ...(10)

(b) $y^T(L^T)^{-1}n(y) \geq 0 \quad \forall\, y \in R^m$ ...(11)

(c) $\lambda_n < 0$ ...(12)

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the (ordered) eigenvalues of $(A_1+A_1^T)/2$.

Proof: The proof is a straightforward application of lemma 1 noting that

$$x^Tf(x,t) = -x^TB_1n(y) = -x^TC_1^T(L^T)^{-1}n(y)$$
$$= -y^T(L^T)^{-1}n(y) \leq 0$$

for all y and hence for all x.

The problem now arises as to the existence of a suitable L matrix. We note in particular that $C_1B_1 = LB_1^TB_1$ must be nonsingular. This condition is not sufficient however to guarantee, for example, that there exists L satisfying equation (10). This situation can be enormously improved by judicious choice of basis in the state space as illustrated by the following result:

Proposition 1: Let $|C_1B_1| \neq 0$ and define the nxn matrix $T_1 = [B_1(K_1C_1B_1)^{-1}, M]$ where K_1 is any real, nonsingular mxm matrix and the columns of M form any basis for the kernel of C_1. Then $|T_1| \neq 0$ and, if $\tilde{A}_1 = T_1^{-1}A_1T_1$, $\tilde{B}_1 = T_1^{-1}B_1$ and $\tilde{C}_1 = C_1T_1$ are the matrices of the transformed system $S(\tilde{A}_1,\tilde{B}_1,\tilde{C}_1)$, we have $\tilde{C}_1 = L\tilde{B}_1^T$ where $L = (K_1^TK_1)^{-1}((C_1B_1)^T)^{-1}$.

Proof: The nonsingularity of T_1 is easily proven. In the defined basis it can be verified that

$$\tilde{B}_1 = \begin{pmatrix} K_1C_1B_1 \\ 0 \end{pmatrix} \quad , \qquad \tilde{C}_1 = [K_1^{-1}, 0] \qquad \ldots(13)$$

and the result follows by inspection.

It would appear therefore that there is great benefit to be obtained by expressing the state-space model in a suitable basis. Using the above basis equation (11) reduces to

$$y^T K_1^T K_1 C_1 B_1 n(y) \geq 0 \qquad \forall\ y \in R^m \qquad \ldots(14)$$

For a given nonlinearity the matrix K_1 provides a degree of flexibility in assessing stability, although it is important to recognize that λ_n is a function of K_1 and hence equation (12) must always be checked for each choice of K_1.

A slightly weaker form of theorem 1 is obtained in terms of the 'matrix gain' of the nonlinearity, by considering nonlinearities of the form

$$n(y) = N(y)y \qquad \ldots(15)$$

where $N(y)$ is bounded in the vicinity of $y = 0$. A sufficient condition for equation (11) to hold is obtained by checking the positive semi-definiteness of a (low-order!) mxm matrix. More precisely, condition (11) will be satisfied if

$$(L^T)^{-1} N(y) \geq 0 \qquad \forall\ y \in R^m \qquad \ldots(16)$$

or, in the case when the basis defined by Proposition 1 is used, it is sufficient that

$$K_1^T K_1 C_1 B_1 N(y) \geq 0 \qquad \forall\ y \in R^m \qquad \ldots(17)$$

Both conditions are matrix generalizations of the infinite sector bounds so well-known in the absolute stability problem. They can be used directly for a given nonlinearity or, conversely, the relationships can be used to derive bounds that the nonlinear gain matrix must satisfy in order to guarantee that equation (11) holds, and hence the possibility of asymptotic stability. More precisely, if we compute the set

$$\Sigma \triangleq \{X \in R^{mxm} : (L^T)^{-1} X \geq 0\} \qquad \ldots(18)$$

then a sufficient condition for equation (11) to hold is that

$$N(y) \in \Sigma_d \quad \forall \quad y \in R^m \qquad \ldots(19)$$

where Σ_d is any subset of Σ. This idea is illustrated by the following example.

