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FIRST AND SECOND ORDER-LIKE STRUCTURES IN LINEAR MULTIVARIABLE-CONTROL-SYSTEMS-DESIGN

by

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Abstract

With the philosophy that many physical multivariable systems can, for the purposes of control systems design, be approximated by much simpler forms and that the analysis of such models can provide valuable insight into the time and frequency domain characteristics of the system, the paper provides an analysis of two types of multivariable structure and derives closed-form solutions for proportional and proportional plus integral feedback controllers. The structures considered are shown to be direct multivariable generalizations of the classical first order lag and the classical unity rank, minimum phase, overdamped second order system.

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List of Symbols

$G(s)$	NxN plant transfer function matrix
$K(s)$	NxN controller transfer function matrix
$T(s)$	NxN return difference matrix
A^+	conjugate transpose of the matrix A
$G_A(s)$	NxN transfer function matrix of the reduced system model
$\{e_j\}_{1 \leq j \leq N}$	natural basis in R^N

1. Introduction

The success of single-input, single-output linear feedback control theory can be partly attributed to the insight available from the time and frequency domain analysis of simple models such as the classical second-order system, and the observation that many high order systems can, for the purpose of control systems design, be approximated by such low order models. Hence, a surprising feature of most frequency domain techniques[1-4,8] for the design of multivariable feedback control systems is the absence of particular results for simple multivariable structures analogous to the classical second order system.

With the philosophy that many physical multivariable systems[5-7] can be approximated by much simpler forms, this paper presents an analysis of two types of multivariable structure and gives derivations of closed-form solutions for proportional and proportional plus integral controllers. The controllers are easily and directly calculated from the system transfer function matrix G(s) and can provide arbitrarily fast and accurate closed-loop transient responses. The structure considered in section 2 is a multivariable generalization of the classical first-order system. The structure considered in section 3 is a multivariable generalization of the single-input, single-output, unity rank, minimum-phase, over-damped second-order system. Illustrative examples of the application of the results are described in each section. An interesting feature is the systematic nature of the synthesis procedure and the absence of direct frequency response considerations such as the use of characteristic loci[3,8] or the inverse Nyquist array[1]. These are not required due to the proven guarantee of closed-loop stability and accuracy and the strong intuitive link between time-domain dynamics and interaction effects and the transfer function matrix analysis at each stage of the design.

2. First-order Type Multivariable Structures

This section considers a unity feedback system for the control of a class of systems described by NxN transfer function matrices of the form

$$G(s) = \sum_{j=1}^{N} \frac{b_j}{s + b_j} \alpha_j \beta_j^{+} \qquad \ldots(1)$$

where $|G(o)| \neq 0$, $\{b_j\}_{1 \leq j \leq N}$ is a set of non-zero real numbers and $\{\alpha_j \beta_j^{+}\}_{1 \leq j \leq N}$ is a set of real dyads (unity rank matrices). Such a transfer function matrix arises in the analysis of systems whose input-output dynamics can be approximately described by a completely controllable state space model with N inputs, N outputs, state-dimension N and a system matrix A having a diagonal canonical form with real eigenvalues. An example of such a system is the pressurized flow-box of a paper-making machine[3].

2.1 Proportional Control

The following result is proved in appendix 6.

RESULT 1

Defining

$$G_{\infty} = \lim_{s \to \infty} s\, G(s) \qquad \ldots(2)$$

then G_{∞} is a non-singular matrix. If K(s) is an NxN forward path controller transfer function matrix of the form

$$K(s) = k\, G_{\infty}^{-1} - G^{-1}(o) \qquad \ldots(3)$$

where k is a real number, then the closed loop transfer function matrix

$$\{I + G(s)K(s)\}^{-1} G(s)K(s) = \frac{k}{s+k} M(k) \qquad \ldots(4)$$

where M(k) is a frequency independent matrix satisfying the relation

$$\lim_{k \to +\infty} M(k) = I_N \text{ (the unit matrix)} \qquad \ldots(5)$$

Also, the return-difference determinant $|T(s)| = |I+G(s)K(s)|$ of the system is

$$|T(s)| = (s+k)^N / \prod_{j=1}^{N} (s+b_j) \qquad \ldots(6)$$

The proposed controller replaces the open-loop poles $\{-b_j\}_{1 \leq j \leq N}$ by N identical simple poles at $s = -k$ (eqn.(6)). The closed-loop system (eqn.(4)) is represented by a single scalar transfer function multiplying a constant matrix (dependent on the choice of k) which represents closed-loop interaction effects and steady state error. As k increases, the speed of response of the system increases and equation (5) indicates that the closed-loop interaction effects and steady state error can be made to be arbitrarily small. For practical applications, this implies that k must be much greater than the maximum of $|b_j|$, $1 \leq j \leq N$.

The physical relationship between the system and the proposed controller can be obtained by noting that (eqn.(3)),

$$\lim_{k \to \infty} k^{-1} K(s) = G_\infty^{-1} \qquad \ldots(7)$$

so that, for large values of k, the controller tends to diagonalize the plant at high frequencies. Such a type of controller has been previously used intuitively. The above analysis provides some theoretical justification for the approach.

2.2 Proportional plus Integral Control

In this section the controller of section 2.1 is augmented by an integral term. The following result is proven in Appendix 7.

RESULT 2

With the notation of Result 1, the use of the forward path controller

$$K(s) = \{k+c + \frac{kc}{s}\} G_\infty^{-1} - G^{-1}(o) \qquad \ldots(8)$$

generates a feedback system with return-difference determinant

$$|T(s)| \quad = \quad (s+k)^N (s+c)^N / s^N \prod_{j=1}^{N} (s+b_j) \qquad \ldots(9)$$

That is, the closed-loop poles are simple poles at -k and -c, each with algebraic multiplicity N. If R(k) and R(c) are the residues of the closed-loop system at s = -k and s = -c respectively, then

$$\lim_{k\to\infty} k^{-1} R(k) \quad = \quad I_N \qquad \ldots(10)$$

and

$$\lim_{k\to\infty} R(c) \quad = \quad 0 \qquad \ldots(11)$$

The dynamic behaviour of the closed-loop system can be assessed by examining relations (10) and (11), from which it follows directly that, for large values of k, the system responds rapidly to step inputs and transient interaction effects can be made to be arbitrarily small. Steady state error is always zero due to the presence of integral action on each loop.

2.3 Summary and Relationship to Classical Theory

Given a system that can be approximated by the structure of equation (1), the previous sections provide a straightforward approach to the design of proportional or proportional plus integral feedback controllers which yields physical insight into closed-loop dynamics at each stage in the design process. A typical procedure could be,

STEP ONE: Calculate G_∞^{-1} and $G^{-1}(o)$.

STEP TWO: If step responses of the form $1-e^{-kt}$ are required from each channel, set the proportional controller to be as in equation (3). Simulate the closed-loop system to check the acceptability of the closed-loop interaction effects and steady state error. An increase in k will reduce any unacceptable transient characteristics.

STEP THREE: If integral action is required, choose an appropriate integral time c^{-1}, set the controller to be as in equation (8) and simulate to check the step response characteristics.

