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University of Sheffield

Department of Control Engineering

Transformations of linear graph theory in multivariable system problems

H. Nicholson

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Summary

A correspondence is established between the multimachine power system, multivariable control system and the classical least squares problem incorporating a priori information. A transformation diagram is developed in order to illustrate the relationships between the system variables which are similar to those associated with a linear graph or electrical network.

The transformations of linear graph theory exist in the solution of the multimachine power system problem, consisting of a series of machine units interconnected with an electrical network. A similar correspondence is shown to exist also in the solution for the transformed outputs of a linear multivariable control system under certain conditions, and also in the solution of the least-squares estimation problem which incorporates a priori information concerning the estimated state. Each solution can be associated with a feedback structure and similar transformation matrices relating 'conjugate' variables also exist in each problem. This correspondence can form the basis for the development of a unified transformation diagram which possesses properties of the Roth-type transformation diagram for a linear graph. Such a diagram can form a framework for the linear system problems incorporating implicit feedback in the system equations. This will permit a physical structure such as an electrical network, and other properties of the control system problem such as those associated with the return-difference matrix, to be introduced into the least-squares estimation problem. Similarly, properties of the estimation problem may be introduced into the multivariable control system problem.

Multimachine power system problem. Consider the interconnection of synchronous generators with an equivalent network containing only generator nodes represented by

$$i_N = Y_N v_N \qquad (1)$$

where Y_N is a symmetrical matrix of driving-point and transfer admittances.

The machine voltages referred to direct and quadrature rotor axes are given by

$$v_k = e_k - Z_{Mk} i_k \qquad i = 1,2..m \tag{2}$$

where Z_{Mk} represents the k-machine transient reactance matrix with components x'_{dk}, x'_{qk}.[1] With load angles δ_k between the field axes of each machine and the common D,Q network reference axes, machine k terminal voltage is related to the i-node network voltage by the transformation

$$v_k = A(\delta_k) v_{Ni} \;, \quad A(\delta_k) = e^{j\delta_k} \;, \quad v_{Ni} = [v_{Di} v_{Qi}]^T \tag{3}$$

With m machines connected to n network nodes

$$v = A(\delta)\, v_N \tag{4}$$

where the connection matrix $A(\delta)$ is of order mxn, $m \geq n$, with elements $A_{ki} = (e^{j\delta_k}, 0)$ if the kth machine is incident or not on the ith node, and corresponds to the tree-branch-node matrix A_T in the electrical network problem with elements (+1,-1,0). The machine and network currents are related by

$$i_N = A^T(\delta)\, i \tag{5}$$

Combining eqns 1-5 for the connected system gives the solution for network voltages[1]

$$v_N = [Y_N + A^T(\delta) Y_M A(\delta)]^{-1} A^T(\delta) Y_M e \tag{6}$$

Then generator voltages

$$v = M(\delta)\, Y_M e \tag{7}$$

where $$M(\delta) = A(\delta)\, [Y_N + A^T(\delta) Y_M A(\delta)]^{-1} A^T(\delta) \tag{8}$$

is a symmetrical mxm matrix, which appears similarly in the general electrical network problem.[2] For the machine system we may also define a conjugate transformation matrix L with

$$Z_M L = I - MY_M \tag{9}$$

Then $$L = Y_M - Y_M M Y_M = (Z_M + A Z_N A^T)^{-1} \tag{10}$$

Also $$M = Z_M - Z_M L Z_M \tag{11}$$

Now machine current

$$i = Y_M(e - v) = Y_M(I - MY_M)e = L\, e \tag{12}$$

The matrix L thus represents a direct transformation between the machine current and internal voltage vectors. Also

$$i_N = A^T L e \qquad v = M Y_M L^{-1} i \tag{13}$$

A transformation diagram illustrating the relationships between the machine and network variables may now be developed as in FIG 1.