Consider the two-input-two-output linear system defined by the matrices

$$A_1 = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix} , \quad B_1 = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} , \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \ldots(20)$$

with a 'diagonal' feedback nonlinearity of the form

$$n(y) = \begin{pmatrix} N_{11}(y_1)y_1 \\ N_{22}(y_2)y_2 \end{pmatrix} = \begin{pmatrix} N_{11}(y_1) & 0 \\ 0 & N_{22}(y_2) \end{pmatrix} y \qquad \ldots(21)$$

satisfying the sector bounds, i = 1,2,

$$\underline{K}_i \leq N_{ii}(y_i) \leq \bar{K}_i \qquad \forall \; y_i \qquad \ldots(22)$$

We pose the absolute stability problem of determining constraints on $\underline{K}_i$, $\bar{K}_i$, i = 1,2 that guarantee the global asymptotic stability of the feedback system.

Comparing (20) with (13) it is seen that the system is in the form defined by Proposition 1 with $K_1 = I_2$. As our nonlinearity is diagonal, we can restrict our attention to the subset $\Sigma_d \subset \Sigma$ of diagonal matrices ie

$$\Sigma_d \triangleq \{ X = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : K_1^T K_1 C_1 B_1 X \geq 0 \}$$

$$= \{ X = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : k_1 \geq 0 , 0.13k_1 \leq k_2 \leq 1.87k_1 \} \quad \ldots(23)$$

In particular $N(y) \in \Sigma_d$ for any nonlinearity satisfying equations (22) if, and only if,

$$0 \leq 0.13\bar{K}_1 \leq \underline{K}_2 \leq \bar{K}_2 \leq 1.87\underline{K}_1 \quad \ldots(24)$$

The eigenvalues of $(A_1+A_1^T)/2$ are $\lambda_1 = -3$, $\lambda_2 = -1<0$. It follows directly from theorem 1 that the closed-loop system is absolutely stable in the 'sector' defined by equation (24).

More information can be obtained by the use of a state-space transformation of basis as defined by Proposition 1. Choosing

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \quad \ldots(25)$$

yields the set $\Sigma_d = \{X = \mathrm{diag}\{k_1,k_2\} : k_1 \geq 0,\ 0.27k_1 \leq k_2 \leq 3.73k_1\}$. The required transformation $T_1 = K_1^{-1}$ leads to

$$\tilde{A}_1 = K_1 A_1 K_1^{-1} = \begin{pmatrix} -2 & \sqrt{2} \\ 0 & -2 \end{pmatrix} \quad \ldots(26)$$

whose symmetric part has eigenvalues $\lambda_1 = -2-\sqrt{2}$, $\lambda_2 = -2+\sqrt{2}<0$. Theorem 1 indicates that the closed-loop system is also absolutely stable in the 'sector'

$$0 \le 0.27\bar{K}_1 \le \underline{K}_2 \le \bar{K}_2 \le 3.73\underline{K}_1 \qquad \ldots(27)$$

Other choices of K_1 will, quite obviously, provide other sectors possessing the absolute stability property.

3. Choice of Basis and the Inverse System

The benefits of a judicious choice of basis in the state space has been demonstrated by Proposition 1 (and by example) in the last section. More precisely, it reduces the stability conditions of theorem 1 to the checking of a low-order positivity condition (eqns (11), (14), (16) or (17)) plus the (numerical well-conditioned) calculation of the largest eigenvalue of a real symmetric matrix. Considering the basis class defined by Proposition 1, it is noted that there is a large freedom in the choice of both K_1 and M. It has been seen that the choice of K_1 can be useful in providing alternative sector condition on the nonlinearity. Intuitively it might be expected that a judicious choice of M and K_1 could provide a simple means of computing λ_n. This possibility is explored in this section.