Finally, the form of the transfer function matrix (eqn(1)) for N = 1 implies that the system can be regarded as a multivariable generalization of the classical first order system. The relationship can be strengthened by considering the forms of the proposed controllers. For example, consider the proportional control of the single-input system $G(s) = b\alpha/(s+b)$. In order to move the closed-loop pole to $s = -k$, the controller must be $K(s) = (k-b)/\alpha b$ which, by direct analogy with equation (2) can be written as $K(s) = (k/G_\infty) - (1/G(o))$. The response speed increases as k increases and the steady state error becomes smaller. By comparing this controller with the multivariable controller of equation (3), it can be seen that the proposed structure is a direct generalization of the single-variable form.

2.4 Application to Gas-turbine Control System Design

The application of the results obtained in the previous sections can be illustrated by a consideration of the gas-turbine model discussed by McMorran[9]. The system is described by the transfer function matrix

$$G(s) = \frac{1}{\Delta_1(s)} \begin{bmatrix} 1.496(s+1.7) & 951.5(s+1.898) \\ 8.52(s+1.44) & 1240.0(s+2.037) \end{bmatrix} \begin{bmatrix} \frac{10}{s+10} & 0 \\ 0 & \frac{100}{s+100} \end{bmatrix} \qquad \ldots(12)$$

with outputs $y_1 = \Delta N_H$ = high-pressure-spool speed, $y_2 = \Delta N_L$ = low-pressure-spool speed and inputs $u_1 = \Delta A_d$ = demanded jet-pipe-nozzle area, $u_2 = \Delta Q_d$ = demanded fuel-flow rate. Also

$$\Delta_1(s) = (s+1.33)(s+1.89) \qquad \ldots(13)$$

The pole at s = -100 is extremely fast and will not affect closed-loop system dynamics if the closed-loop gain is not too high. The phase lag introduced by the pole at s = -10 could however cause closed-loop oscillatory behaviour even at low gains. This suggests synthesizing a controller as the product of two factors

$$K(s) = K_1(s)K_2(s) \qquad \ldots(14)$$

where $K_1(s)$ introduces phase advance into loop one,

$$K_1(s) = \begin{bmatrix} \frac{1+0.1s}{1+0.01s} & 0 \\ 0 & 1 \end{bmatrix} \qquad \ldots(15)$$

That is

$$G(s)K_1(s) = \frac{1}{\Delta_1(s)} \begin{bmatrix} 1.496(s+1.7) & 951.5(s+1.898) \\ 8.52(s+1.44) & 1240.0(s+2.037) \end{bmatrix} \frac{100}{s+100} \qquad \ldots(16)$$

The term 100/(s+100) can now be approximated by unity over the frequency range of interest and the controller factor $K_2(s)$ designed using the approximate model for $G(s)K_1(s)$ which is a multivariable first order system in the sense defined in previous sections. Using Result 2, the controller structure becomes

$$K_2(s) = \{k + c + \frac{kc}{s}\} \begin{bmatrix} -0.1983 & 0.1522 \\ 0.001363 & -0.000239 \end{bmatrix}$$

$$- \begin{bmatrix} -0.4054 & 0.2898 \\ 0.001969 & -0.000408 \end{bmatrix} \qquad \ldots(17)$$

Choosing k = 30 to ensure a fast closed-loop response with small interaction effects, and c = 5 for a reasonably short reset time, the responses of the closed-loop system to step demands in loops one and two are as given in Fig.1. It is seen that the system responds rapidly to demands with 8% overshoot and less than 4% interaction.

3. A second-order Type Multivariable Structure

The majority of systems will, in general, have a more complex interaction structure than that discussed in section 2. This section takes a step towards the analysis of more complex structures by considering the feedback control analysis of systems described by an NxN transfer function matrix of the form

$$G(s) = \sum_{j=1}^{N+1} \frac{b_j}{s+b_j} \alpha_j \beta_j^+ \qquad \ldots(18)$$

where $|G(o)| \neq 0$, $\{b_j\}_{1\leq j\leq N+1}$ is a set of non-zero, positive real numbers, $\{\alpha_j \beta_j^+\}_{1\leq j\leq N+1}$ is a set of real dyads and the polynomial

$$P_G(s) = |G(s)| \prod_{j=1}^{N+1} (s+b_j) \qquad \ldots(19)$$

has a zero in the open left-half of the complex plane. Such a transfer function matrix can arise in the control analysis of systems whose dynamics can be approximated by a completely controllable and completely observable state-space model with N-inputs, N outputs, state dimension N+1 and a system matrix A having a diagonal form with negative real eigenvalues. Closed-form solutions are obtained for suitable proportional and proportional plus integral controllers and a comparison indicates that the system G(s) is a direct multivariable generalization of single-input, single-output systems of the form

$$G(s) = g \frac{(s+\lambda_1)}{(s+\lambda_2)(s+\lambda_2)} \qquad \ldots(20)$$

where $\lambda_1, \lambda_2, \lambda_3$ are non-zero, distinct, positive real numbers.

3.1 Summary of the Design Procedure

In this section the design procedure is stated in the form of a result, the proof of which is outlined in Appendix 8. The idea underlying the technique is the design of the control based on a first order type reduced model of G(s).

Result 3

For the system defined by equation (18), it is possible, for the purposes of control system design, to replace G(s) by a reduced multivariable first order system of the form

$$G_A(s) = \sum_{j=1}^{N} \frac{b_j}{s+b_j} \alpha_j \beta_j^{+} \qquad \ldots(21)$$

where $|G_A(o)|/|G(o)| > 0$ (see Appendix 9). Defining

$$C_A = I_N + b_{N+1}\alpha_{N+1}\beta_{N+1}^{+}G_{A\infty}^{-1} \qquad \ldots(22)$$

$$K(s) = kG_{A\infty}^{-1} - G_A^{-1}(o) \qquad \text{(c.f. section 2.1)} \qquad \ldots(23)$$

$$p_1 = 1 + b_{N+1}\beta_{N+1}^{+}G_{A\infty}^{-1}\alpha_{N+1} \qquad \ldots(24)$$

$$p_2 = b_{N+1}\{1 + \beta_{N+1}^{+}G_A^{-1}(o)\alpha_{N+1}\} \qquad \ldots(25)$$

$$c_1(k) = kp_1 + 2b_{N+1} - p_2 \qquad \ldots(26)$$

$$c_2(k) = kp_2 - b_{N+1}\beta_{N+1}^{+}G_A^{-1}(o)G_{A\infty}G_A^{-1}(o)\alpha_{N+1} \qquad \ldots(27)$$

then,

(a) $p_1 > 0$, $p_2 > 0$ and $-p_2/p_1$ is a zero of $P_G(s)$.