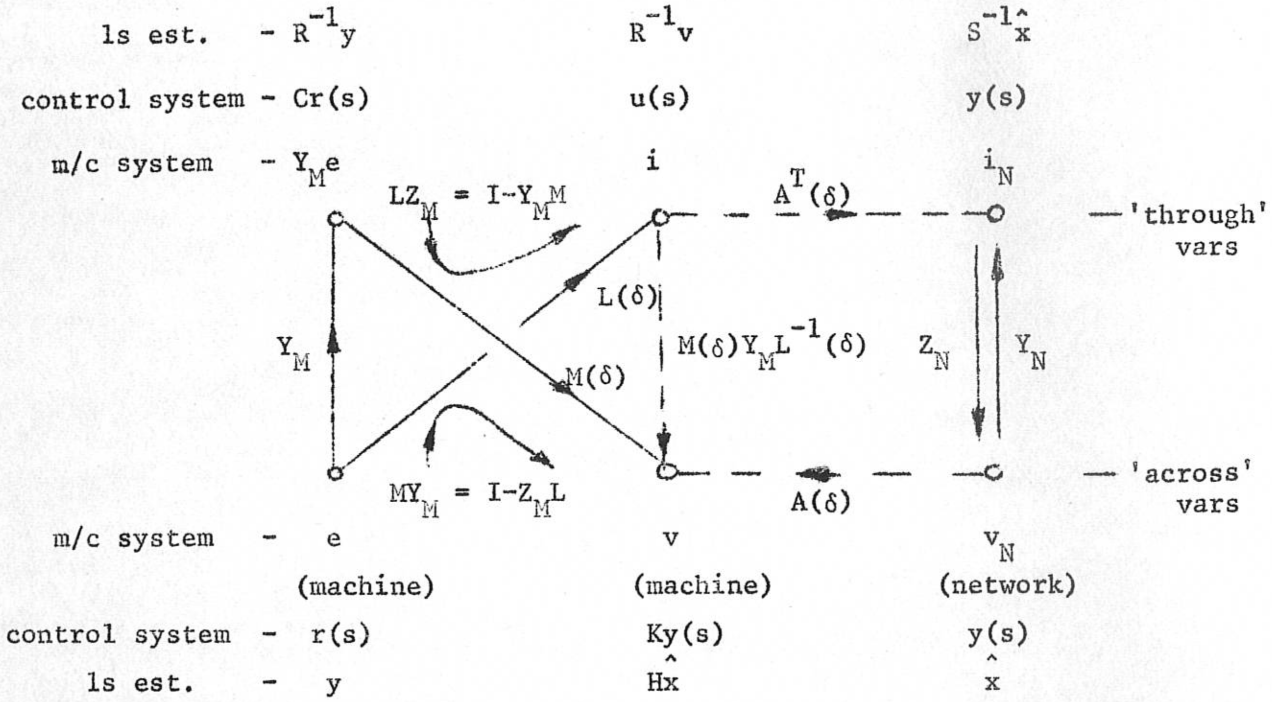

FIG 1 Transformation diagram for the multimachine power system, multivariable control and least-squares estimation problems.

The transformation diagram includes the characteristics of the corresponding Roth-type diagram for the general electrical network problem[3], with the impedance- and admittance-type operators $M(\delta)$ and $L(\delta)$ respectively directed across the diagram between the conjugate 'through' and 'across' variables. The matrix $L(\delta)$ defines the effect of feedback introduced by the machine-network interconnection, and possesses properties of a return-difference matrix associated with the multivariable control problem.

In the multimachine power system problem the concepts of a generalised inverse matrix are not evident, and the transformation operators do not possess an orthogonality property, as in the general electrical network problem[4], with

$$Z_M L M Y_M \neq 0 \tag{14}$$

and the matrix MY_M is not idempotent. This results from the conditions imposed by the form of the inverse matrices in $M(\delta)$ and $L(\delta)$, which are determined by the implicit feedback effects in the defining system equations.

Multivariable control problem. The linear multivariable control system includes transfer function matrix controllers C(s), K(s) of orders (mxn), (nxn) respectively and a process G(s) of order (nxm) as in the general feedback structure of FIG 2. The transformed reference input variables r(s) specify the required behaviour of the output variables y(s). The solution for network voltages in the machine system problem is similarly represented by the form of FIG 2, with feedback introduced by the voltage-current relationship of eqn 2. The diagram illustrates the particular difficulties of the multivariable control problem, in which the dynamics are immersed within the structure of the system compared to the machine problem in which the units exist external to the structure of the network, together with a diagonal machine admittance matrix compared to the more general forward controller matrix C(s).