3.1 A Nested Feedback Structure for System Representation:

Suppose that the basis change defined by Proposition 1 has been implemented yielding

$$A_1 = \begin{pmatrix} K_1A_{11}K_1^{-1} & -K_1C_1B_1C_2 \\ B_2K_1^{-1} & A_2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} K_1C_1B_1 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{bmatrix} K_1^{-1} & 0 \end{bmatrix} \qquad \ldots(28)$$

where C_1B_1, A_{11}, C_2, B_{21}, A_2 are independent of the choice of K_1. The physical significance of this system representation is illustrated in Fig. 2(a) or, defining the system transfer function matrix $G(s) = C_1(sI_n-A_1)^{-1}B_1$ and the transfer function matrices $G_1(s)$ and $H_2(s)$ by the inverse system

$$G^{-1}(s) = G_1^{-1}(s) + H_2(s) \qquad \ldots(29)$$

where

$$G_1^{-1}(s) = s(C_1B_1)^{-1} - (C_1B_1)^{-1}A_{11}$$

$$H_2(s) = C_2(sI_{n-m} - A_2)^{-1} B_2 \quad , \qquad \ldots(30)$$

by the configuration of Fig. 2(b). That is, a forward path system with a strictly proper dynamic output feedback system. It is easily verified that

$$\begin{vmatrix} sI_m - A_{11} & -C_1B_1 \\ I_m & 0 \end{vmatrix} \equiv |C_1B_1| \qquad \ldots(31)$$

$$\begin{vmatrix} sI_n - A_1 & -B_1 \\ C_1 & 0 \end{vmatrix} \equiv |C_1B_1|\cdot|sI_{n-m}-A_{22}| \qquad \ldots(32)$$

and hence that (a) the forward path system has no zeros and (b) that the poles of the feedback system (Owens 1977, 1978a) are the zeros of $S(A_1,B_1,C_1)$. If we note (Owens, 1978a,b) that the asymptotic directions and pivots of the root-locus of $S(A_1,B_1,C_1)$ are governed completely by the structure of $G_1(s)$, it can be seen that the system decomposition has a direct relationship to the root-locus plot.

Consider now the possibility of continuing with this decomposition by induction. More precisely, suppose that $|C_jB_j| \neq 0$ for some $j \geq 1$ and, applying Proposition 1 to $S(A_j,B_j,C_j)$, a suitable choice of basis yields a representation of the form

$$A_j = \begin{pmatrix} K_jA_{jj}K_j^{-1} & -K_jC_jB_jC_{j+1} \\ B_{j+1}K_j^{-1} & A_{j+1} \end{pmatrix} , \quad B_j = \begin{pmatrix} K_jC_jB_j \\ 0 \end{pmatrix}$$

$$C_j = \left[K_j^{-1} \;,\; 0\right] \qquad \text{...(33)}$$

where K_j is an arbitrary real, nonsingular mxm matrix. In algebraic terms, taking $H_o(s) \triangleq G(s)$, this operation is equivalent to inversion of $H_j(s)$

$$H_j^{-1}(s) = G_j^{-1}(s) + H_{j+1}(s) \qquad \text{...(34)}$$

where $G_j(s)$ is a system with polynomial inverse of the form

$$G_j^{-1}(s) = s(C_jB_j)^{-1} - (C_jB_j)^{-1}A_{jj}$$

(and hence with no zeros) and $H_{j+1}(s)$ is strictly proper of the form

$$H_{j+1}(s) = C_{j+1}(sI_{n-jm} - A_{j+1})^{-1}B_{j+1} \qquad \text{...(35)}$$

In generic terms these operations can be continued up to any $q \leq n/m$ but this is not always the case. Suppose therefore that these operations are valid for $1 \leq j \leq q$, then the system has the nested feedback structure illustrated in Fig. 3. Alternatively, in state-space terms, the system can be expressed in the form