(b) The return difference of the closed-loop system using the controller K(s) is

$$|T(s)| = |I+G(s)K(s)| = \frac{(s+k)^{N-1}}{\prod_{j=1}^{N}(s+b_j)} \frac{\{s^2+c_1(k)s+c_2(k)\}}{(s+b_{N+1})} \qquad \ldots(28)$$

(c) The closed-loop system is stable (eqn.(28)) if, and only if, $c_1(k)>0$ and $c_2(k)>0$. This is always possible to obtain by using large enough values of k and the closed-loop poles are $s = -k$ (N-1 times) and, for large values of k,

$$\mu_1(k) = \frac{-p_2}{p_1} + O(k^{-1})$$

$$\mu_2(k) = -kp_1 + \frac{p_2}{p_1} - O(k^{-1}) \qquad \ldots(29)$$

so that N closed-loop poles tend to $-\infty$ as $k\to+\infty$. The other pole tends to the zero of $P_G(s)$.

(d) $$\lim_{k\to+\infty} \lim_{s\to\mu_1(k)} (s-\mu_1(k))T^{-1}(s) = 0 \qquad \ldots(30)$$

so that, at high gains, the pole at $\mu_1(k)$ is insignificant in the closed-loop response and the system will behave very much like a first-order type multivariable structure.

(e) $$\lim_{k\to\infty} \lim_{s\to\infty} k^{-1}sG(s)K(s) = C_A$$

Finally, if integral action is required, the controller of equation (23) can be augmented by an integral term, based on the first order approximation $G_A(s)$, (c.f. section 2.2)

$$K(s) = \{k + c + \frac{kc}{s}\} G_{A\infty}^{-1} - G_A^{-1}(o) \qquad \ldots(31)$$

Closed loop stability is maintained if

$$c > 0 \quad , \quad c_1(k)c_2(k) > kcp_2 \qquad \ldots(32)$$

The use of large values of k will ensure small steady state errors and (see (c)) a fast response with interaction effects confined to a time interval of length of order k^{-1}. Some insight can be obtained into the degree of interaction by examining the matrix C_A (eqn.(22)). That is, using the proposed controller, the normalized initial derivative of output j in response to a step demand in output ℓ is $(C_A)_{j\ell}$ (see (e)). In general (Appendix 9), the reduced model is non-unique and hence $G_A(s)$ can be chosen to ensure a most nearly diagonal or diagonally dominant C_A, representing small closed-loop effects.

The application of the above results is illustrated in section 3.3. using an example.

3.2 Relationship to Classical Theory

By considering the case of $N = 1$, equation (18) immediately takes the form of equation (20). In this case, the closed-loop system is stable for arbitrarily high proportional gains, the response speed increases as gain increases and the steady state error becomes arbitrarily small. The closed-loop poles tend to $-\lambda_1$ and $-\infty$ at high gains and the closed-loop system behaves very much like a first order system. These observations are remarkably similar to the results obtained in section 3.1 for the multivariable structure (eqn (18)) and leads directly to the classification of the structure as a multivariable generalization of equation (20).

The approximation method of section 9 has a direct interpretation in terms of single-variable concepts. Using a partial fraction expansion of eqn (20)

$$G(s) = B_2(s+\lambda_2)^{-1} + B_3(s+\lambda_3)^{-1} \qquad \text{...(33)}$$

and, as both λ_2,λ_3 are positive, at least one of B_2,B_3 (say B_2) has the same sign as g. Defining $G_A(s) = B_2(s+\lambda_2)^{-1}$, it is easily seen that a proportional controller based on the first order approximation $G_A(s)$ to $G(s)$ will lead to a stable feedback configuration.

3.3 Illustrative Example

The straightforward nature of the techniques described in section 3.1 and the direct insight gained into the design process for the class of systems considered is illustrated in this section by a consideration of the feedback control of a system whose input-output dynamics can be approximated by the transfer function matrix

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} 27s^2+77s+48 & -s^2-3s \\ 15s^2+51s+32 & 3s^2+13s+12 \end{bmatrix}$$

$$= \frac{1}{s+1}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + \frac{2}{s+2}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \frac{3}{s+3}\begin{bmatrix}10\\6\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \quad \ldots(34)$$

Applying the procedure of appendix 9, define

$$U = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1} = \tfrac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \quad \ldots(35)$$

$$V = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix} \quad \ldots(36)$$

so that

$$UG(s)V = \frac{1}{s+1}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & -2\end{bmatrix} + \frac{2}{(s+2)}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}$$

$$+ \frac{3}{(s+3)}\begin{bmatrix}8\\-2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \quad \ldots(37)$$

and hence

$$UG(o)V = \begin{bmatrix}1 & -2+8\\1 & -2\end{bmatrix} \quad \ldots(38)$$

and $(-1)\,|UG(o)V| = 8>0$. Noting that $-(1)\times(-2)>0$, $8>0$, $-2<0$, it follows that the only acceptable approximations are

$$G_A(s) = \frac{1}{(s+1)}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + \frac{3}{(s+3)}\begin{bmatrix}10\\6\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \quad \ldots(39)$$

and

$$G_A(s) = \frac{2}{(s+2)}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \frac{3}{(s+3)}\begin{bmatrix}10\\6\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \quad \ldots(40)$$

To choose between the two options consider the expression for the closed-loop interaction effects as given in eqn.(22). For system (39), interaction effects are represented by the matrix

$$C_A = \frac{1}{3}\begin{bmatrix} 11 & -14 \\ -8 & 17 \end{bmatrix} \qquad \ldots(41)$$

and, for the system (40)

$$C_A = \frac{1}{96}\begin{bmatrix} 74 & 26 \\ -22 & 122 \end{bmatrix} \qquad \ldots(42)$$

An examination of these matrices indicates that interaction effects will be smaller for the choice of approximation (40), but the choice of approximation (39) will, for a given value of k, produce a faster response. As speed of response can always be increased by increasing the chosen value of k, the design requirement of low interaction leads to the choice of approximation (40).

The design of a suitable proportional controller can now proceed using the analysis of section 3.1 and $G_A(s)$ as an approximation to G(s). From equations (24), (34), (40), (as),

$$p_1 = \frac{25}{24} \quad , \quad p_2 = 1 \qquad \ldots(43)$$

from which, using equations (26), (27),

$$c_1(k) = \frac{25}{24}k + 2 - 1$$

$$c_2(k) = k - (-\tfrac{1}{4}) \qquad \ldots(44)$$

so that the closed-loop system will be stable provided $k>0$. Choosing $k = 20$ for a reasonably fast response, the proposed controller is (eqn.23)

$$K(s) = 20\begin{bmatrix} 28 & -2 \\ 20 & 2 \end{bmatrix}^{-1} - \begin{bmatrix} 9 & -1 \\ 7 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{48}\begin{bmatrix} 17 & 17 \\ -179 & 253 \end{bmatrix} \qquad \ldots(45)$$

The closed-loop responses of the approximate closed-loop system $\{I+G_A(s)K(s)\}^{-1}G_A(s)K(s)$ are given in Fig.2 where it is seen that steady-state errors are predicted to be of the order of 15% and interaction negligible. The responses of the actual closed-loop system are given in Fig.3, where it is seen that interaction effects are greater than predicted by the approximation (but still acceptable) and steady state errors in channel one in response to a step demand are increased to approximately 20%.