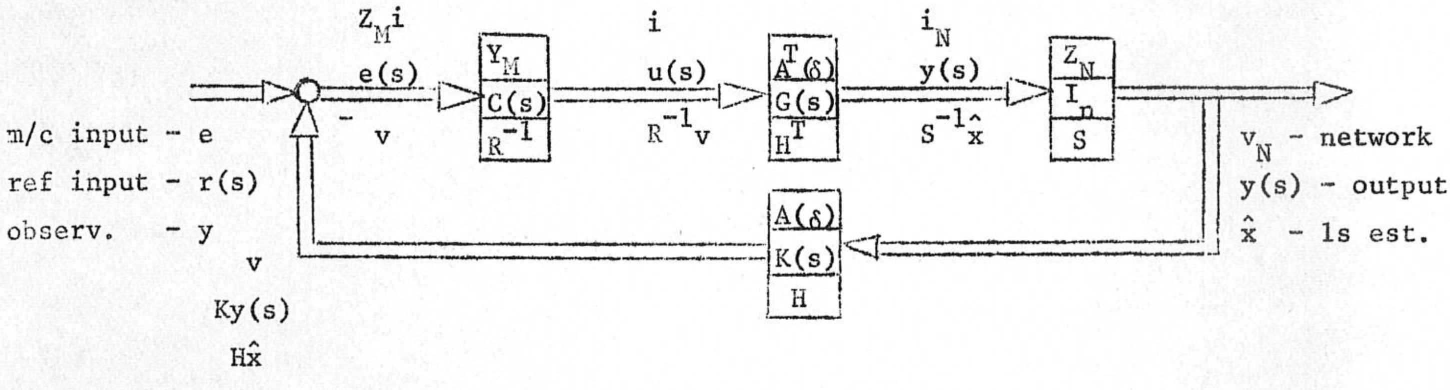

FIG 2 Basic feedback structure representing multimachine power system, multivariable control and least-squares estimation problems.

The feedback control system of FIG 2 has a closed-loop response given by

$$y(s) = (I_n + GCK)^{-1} GC\, r(s) \tag{15}$$

A direct correspondence exists between the solution of the multimachine power system and the multivariable control system problems if the feedback controller is defined by $K(s) \equiv G^T(s)$, with m reference inputs. The solution of eqn 15 can then also be associated with a least-squares estimation problem.[2] We may now define, as in the machine system problem, the transformation matrices

$$M(s) = K(I_n + GCK)^{-1}G \tag{16}$$

$$L(s) = C - CMC = C(I_n + KGC)^{-1} = CF^{-1} \tag{17}$$

where $$F(s) = I_n + KGC \tag{18}$$

represents a return-difference-type matrix. Also from eqn 18

$$F^{-1} + MC = I_n \tag{19}$$

The transformation diagram of FIG 1 may now be used to illustrate the solution of the multivariable control system problem and the inter-relationships between the variables based on the correspondence in Table 1. The matrices M(s), L(s) again possess properties of 'impedance'- and 'admittance'-type operators respectively, compared to the dimensionless properties of the return-difference matrix F(s), which will be directed horizontally in the transformation diagram.

The state-adjoint variable relationships in the linear optimal control problem incorporating a terminal constraint[5] possess properties which can also be illustrated in a Roth-type transformation diagram. The problem is defined by the matrix differential equations

$$\begin{bmatrix} \dot{G} & \dot{P} \\ \dot{N} & \dot{S} \end{bmatrix} = \begin{bmatrix} (PBR^{-1}B^T-A^T)G, & -C^TQC-PA-A^TP+PBR^{-1}B^TP \\ SBR^{-1}B^TG\ , & S(BR^{-1}B^TP-A) \end{bmatrix} \tag{20}$$

together with the state-adjoint variable and terminal constraint relationships given by

$$\begin{bmatrix} \lambda \\ p(t) \end{bmatrix} = \begin{bmatrix} -N^{-1}G^T & N^{-1} \\ P-GN^{-1}G^T\ , & GN^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ z \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} p(t_1) \\ z \end{bmatrix} = \begin{bmatrix} C^TZ^T & C^TFC \\ 0 & ZC \end{bmatrix} \begin{bmatrix} \lambda \\ x(t_1) \end{bmatrix} \tag{22}$$

The terminal constraint is defined by the vector z and λ is a vector multiplier used to incorporate the terminal constraint in the transversality condition. Eqns 21, 22 are also associated with a scattering-type representation of the optimal control problem.[5] It is now of interest to find that the same relationships can be illustrated in a time-domain transformation diagram, as in FIG 3.

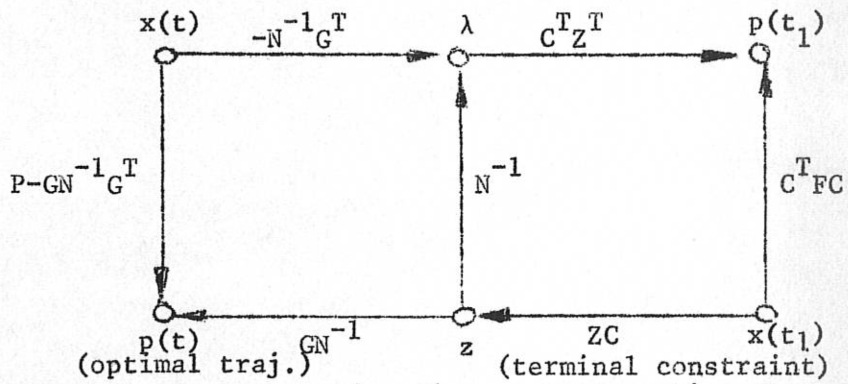

FIG 3 Transformation diagram representing the trajectory and terminal constraint relationships in the linear optimal control problem.