$$A_1 = \begin{pmatrix} K_1A_{11}K_1^{-1} & -K_1C_1B_1K_2^{-1} & 0 & \cdots\cdots & 0 \\ K_2C_2B_2K_1^{-1} & K_2A_{22}K_2^{-1} & -K_2C_2B_2K_3^{-1} & 0 \cdots & 0 \\ 0 & K_3C_3B_3K_2^{-1} & K_3A_{33}K_3^{-1} & & \cdot \\ \cdot & 0 & & & \\ \cdot & \cdot & & & 0 \\ \cdot & \cdot & & & -K_qC_qB_qC_{q+1} \\ \cdot & \cdot & & & \\ 0 & 0 & \cdots\cdots\cdots\; 0 & B_{q+1}K_q^{-1} & A_{q+1} \end{pmatrix}$$

$$B_1 = \begin{bmatrix} K_1C_1B_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad , \quad C_1 = [K_1^{-1} \; 0 \; . \; . \; . \; 0] \qquad ...(36)$$

where A_1 has a partial block band structure. Although this structure gives a formal simplicity to A (and hence to its symmetric part) it does not allow an explicit simplification of the calculation of λ_n except in very special cases, namely when there exists real nonsingular mxm matrices K_j, $1 \le j \le q$, and a suitable state space basis for $S(A_{q+1}, B_{q+1}, C_{q+1})$ such that

$$K_{j+1}C_{j+1}B_{j+1}K_j^{-1} = (K_jC_jB_jK_{j+1}^{-1})^T \quad , \quad 1 \le j \le q-1$$

$$B_{q+1}K_q^{-1} = (K_qC_qB_qC_{q+1})^T \qquad ...(37)$$

when it is easily verified that

$\lambda_n = \max \{\lambda : \lambda$ is an eigenvalue of the symmetric part of one of the mxm matrices $K_jA_{jj}K_j^{-1}$, $1 \le j \le q$, or the $(n-qm)x(n-qm)$ matrix $A_{q+1}\}$ ...(38)

Equations (37) are of a fairly complex form and, most certainly, do not always have solutions. There are however a case when an explicit solution can be constructed, as in the following section.

3.2 Symmetric Systems

Consider now the case of symmetric systems satisfying the constraint

$$G(s) \equiv G^T(s) \qquad \forall \; s \qquad ...(39)$$

Although a rather restrictive assumption for multivariable systems it does occur in practice and, in particular, encompasses the important case of scalar systems quite trivially.

The assumption of symmetry ensures that $G_j(s)$, $1\le j\le q$, and $H_{q+1}(s)$ are all symmetric and hence that C_jB_j, $1\le j\le q$, $(C_jB_j)^{-1}A_{jj}$, $1\le j\le q$, and $C_{q+1}B_{q+1}$ are symmetric. Suppose now that the system is positive (loop-wise) in the sense that

$$C_jB_j > 0 \quad , \quad 1\le j\le q+1 \qquad \ldots(40)$$

and that $S(A_{q+1},B_{q+1},C_{q+1})$ is in the form defined by equation (33) with $j = q+1$.

Write

$$K_j = (C_jB_j)^{-\frac{1}{2}} = K_j^T \quad , \quad 1\le j\le q+1 \qquad \ldots(41)$$

It follows immediately that

$$K_{j+1}C_{j+1}B_{j+1}K_j^{-1} = K_{j+1}^{-1}K_j^{-1} = (K_jC_jB_jK_{j+1}^{-1})^T \ , \quad 1\le j\le q \qquad \ldots(42)$$

$$K_jA_{jj}K_j^{-1} = K_j^{-1}(C_jB_j)^{-1}A_{jj}K_j^{-1} = (K_jA_{jj}K_j^{-1})^T \ , \quad 1\le j\le q \qquad \ldots(43)$$

Using this notation, we deduce the following result concerning absolute stability from theorem 1:

Theorem 2

The system defined by equations (6) and (7) is globally asymptotically stable if

(a) $y^Tn(y) \ge 0 \quad \forall \; y\in R^m$ ...(44)

(b) the linear system is symmetric and positive (loop-wise) in the sense defined above

(c) $0>\lambda_n = \max\{\lambda : \lambda$ is an eigenvalue of one or more of the $m\times m$ symmetric matrices $K_jA_{jj}K_j^{-1}$, $1\le j\le q$, or the $(n-qm)\times(n-qm)$ matrix $(A_{q+1}+A_{q+1}^T)/2\}$

(Remark: we can obviously replace the matrices $K_jA_{jj}K_j^{-1}$, $1\le j\le q$, by A_{jj}, $1\le j\le q$, in the calculation of λ_n).