Residual errors can now be removed by the application of integral action. Using equations (32), (44) and (43), closed-loop stability will be retained if

$$0 < c < 20 \qquad \ldots(46)$$

Choosing c = 5, the proportional plus integral controller becomes (eqn(31))

$$K(s) = \left\{25 + \frac{100}{s}\right\}\begin{bmatrix} 28 & -2 \\ 20 & 2 \end{bmatrix}^{-1} - \begin{bmatrix} 9 & -1 \\ 7 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{48}\begin{bmatrix} 22 & 22 \\ -229 & 323 \end{bmatrix} + \frac{100}{48s}\begin{bmatrix} 1 & 1 \\ -10 & 14 \end{bmatrix} \qquad \ldots(47)$$

The closed-loop responsesof the system $\{I+G(s)K(s)\}^{-1}G(s)K(s)$ are given in Fig.4.

In summary, the analytical techniques discussed in section 3 has provided a straightforward and efficient procedure for the design of a proportional plus integral controller for the given system. The controller structure is easily calculated from the transfer function matrix G(s) and, at each stage in the design process, insight can be obtained into the required proportional gains and integral times required to attain desired response characteristics and interaction effects. An important observation is the absence of any frequency response analysis analogous to those of the characteristic locus[3,8] and inverse Nyquist array techniques[1]. These are not explicitly required in the synthesis procedure due to the guarantee of stability and the strong intuitive link between the time-domain dynamics and transfer function matrix analysis.

4. Conclusions

Using the observation that many physical multivariable systems can, for the purposes of control systems design, be approximated by much simpler forms, and that the analysis of such structures can provide a valuable link between the time and frequency domain analysis of such systems, the paper has provided a theoretical control analysis of two such simple structures. The first (eqn.(1)) has been shown to be a direct multivariable generalization of the classical first order system and can be suitably controlled using a PI controller (eqn.(8)) which is a direct generalization of the equivalent classical controller. The second structure (eqn.(18)) is a multivariable generalization of the classical unity rank, minimum phase, overdamped second order system and can be controlled by firstly reducing the system to a first order type using the procedure of appendix 9 and then applying a multivariable feedback control based upon this approximation. In both cases the analysis provides closed-form solutions for controllers capable of ensuring closed-loop stability, accuracy and low interaction effects.

The analysis of section 2 is easily extended to incorporate the use of complex conjugate poles in G(s), but the model reduction philosophy inherent in the proposed analysis of second-order type structures, in general, precludes its application to systems with complex poles unless b_{N+1} is real (when $G_A(s)$ is then a physically realizable transfer function matrix). Despite this limitation, the success of the approach used in the paper indicates that the analysis of more general multivariable structures could be a useful tool in practical applications and that such results could suggest trial controllers to initiate the analysis of more complex systems using general design techniques.

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Appendices

6. Proof of Result 1

As $|G(o)| \neq 0$, then $\{\alpha_j\}_{1\leq j\leq N}$ and $\{\beta_j\}_{1\leq j\leq N}$ are sets of linearly independent vectors and hence it is possible to define sets of vectors $\{\gamma_j\}_{1\leq j\leq N}$ and $\{\psi_j\}_{1\leq j\leq N}$ such that, $1\leq j,\ k\leq N$,

$$\gamma_j^+ \alpha_k = \beta_j^+ \psi_k = \delta_{jk} \qquad \ldots(48)$$

Writing (eqn.(2)) $G_\infty = \sum_{j=1}^{N} b_j \alpha_j \beta_j^+ = \sum_{j=1}^{N} \alpha_j \beta_j^+ \sum_{k=1}^{N} b_k \psi_k \beta_k^+$ then (eqn.(48))

$|G_\infty| = |G(o)| \prod_{j=1}^{N} b_j \neq 0$ and hence G_∞ is invertible, and

$$G_\infty^{-1} = \sum_{j=1}^{N} b_j^{-1} \psi_j \gamma_j^+ \qquad \ldots(49)$$

In a similar manner,

$$G^{-1}(o) = \sum_{j=1}^{N} \psi_j \gamma_j^+ \qquad \ldots(50)$$

so that (eqn.3)

$$K(s) = \sum_{j=1}^{N} \frac{k-b_j}{b_j} \psi_j \gamma_j^+ \qquad \ldots(51)$$

Noting that $\sum_{j=1}^{N} \alpha_j \gamma_j^+ = I_N$, eqns (1),(51) and (48) give

$$G(s)K(s) = \sum_{j=1}^{N} \frac{k-b_j}{s+b_j} \alpha_j \gamma_j^+ \qquad \ldots(52)$$

$$T(s) = I + G(s)K(s) = \sum_{j=1}^{N} \{1 + \frac{k-b_j}{s+b_j}\} \alpha_j \gamma_j^+ \qquad \ldots(53)$$

$$T^{-1}(s) = \sum_{j=1}^{N} \frac{s+b_j}{s+k} \alpha_j \gamma_j^+ \qquad \ldots(54)$$

and the return difference determinant $|T(s)|$ is

$$|T(s)| = (s+k)^N / \prod_{j=1}^{N} (s+b_j) \qquad \ldots(55)$$

The closed-loop transfer function matrix is (eqns(54),(52))

$$T^{-1}(s)G(s)K(s) = \frac{k}{s+k} M(k) \qquad \ldots(56)$$

where

$$M(k) = \sum_{j=1}^{N} \frac{k-b_j}{k} \alpha_j \gamma_j^+ \qquad \ldots(57)$$

and hence $\lim_{k\to\infty} M(k) = \sum_{j=1}^{N} \alpha_j \gamma_j^+ = I_N$.

7. Proof of Result 2

From eqns (8), (49), (51)

$$K(s) = \sum_{j=1}^{N} \left\{ \frac{(k+c-b_j)}{b_j} + \frac{kcb_j^{-1}}{s} \right\} \psi_j \gamma_j^+ \qquad \ldots(58)$$

and hence

$$G(s)K(s) = \sum_{j=1}^{N} \frac{\{(k+c-b_j)s + kc\}}{s(s+b_j)} \alpha_j \gamma_j^+ \qquad \ldots(59)$$

so that the return-difference matrix is

$$T(s) = \{s^2 + (k+c)s + kc\} \sum_{j=1}^{N} \{s(s+b_j)\}^{-1} \alpha_j \gamma_j^+ \qquad \ldots(60)$$

and the closed-loop transfer function matrix

$$T^{-1}(s)G(s)K(s) = \frac{1}{(s+k)(s+c)} \sum_{j=1}^{N} \{(k+c-b_j)s + kc\} \alpha_j \gamma_j^+ \qquad \ldots(61)$$