The diagram illustrates particularly the signal flows from the state x(t) and $x(t_1)$ to the adjoint variable p(t) and $p(t_1)$, with the terminal constraint introducing a feedback-type path in the right-hand section. The horizontal operators define transmission components and the operators directed vertically between the corresponding conjugate variables represent reflection components in the 'scattering' matrices of eqns 21 and 22.

Least-squares estimation problem. The solutions of the linear system problems which can be associated with a linear graph suggest the possibility of introducing a physical structure, such as an electrical network or machine system, into the abstract least-squares estimation problem. In a statistical framework the estimation problem is associated with a measurement process represented by

$$y = Hx + v \tag{23}$$

where y is an observed m-vector, x is an unknown parameter or state n-vector and v is an uncorrelated random error m-vector defined with zero mean $E[v] = 0$ and with a covariance matrix $E[v_i v_j^T] = R\delta_{ij}$, $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$, where R is a positive definite symmetrical matrix. With a priori information concerning the vector x, represented by $E[xx^T] = S$, $E[vx^T] = 0$, the linear estimate associated with

$$\min \{ \| y - Hx \|^2_{R^{-1}} + \| x \|^2_{S^{-1}} \}$$

is given by

$$\hat{x} = (S^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1} y = P H^T R^{-1} y \tag{24}$$

and $$\hat{y} = H\hat{x} = M R^{-1} y \tag{25}$$

where $$M = H(S^{-1} + H^T R^{-1} H)^{-1} H^T \tag{26}$$

We also define

$$L = R^{-1} - R^{-1} M R^{-1} = (R + HSH^T)^{-1} \tag{27}$$

then $$M = R - RLR \tag{28}$$

or $$RL + MR^{-1} = I \tag{29}$$

and $$P = (S^{-1} + H^T R^{-1} H)^{-1} = S - SH^T LHS \tag{30}$$

The matrix RL represents a transformation between the residual vector $y-\hat{y}$ and the measurement y with

$$y - \hat{y} = RLy \tag{31}$$

The least-squares solution of eqn 24 may be represented in the block diagram form of FIG 2, based on the correspondence of variables in Table 1. The problem solution also fits within the framework of the transformation diagram of FIG 1 with the matrix operators M,L defined by eqns 26 and 27.

Table 1 - correspondence of variables

m/c system	control system	least squares
Y_N	I_n	S^{-1}
$A^T(\delta)$	$G(s)$	H^T
$A(\delta)$	$K(s)$	H
Y_M	$C(s)$	R^{-1}
v_N	$y(s)$	$\hat{x}$
i_N	$y(s)$	$S^{-1}\hat{x}$
e	$r(s)$	y
v	$K(s)y(s)$	$H\hat{x}$
i	$u(s)$	$R^{-1}v$
$Z_M i$	$e(s)$	v
$i^T Z_M i + v_N^T Y_N v_N$	$e^T Ce + y^T y$	$v^T R^{-1} v + \hat{x}^T S^{-1} \hat{x}$

Table 1 includes also the potential function or performance criterion associated with the defining system equations.

A comparison of the solution for each system problem indicates that in the machine system problem the network admittance matrix Y_N introduces a priori information concerning the 'estimate' for network voltages, and appears as an inverse post-operator in the system structure of FIG 2. It is of interest to note that such operators may be similarly introduced into the multivariable control problem to effect a certain measure of uncoupling and diagonal dominance. It may also be significant to note that the controller matrix C(s) may be associated with reciprocal covariance properties of the error signal by comparison with the estimation problem. The basic properties of a linear graph incorporating a topological and algebraic structure can be associated naturally with the formulation of the multimachine power system problem. The solution of the machine problem, including the signal flows from source to response variables, and the inherent constraints of the connecting network can then be illustrated in a Roth-type transformation diagram. The multivariable control system and least-squares estimation problems have been shown to possess similar structural and algebraic properties which permit the problem solution and the associated transformation operators to be represented by a transformation diagram. Such a diagram defines particularly the interrelationships between conjugate sets of variables and appears to have an important role in illustrating the available forms of solution for system problems which can be associated with a linear graph. An understanding of the properties of such diagrams is of fundamental importance for extending the concepts to higher-dimensional network problems.

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