Proof of Theorem 2

The definitions of K_1, equation (44) and Proposition 1 imply equations (10) and (11) of theorem 1. Also, equations (42) and (43) yield $(A_1+A_1^T)/2$ = block diag$\{K_1A_{11}K_1^{-1},\ldots,K_qA_{qq}K_q^{-1},(A_{q+1}+A_{q+1}^T)/2\}$ which implies equation (12) of theorem 1.

The following special case follows trivially:

Corollary

With the assumptions of theorem 2, the closed-loop system is globally asymptotically stable in the presence of any 'diagonal' nonlinearity of the form $n(y) = (n_1(y_1),\ldots,n_m(y_m))^T$, $n_j(y_j) = N_{jj}(y_j)y_j$, $1\le j\le m$, where $N_{jj}(y_j) \ge 0 \;\forall y_j$, $1\le j\le m$.

It is interesting to note that the stability conditions can be expressed in terms of the 'positivity' of the loops in Fig. 3 (as expressed by equation (40)), the stability of the q loops $G_j(s)$, $1\le j\le q$, and the stability of the 'symmetric part' of the (q+1)th loop. A particularly interesting case occurs when $S(A_{q+1},B_{q+1},C_{q+1})$ has a realization of the required form with $A_{q+1} = A_{q+1}^T$. In this situation theorem 2(c) reduces to the requirement that all loops in Fig. 3 are asymptotically stable. Conditions for this are stated below:

Proposition 2: If the symmetric system $S(A_{q+1},B_{q+1},C_{q+1})$ is both controllable and observable, then there is a choice of basis in the state space such that $C_{q+1} = \left[(C_{q+1}B_{q+1})^{\frac{1}{2}} \;\; 0\right] = B_{q+1}^T$ and $A_{q+1} = A_{q+1}^T$ if, and only if, $H_{q+1}(s)$ takes the form

$$H_{q+1}(s) \quad = \quad \sum_{j=1}^{\ell} \frac{1}{s-\mu_j} R_j \qquad \ldots(45)$$

where the system poles $\{\mu_j\}$ are real and the mxm real, constant 'residue' matrices are symmetric and positive semi-definite.

Proof: Necessity is obvious. To prove sufficiency, note that H_{q+1} has a minimal realization of the form $\tilde{C}_{q+1} = [P_1, P_2, \ldots, P_\ell] = \tilde{B}_{q+1}^T$, $\tilde{A}_{q+1} = \text{block diag}\{\mu_j I_{r_j}\}_{1 \le j \le \ell}$ where $r_j = \text{rank } R_j$ and $R_j = P_j P_j^T$, $1 \le j \le \ell$ where the matrices P_j are $m \times r_j$, $1 \le j \le \ell$. The controllability and observability assumption ensures that the two realizations $S(A_{q+1}, B_{q+1}, C_{q+1})$ and $S(\tilde{A}_{q+1}, \tilde{B}_{q+1}, \tilde{C}_{q+1})$ are related by a state transformation T_o. Let $T_o = [\tilde{B}_{q+1}(\tilde{C}_{q+1}\tilde{B}_{q+1})^{-\frac{1}{2}}, M]$ where M is a matrix with columns defining an orthonormal basis for the kernel of $\tilde{C}_{q+1}$. Note that T_o is orthogonal and write $A_{q+1} = T_o^T \tilde{A}_{q+1} T_o$, $C_{q+1} = \tilde{C}_{q+1} T_o = [(\tilde{C}_{q+1}\tilde{B}_{q+1})^{\frac{1}{2}} \; 0]$ and $B_{q+1} = T_o^T \tilde{B}_{q+1} = (C_{q+1} T_o)^T = C_{q+1}^T$. This proves the proposition.