The return-difference determinant is (eqns (60),(48))

$$|T(s)| = (s+k)^N (s+c)^N / s^N \prod_{j=1}^{N} (s+b_j) \qquad \ldots(62)$$

so that the closed-loop poles are simple poles at s = -k, s = -c, each with algebraic multiplicity N. From eqn.(61), if the residue at the pole s = -k is R(k), then

$$k^{-1}R(k) = \sum_{j=1}^{N} (k-b_j)(k-c)^{-1}\alpha_j\gamma_j^+$$

$$\rightarrow \sum_{j=1}^{N} \alpha_j\gamma_j^+ = I_N \qquad (k\to\infty) \qquad \ldots(63)$$

and, in a similar manner,

$$R(c) = \sum_{j=1}^{N} (b_j-c)(k-c)^{-1}\alpha_j\gamma_j^+$$

$$\rightarrow 0 \qquad (k\to\infty) \qquad \ldots(64)$$

8. Analysis of Second-order Type Multivariable Structures: Result 3

8.1 Relationship between $|G(s)|$, $|G_A(s)|$ and $P_G(s)$

Using the result of appendix 9, construct $G_A(s)$ (eqn(21)) such that $|G_A(o)|/|G(o)|>0$. Hence

$$|G(s)| = \left|G_A(s) + \frac{b_{N+1}}{s+b_{N+1}} \alpha_{N+1}\beta_{N+1}^+\right|$$

$$= |G_A(s)|\cdot\left|I + \frac{b_{N+1}}{s+b_{N+1}} G_A^{-1}(s)\alpha_{N+1}\beta_{N+1}^+\right| \qquad \ldots(65)$$

so that, using a well-known determinental identity[5] and equations (21), (48), (49), (50),

$$\frac{|G(s)|}{|G_A(s)|} = 1 + \frac{b_{N+1}}{s+b_{N+1}} \beta_{N+1}^{+}\{\sum_{j=1}^{N} \frac{(s+b_j)}{b_j} \psi_j \gamma_j^{+}\}\alpha_{N+1}$$

$$= 1 + \frac{b_{N+1}}{s+b_{N+1}} \beta_{N+1}^{+}\{sG_{A\infty}^{-1} + G_A^{-1}(o)\}\alpha_{N+1} \quad \ldots(66)$$

or, using equation (24) and (25),

$$\frac{|G(s)|}{|G_A(s)|} = \frac{p_1 s + p_2}{s+b_{N+1}} \quad \ldots(67)$$

As $|G_A(o)|/|G(o)|>0$ and $b_{N+1}>0$, it follows that $p_2>0$.
Using the relation (eqns(48), (21))

$$|G_A(s)| = |G_A(o)| \cdot \prod_{j=1}^{N} b_j(s+b_j)^{-1} \quad \ldots(68)$$

then (eqns(19),(67))

$$P_G(s) = \{p_1 s + p_2\} \; |G_A(o)| \cdot \prod_{j=1}^{N} b_j \quad \ldots(69)$$

By assumption $P_G(s)$ has a zero in the open left-half complex plane at $z = -p_2/p_1<0$ and hence $p_1>0$.

8.2 Proportional Control Analysis

If $T_A(s) = I + G_A(s)K(s)$ then the return difference determinant of the system $|T(s)|$ is

$$|T(s)| = |T_A(s) + \frac{b_{N+1}}{s+b_{N+1}} \alpha_{N+1}\beta_{N+1}^{+} K(s)| \quad \ldots(70)$$

so that, after some manipulation

$$|T(s)|/|T_A(s)| = 1 + \beta_{N+1}^{+}K(s)T_A^{-1}(s)\alpha_{N+1} \frac{b_{N+1}}{s+b_{N+1}} \quad \ldots(71)$$

Using equations (54),(23)

$$\frac{|T(s)|}{|T_A(s)|} = 1 + \beta_{N+1}^{+}\{kG_{A\infty}^{-1} - G_A^{-1}(o)\} \sum_{j=1}^{N} \frac{(s+b_j)}{s+k} \alpha_j \gamma_j^{+} \alpha_{N+1} \frac{b_{N+1}}{s+b_{N+1}} \quad \ldots(72)$$

or, using equations (26),(27) and noting that $\sum_{j=1}^{N} (s+b_j)\alpha_j\gamma_j^{+} = sI_N + G_{A\infty}G_A^{-1}(o)$ (eqns (2),(48) and (50)) then

$$\frac{|T(s)|}{|T_A(s)|} = \frac{s^2 + c_1(k)s + c_2(k)}{(s+k)(s+b_{N+1})} \quad \ldots(73)$$

Relation (28) now follows by substitution for $T_A(s)$ from equation (55), and relations (29) follow by expanding the roots of the numerator of eqn.(73) as a Taylor series.

Insight can be gained into the closed-loop transient response by a residue analysis of the error matrix

$$T^{-1}(s) = \{T_A(s) + \frac{b_{N+1}}{s+b_{N+1}} \alpha_{N+1}\beta_{N+1}^{+} K(s)\}^{-1}$$

$$= \{I + \frac{b_{N+1}}{s+b_{N+1}} T_A^{-1}(s)\alpha_{N+1}\beta_{N+1}^{+}K(s)\}^{-1} T_A^{-1}(s)$$

$$= \{I - \frac{\frac{b_{N+1}}{s+b_{N+1}} T_A^{-1}(s)\alpha_{N+1}\beta_{N+1}^{+}K(s)}{1 + \frac{b_{N+1}}{s+b_{N+1}} \beta_{N+1}^{+}K(s)T_A^{-1}(s)\alpha_{N+1}}\} T_A^{-1}(s) \quad \ldots(74)$$

which, using equations (71),(73),(29),(54),(23) becomes

$$T^{-1}(s) = \{I - b_{N+1} \sum_{j=1}^{N} \frac{(s+b_j)\alpha_j\gamma_j^{+}\alpha_{N+1}\beta_{N+1}^{+}\{kG_{A\infty}^{-1} - G_A^{-1}(o)\}}{(s - \mu_1(k))(s - \mu_2(k))}\}$$

$$\times \sum_{j=1}^{N} \frac{(s+b_j)}{(s+k)} \alpha_j\gamma_j^{+} \quad \ldots(75)$$

Equation (30) follows directly from eqns (75) and (29).

Closed-loop interaction effects can be established by using the well-known initial value theorem to evaluate the output derivatives at $t = 0+$ in response to step demands in output. These are represented by the matrix, normalized with respect to k,

$$\lim_{k\to+\infty} \lim_{s\to\infty} k^{-1} s\, G(s)K(s) = \lim_{k\to\infty} k^{-1}\{G_{A\infty} + b_{N+1}\alpha_{N+1}\beta_{N+1}^{+}\}\{kG_{A\infty}^{-1} - G_A^{-1}(o)\}$$

$$= C_A \qquad \text{(see eqn (22))} \qquad \ldots(76)$$

This matrix is non-diagonal in general so closed-loop interaction effects will be present, but, bearing in mind the result of eqn (30) and the fact that all closed-loop poles tend to $-\infty$ as $k\to+\infty$ (except $\mu_1(k)$), the interaction can only be of significance in an interval of length of order k^{-1}.

