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MODAL DECOUPLING AND DYADIC TRANSFER FUNCTION MATRICES

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Abstract

Previous results on the feedback control analysis of dyadic transfer function matrices are related to the general concept of modal decoupling and the techniques extended to cope with the case of unbounded or singular D.C. matrices.

The concept of dyadic approximation[1] has been extended[2,3] to provide a systematic approach to the manipulation and compensation of the characteristic loci of a system described by an NxN transfer function matrix G(s). This letter uses the concepts and notation of Ref. 3 to extend the definition and analysis of dyadic transfer function matrices[1] to include the possibility of unbounded or singular G(o).

For the purpose of this letter, on NxN dyadic transfer function matrix G(s) takes the form

$$G(s) = \sum_{j=1}^{N} g_j(s)\alpha_j \beta_j^+ \qquad \ldots.(1)$$

where the previous requirement[1] that $|G(o)| \neq o$ and finite is replaced by the requirement $|G(s)| \not\equiv o$. As before $\{g_j(s)\}_{1\leq j\leq N}$ are rational scalar transfer functions and $\{\alpha_j\}_{1\leq j\leq N}, \{\beta_j\}_{1\leq j\leq N}$ are sets of linearly independent vectors such that if $\alpha_\ell = \overline{\alpha}_j$ then $\beta_\ell = \overline{\beta}_j$. The possibility of complex dyads is allowed mathematically but, if a physical interpretation is desired[1], it is necessary that the dyads $\{\alpha_j\beta_j^+\}_{1\leq j\leq N}$ be real.

The decoupling matrix [3] of G(s) at any point $s=iw_1$ is

$$K_D(\omega_1) = \{\sum_{j=1}^{N} \alpha_j \beta_j^+\}^{-1} \qquad \ldots.(2)$$

which is non- singular real and independent of frequency ω_1. Moreover[3] $K_D^{-1}(\omega_1)\, K_D(\omega_1) = I_N$ implies that $\beta_j^{+} K_D(\omega_1)\alpha_k = \delta_{jk}$, or , for every s,

$$G(s)\, K_D(\omega_1)\, \alpha_j = g_j(s)\, \alpha_j \qquad \ldots.(3)$$

so that $\{\alpha_j\}_{1\leq j\leq N}$ and $\{g_j(s)\}_{1\leq j\leq N}$ are the eigenvectors and eigenvalues respectively of $G(s)\, K_D(\omega_1)$. Equivalently, if[3]

$$T(\omega_1) = [\alpha_1, \alpha_2, \ldots, \alpha_N] \qquad \ldots.(4)$$

then $T^{-1}(\omega_1)$ exists, is independent of frequency and

$$T^{-1}(\omega_1)\, G(s)\, K_D(\omega_1)\, T(\omega_1) = \mathrm{diag}\{g_1(s), \ldots, g_N(s)\} \qquad \ldots.(5)$$

Consider the unity negative feedback control configuration for the system $G(s)$ and let the forward path controller transfer function matrix $K(s)$ take the form[3]

$$K(s) = K_D(\omega_1)\, T(\omega_1)\, \mathrm{diag}\{k_1(s), \ldots, k_N(s)\}\, T^{-1}(\omega_1) \qquad \ldots.(6)$$

where $\{k_j(s)\}_{1\leq j\leq N}$ are rational scalar controller transfer functions. It follows directly that

$$T^{-1}(\omega_1)\, G(s)\, K(s)\, T(\omega_1) = \mathrm{diag}\{g_1(s)k_1(s), \ldots, g_N(s)k_N(s)\} \qquad \ldots.(7)$$

so that the closed-loop transfer function matrix of the system takes the form

$$\{I_N + G(s)K(s)\}^{-1}G(s)K(s) = T(\omega_1)\,\mathrm{diag}\left[\frac{g_j(s)k_j(s)}{1+g_j(s)k_j(s)}\right]T^{-1}(\omega_1) \quad \ldots.(8)$$

The choice of controller (eqn (6)) reduces the feedback control analysis to the analysis of N classical feedback systems $\{g_j(s)k_j(s)/(1+g_j(s)k_j(s))\}_{1\leq j\leq N}$ in a manner directly analogous to previous results[1]. Note the use of the decoupling matrix $K_D(\omega_1)$ (eqn.3) in the place of $G^{-1}(o)$ (see ref.1) which makes possible the control analysis of dyadic plants where G(o) is unbounded or singular.