To illustrate these results, consider the symmetric invertible system of state dimension n = 4 with inverse transfer function matrix factored into polynomial and strictly proper form as follows

$$G^{-1}(s) \equiv \underbrace{s\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 9 & 3 \\ 3 & 9 \end{pmatrix}}_{G_1^{-1}(s)} + \underbrace{\frac{1}{(s+1)(s+2)}\begin{pmatrix} 4s+8 & 2s+4 \\ 2s+4 & 2s+3 \end{pmatrix}}_{H_2(s)} \qquad \ldots(46)$$

By inspection

$$(C_1B_1)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad , \quad (C_1B_1)^{-1}A_{11} = \begin{pmatrix} -9 & -3 \\ -3 & -9 \end{pmatrix} \qquad \ldots(47)$$

from which it is easily verified that $C_1B_1 = C_1B_1^T > 0$ and A_{11} has eigenvalues of $-4,-6$. Note also that $H_2(s)$ has the form defined in Proposition 2,

$$H_2(s) = \frac{1}{(s+1)} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} + \frac{1}{(s+2)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \ldots(48)$$

with eigenvalues -1, -2, and that,

$$C_2B_2 = \lim_{s\to\infty} s\, H_2(s) = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} = (C_2B_2)^T > 0 \qquad \ldots(49)$$

The system is hence symmetric and positive (loop-wise) with $\lambda_4 = \max\{-4,-6,-1,-2\} = -1 < 0$. The system is hence absolutely stable with respect to all negative feedback nonlinearities of the form defined by equation (44).

3.3 Scalar Systems

As might be expected, there is further simplification to be obtained for the case of m = 1 (ie single-input/single-output systems) if we work directly in terms of the system transfer function. In particular, the sequential inversion operation defined by equation (34) is easily undertaken and we obtain the following result,

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Theorem 3

If m = 1, the system defined by equations (6) and (7) is globally asymptotically stable if it can be decomposed into the form shown in Fig. 3 and

(a) $yn(y) \geq 0 \;\forall\; y$

(b) each $G_j(s)$ is a stable first order lag of the form $G_j(s) = a_j/(s+b_j)$ with $a_j>0$, $b_j>0$, $1\leq j\leq q$

(c) $H_{q+1}(s)$ is a stable first order lag of the form $H_{q+1}(s) = a_{q+1}/(s+b_{q+1})$ with $a_{q+1}>0$, $b_{q+1}>0$ or a stable transfer function of the form

$$H_{q+1}(s) = \frac{k(s-z_1)(s-z_2)\ldots(s-z_{n-q-1})}{(s-p_1)(s-p_2)\ldots(s-p_{n-q})} \qquad \ldots(50)$$

with k>0 and real poles and zeros with the 'interlacing' property $p_1<z_1<p_2<z_2<\ldots<z_{n-q-1}<p_{n-q}<0$ ...(51)

Proof

Note that $C_jB_j = a_j$, $1\leq j\leq q$ and that $G(s) \equiv G^T(s)$ trivially. Note also that the form of $H_{q+1}(s)$ yields $C_{q+1}B_{q+1} = k>0$ and that the interlacing property guarantees that $H_{q+1}(s)$ has an expansion of the form given in Proposition 2. In particular we see that $\lambda_n = \max\{-a_1,-a_2,\ldots,-a_q,-p_1,\ldots,-p_{n-q}\}<0$. The theorem follows directly from theorem 2.