8.3 Proportional plus Integral Control

In a similar manner to the analysis of section 8.2, the return difference determinant is given by equation (71) which using equations (31),(60) becomes

$$\frac{|T(s)|}{|T_A(s)|} = 1 + \frac{b_{N+1}}{s+b_{N+1}}\beta_{N+1}^{+} \frac{\{\{(k+c)G_{A\infty}^{-1} - G_A^{-1}(o)\}s + kcG_{A\infty}^{-1}\}}{(s+k)(s+c)} \sum_{j=1}^{N}(s+b_j)\alpha_j\gamma_j^{+}\alpha_{N+1} \qquad \ldots(77)$$

Simple manipulations, the use of the identity $\sum_{j=1}^{N}(s+b_j)\alpha_j\gamma_j^{+} = sI + G_{A\infty}G_A^{-1}(o)$ and equations (26),(27) yields the form

$$\frac{|T(s)|}{|T_A(s)|} = \frac{s^3 + c_1(k+c)s^2 + \{kcp_1 + c_2(k+c)\}s + kcp_2}{(s+b_{N+1})(s+k)(s+c)} \qquad \ldots(78)$$

Hence, if $|T_A(s)|$ is stable, elementary stability tests[10] imply that the system is closed-loop stable if, and only if,

$$c_1(k+c)\{kcp_1 + c_2(k+c)\} - kcp_2 > 0 \qquad \ldots(79)$$

$$c_1(k+c) > 0 \qquad \ldots(80)$$

$$kcp_2 > 0 \qquad \ldots(81)$$

Equation (81) is equivalent to c>0 and hence, if the closed-loop system is stable with proportional control only, equation (80) is automatically satisfied as $c_1(k+c) > c_1(k) > 0$. Relations (36) follow directly by noting that $c_1(k+c)\{kcp_1+c_2(k+c)\} > c_1(k)c_2(k)$.

9. Result 4

Given a transfer function matrix of the form of equation (18) where $|G(o)| \neq 0$ then, by suitable reordering of terms, it is possible to define a reduced model

$$G_A(s) = \sum_{j=1}^{N} \frac{b_j}{s+b_j} \alpha_j \beta_j^+ \qquad \ldots(82)$$

where $|G_A(o)|/|G(o)| > 0$.

Proof

Note firstly that, using simple column operations and extraction of scalar factors, if $s_j \neq 0$, $1 \leq j \leq N-1$,

$$\begin{vmatrix} q_1 & q_2 & \cdots & \cdots & q_N + t_1 \\ s_1 & 0 & \cdots & \cdots & t_2 \\ 0 & s_2 & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ \vdots & & & 0 & t_{N-1} \\ 0 & \cdots & 0 & s_{N-1} & t_N \end{vmatrix}$$

$$= (-1)^{N-1} \prod_{j=1}^{N-1} s_j \{q_N + t_1 - \sum_{j=2}^{N} \frac{t_j q_{j-1}}{s_{j-1}}\}$$

$$= (-1)^{N-1}\{\{q_N+t_1\} \prod_{j=1}^{N-1} s_j - \sum_{j=2}^{N} t_j q_{j-1} \prod_{\substack{k=1 \\ k\neq j-1}}^{N-1} s_k\} \qquad \ldots(83)$$

and, by continuity arguments, the result still holds if any of the $\{s_k\}_{1\leq k\leq N-1}$ are zero.

As $|G(s)| \neq 0$, it is possible to assume that $\{\alpha_j\}_{1\leqslant j\leqslant N}$ and $\{\beta_j\}_{2\leqslant j\leqslant N+1}$ are sets of linearly independent vectors and hence to define non-singular matrices U,V such that $U\alpha_j = e_j$, $1\leqslant j\leqslant N$ and $\beta_j{}^+V = e_{j-1}{}^T$, $2\leqslant j\leqslant N+1$. Hence, if $\beta_1{}^+V = [q_1,q_2,..,q_N]$ and $U\alpha_{N+1} = [t_1,t_2,..,t_N]^T$

$$UG(o)V = e_1\beta_1{}^+V + \sum_{j=2}^{N} e_j e_{j-1}{}^T + U\alpha_{N+1}e_N{}^T$$

$$= \begin{vmatrix} q_1 & q_2 & \cdots & q_{N-1} & q_N + t_1 \\ 1 & 0 & \cdots & 0 & t_2 \\ 0 & 1 & & \vdots & \vdots \\ & & & 0 & \vdots \\ 0 & \cdots & 0 & 1 & t_N \end{vmatrix} \qquad \ldots(84)$$

and, by comparison with (83),

$$|UG(o)V| = (-1)^{N-1}\{q_N+t_1 - \sum_{j=2}^{N} t_j q_{j-1}\} \qquad \ldots(85)$$

Without loss of generality, take $(-1)^{N-1}|G(o)|>0$ so that at least one of the terms $q_N, t_1, -t_2q_1, \ldots, -t_Nq_{N-1}$ must be strictly positive.

(i) If $q_N>0$, define $G_A(s)$ by eliminating the pole of G(s) at $s = -b_{N+1}$. The value of $|UG_A(o)V|$ is obtained directly from (85) by setting $t_1 = t_2 = \ldots = t_N = 0$. That is $|UG_A(o)V| = (-1)^{N-1}q_N$ or $|UG_A(o)V|/|UG(o)V| = |G_A(o)|/|G(o)| > 0$ as required.

(ii) If $t_1>0$, define $G_A(s)$ by eliminating the pole of G(s) at $s = -b_1$. The value of $|UG_A(o)V|$ is obtained from (85) by setting $q_1 = q_2 = .. = q_N = 0$ so that $|UG_A(o)V| = (-1)^{N-1}t_1$ and hence $|G_A(o)|/|G(o)| > 0$.

(iii) If $-t_\ell q_{\ell-1}>0$, define $G_A(s)$ by eliminating the pole of G(s) at $s = -b_\ell$.

$|UG_A(o)V|$ is obtained directly from (83) by setting $s_i = 1$, $i \neq \ell-1$, $s_{\ell-1} = 0$. That is $|UG_A(o)V| = (-1)^{N-1}(-t_\ell q_{\ell-1})$ and hence $|G_A(o)|/|G(o)| > 0$.

QED

The proof of the result provides a direct method for the construction of $G_A(s)$. In practice however, the most practical technique is probably simple trial and error.