A more convenient formula for the decoupling matrix can be obtained by noting that $K_D(\omega_1)$ is non-unique. To illustrate this point, let $\hat{g}_j(s) = \lambda_j g_j(s)$ and $\hat{\alpha}_j = \lambda_j^{-1}\alpha_j$, $1\leq j\leq N$, where $\{\lambda_j\}_{1\leq j\leq N}$ are non-zero real scalars. Then $G(s) = \sum_{j=1}^{N} \hat{g}_j(s)\hat{\alpha}_j\beta_j^+$ and hence $\{\sum_{j=1}^{N} \hat{\alpha}_j\beta_j^+\}^{-1}$ is a decoupling matrix for G(s). Let s_1 be a real number such that $|G(s_1)| \neq 0$ and finite. The existance of such an s_1 is guaranteed by the assumption that $|G(s)| \not\equiv 0$. It follows that $g_j(s_1) \neq 0$, $1\leq j\leq N$ and the above argument, with $\lambda_j = (g_j(s_1))^{-1}$, indicates that $G^{-1}(s_1)$ is a decoupling matrix for the system.

A systematic approach to the design of a unity negative feedback controller K(s) for the dyadic plant G(s) (eqn.(1)) could proceed as follows:-

STEP 1: Choose a real number s_1 such that $|G(s_1)| \neq 0$ and finite. If the system is minimum phase and stable then any $s_1 > o$ will suffice. In more general situations, trial and error techniques soon yield a suitable value.

STEP 2: Compute the eigenvectors $\{\alpha_j\}$ of $G(s)\,G^{-1}(s_1)$, the similarity transformation $T(\omega_1)$ and hence $\{g_j(s)\}_{1\leq j\leq N}$ from equation (5) with $K_D(\omega_1) = G^{-1}(s_1)$. For computational purposes, the calculation of $\{\alpha_j\}$ is best achieved by noting that $\{\alpha_j\}$ are the eigenvectors of

$G(s_2)G^{-1}(s_1)$ for any real number s_2.

STEP 3: Choose N scalar transfer functions $\{k_j(s)\}_{1 \le j \le N}$ so that the subsystems $\{g_j(s)k_j(s)/(1+g_j(s)k_j(s))\}_{1 \le j \le N}$ have satisfactory transient response and stability properties.

STEP 4: Setting $K_D(\omega_1) = G^{-1}(s_1)$, evaluate K(s) from equation (6).

To illustrate the simplicity of the technique, consider the transfer function matrix

$$G(s) = \frac{1}{s(s+1)} \begin{bmatrix} 1-s & 3s+1 \\ 1 & 2s+1 \end{bmatrix} \qquad \ldots.(9)$$

Note that G(o) is unbounded so previous results[1] do not apply. Choose $s_1 = 1.0$, then $|G(1)| \neq 0$ and

$$G(s)G^{-1}(1) = \frac{1}{s(s+1)} \begin{bmatrix} -1+3s & 2-2s \\ -1+s & 2 \end{bmatrix} \qquad \ldots(10)$$

Taking $s_2 = 2.0$, the eigenvectors of $G(2)G^{-1}(1)$ are $\alpha_1 = \{1\ ,\ 1\}^T$ and $\alpha_2 = \{2\ ,\ 1\}^T$. Defining $T = \{\alpha_1, \alpha_2\}$ then

$$T^{-1}G(s)G^{-1}(1)T = \text{diag}\left\{\frac{1}{s}\ ,\ \frac{2}{s+1}\right\} \qquad \ldots(11)$$

Hence G(s) is, in fact, dyadic and the design can proceed in a straightforward manner.

References

1. D. H. Owens: 'Dyadic Approximation Method for Multivariable Control Systems Analysis with a Nuclear Reactor Application', Proc. IEE, 120 (7), 1973, pp.801-809.

2. D. H. Owens: 'Dyadic Expansion for the Analysis of Linear Multivariable Systems', Proc. IEE, 121 (7), 1974, pp.713-716.

3. D. H. Owens: 'Dyadic Expansion, Characteristic Loci and Multivariable-control-systems design', Proc. IEE, 122 (3), 1975, pp.315-320.