To illustrate the application of the result consider the system with transfer function

$$G(s) = \frac{(s+2)^3}{(s+1)(s^3+6s^2+24s+38)} \qquad \ldots(52)$$

Sequential inversion with q = 2 yields

$$G_1(s) = \frac{1}{(s+1)} \quad , \qquad G_2(s) = \frac{10}{(s+2)}$$

$$H_3(s) = \frac{0.1(s+2)}{(s+1)(s+3)} \qquad \ldots(53)$$

Both G_1 and G_2 are stable with positive gain and H_3 is stable with a positive gain and possesses the interlacing property. The system is hence absolutely stable in the presence of all first and third quadrant feedback nonlinearities.

4. Discussion

The stability criteria presented in this paper are based on the search for a suitable basis in the state space and suitable constraints on system structure such that the elementary Lyapunov function $V(x) = \frac{1}{2}x^T x$ can be used as a basis for absolute stability studies. This leads naturally to the structural constraint expressed by lemma 2 and hence to the basic and easily applied result of theorem 1. The logical consequence of this result is (Proposition 1) the idea of decomposing the system into the feedback representations illustrated in Fig. 2. The intuitive justification of this step is the observation that the decomposition has a direct connection with the pole-zero or root-locus structure of the system (Owens 1978a,b). Extension of these considerations does lead to some simplification of the stability criteria in quite general cases, the real advantages being seen in applications to the (defined) symmetric, positive (loop-wise) systems and the special case of scalar systems. In these cases the nested feedback representation illustrated in Fig. 3 is seen (theorems 2 and 3) to be a natural and particularly simple tool for the determination of absolute stability properties.

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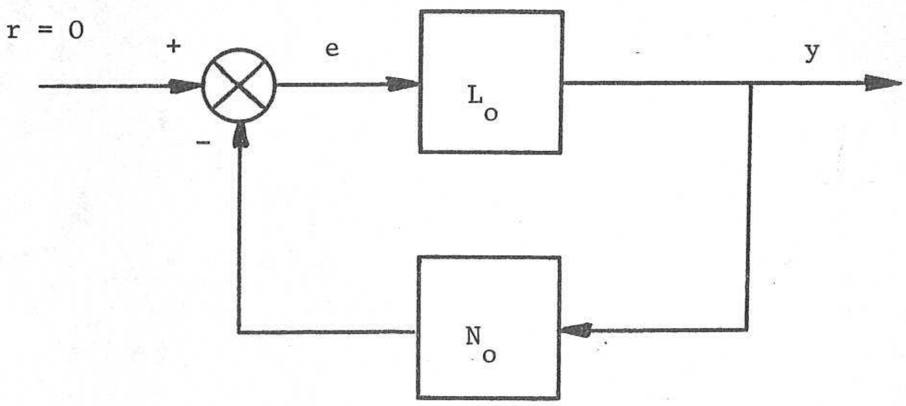
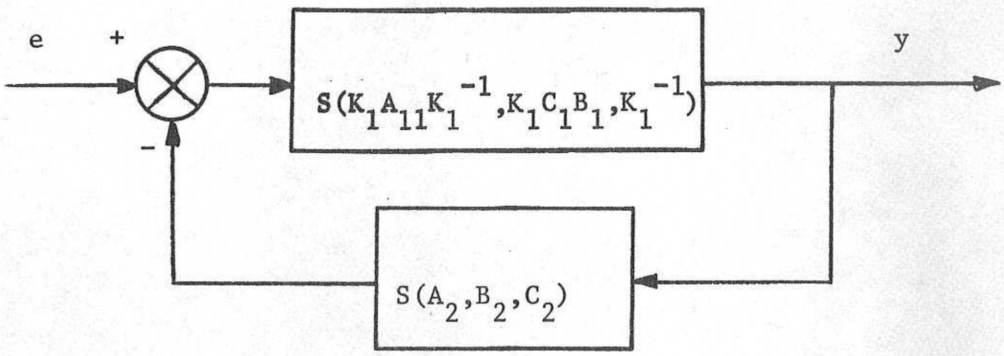
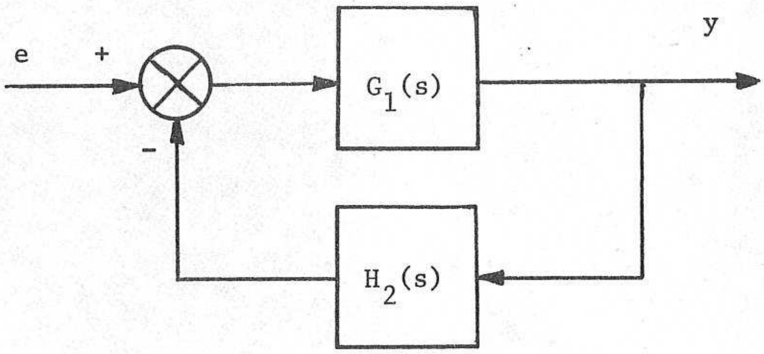