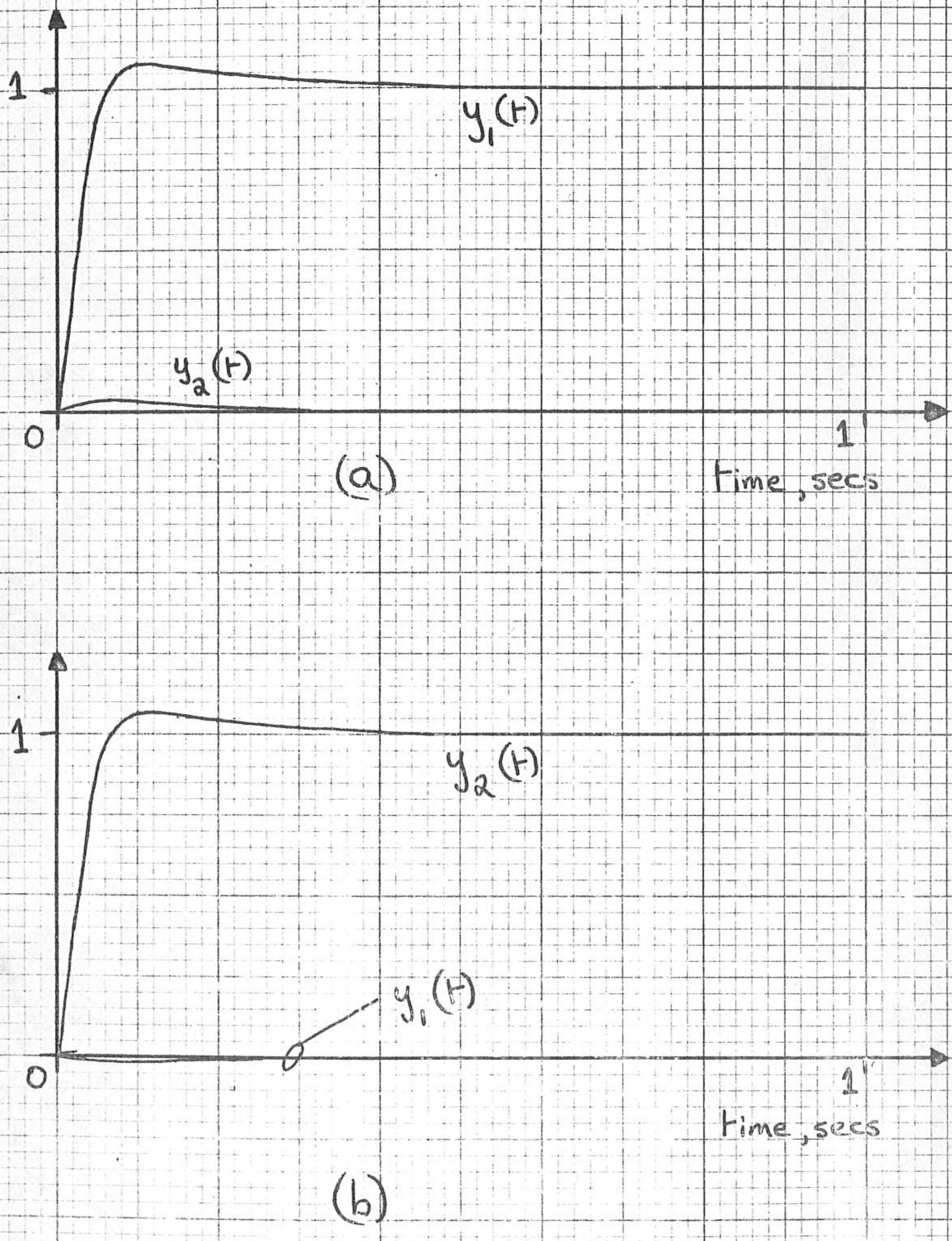

Fig. 1.

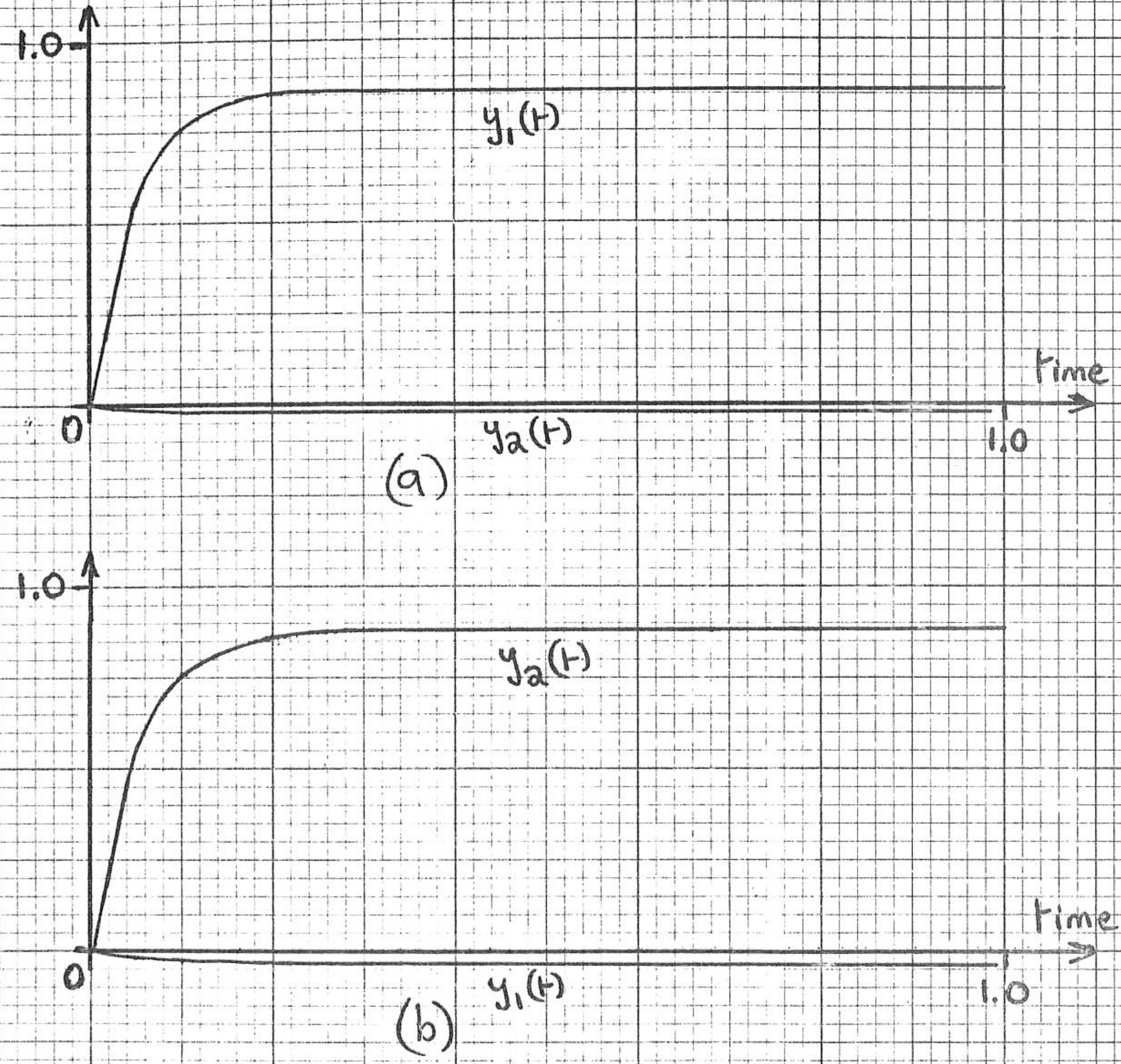

Fig.2

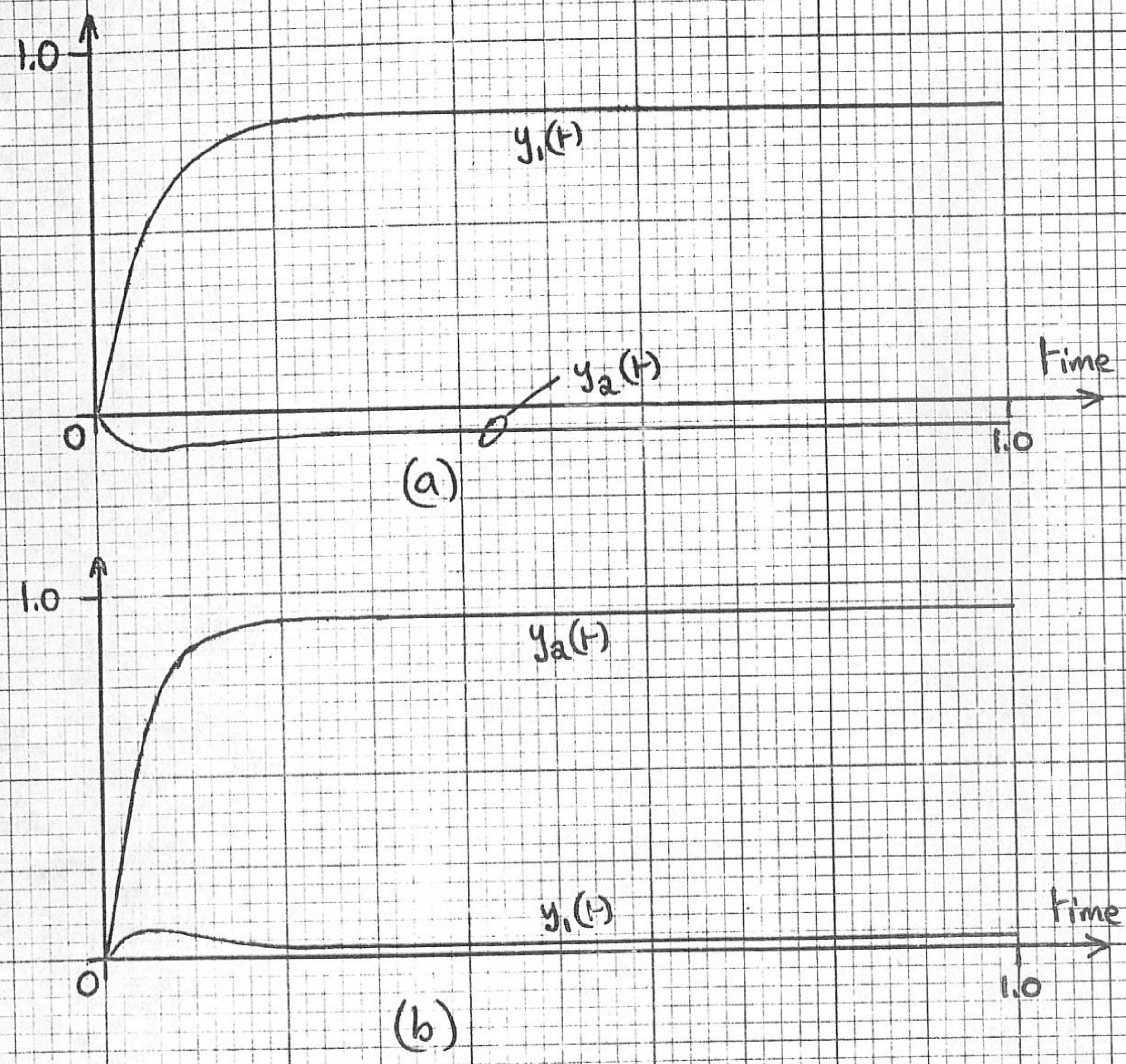

Fig. 3.

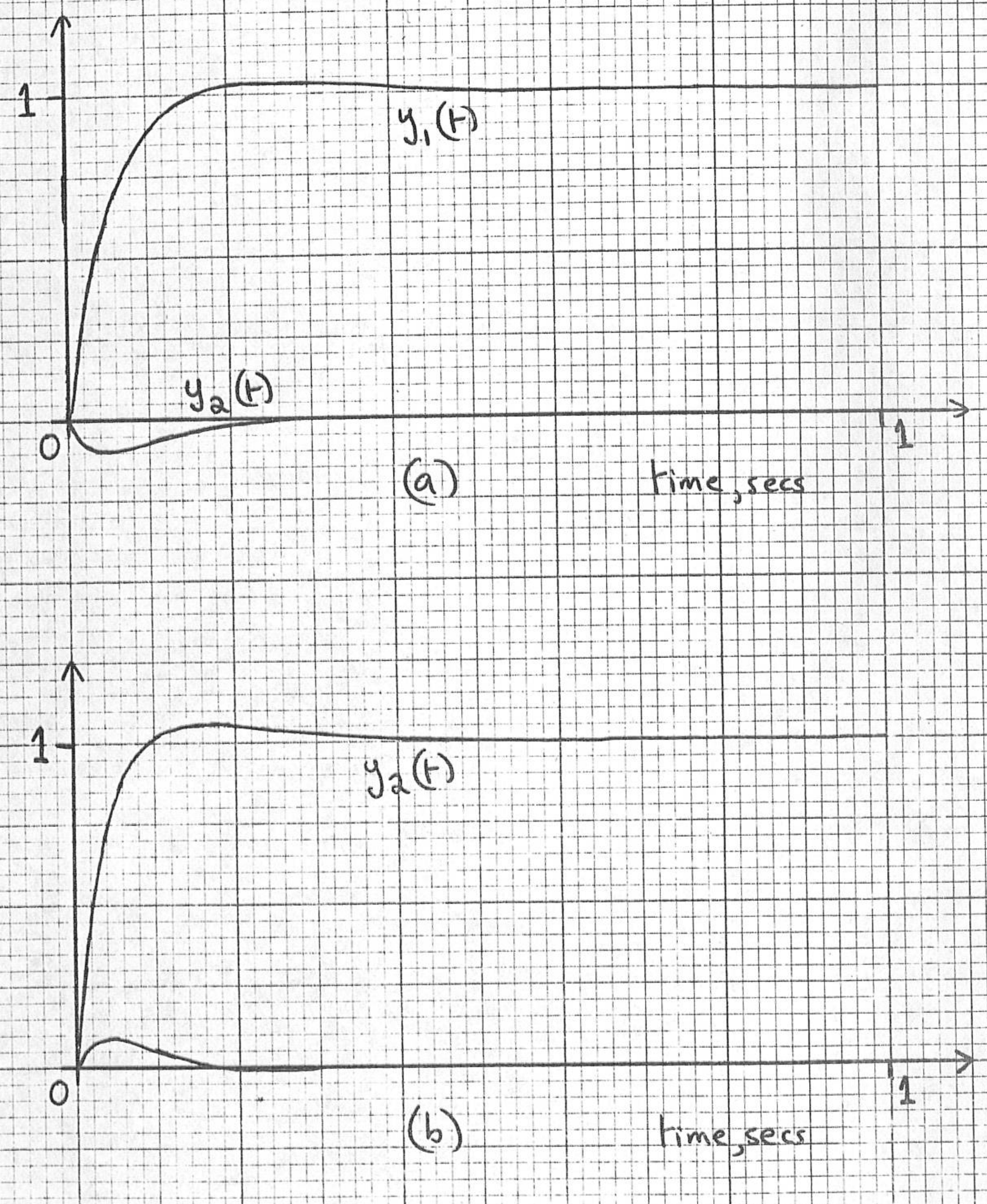

Fig. 4

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