Fig. 1 Nonlinear Feedback System



(a)



(b)

Fig. 2 Representations of L_o

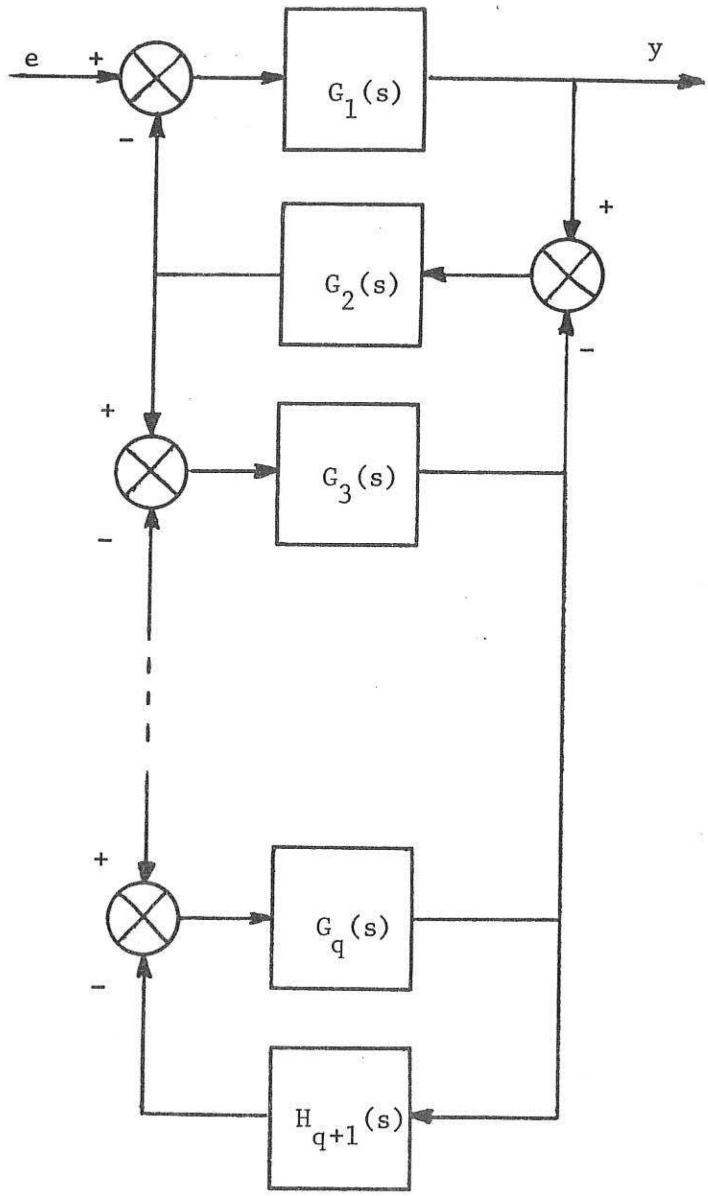


Fig. 3 Nested Feedback Representation of L_o