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Properties of the generalised inverse matrix in the electrical network problem

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Summary

A modified form of Roth's transformation diagram for a linear graph is used to illustrate the solution of the electrical network problem. The diagram illustrates particularly the significance of the orthogonal projections of the branch space into the branch voltage and current subspaces which are defined by Kirchoff's laws, and also the existence of the constrained matrix inverse which forms a basis for the solution of the electrical network problem. Properties of the generalised inverse matrix are also discussed in relation to the network problem.

The electrical network problem includes an algebraic structure relating the physical variables together with a topological or graph representing the interconnection of the network elements. The conjugate variables of voltage and current are related by Ohm's law, and Kirchoff's laws constrain the branch voltages and currents to orthogonal complementary subspaces. The solution of the electrical network problem introduces properties of a constrained matrix inverse[1] and, in the general case, includes the concept of a minimum-norm generalised inverse matrix.

The algebraic relationships forming the solution of the network problem can be represented by means of Roth's transformation diagram[2], and a modified diagram is developed which illustrates, particularly, the existence of the constrained inverse matrix and the orthogonal projections associated with Kirchoff's laws. The transformation diagram has an important application in illustrating the various forms of solution available for problems which can be identified with a linear graph, and is particularly important in an extended form in the study of higher-dimensional networks.

The general electrical network problem includes the interconnection of a set of branches, and with the voltage and current variables defined as in FIG 1, b-branch equations are specified by[2]

$$\begin{aligned} E + e &= Z(I + i) & \qquad I + i &= Y(E + e) \\ \text{or} \quad V &= ZJ & J &= YV \end{aligned} \tag{1}$$

where Z, Y represent symmetrical impedance and admittance matrices respectively for the primitive network.

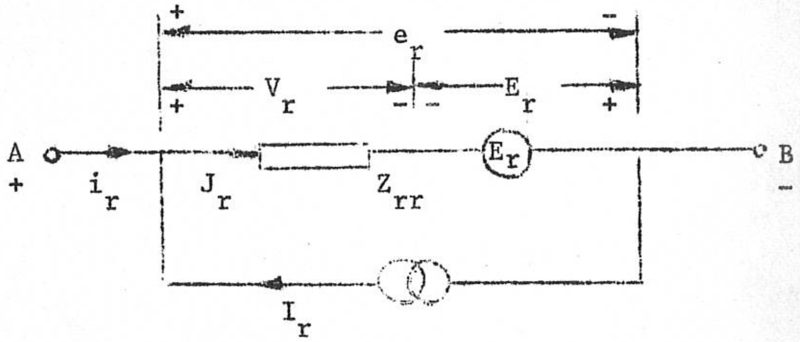

FIG 1 rth network branch

The structure of the connected network is defined by the branch-node-pair matrix A and branch-mesh matrix C, which are related by the orthogonality condition

$$A^T C = 0 \qquad\qquad C^T A = 0 \tag{2}$$

With m meshes and p node pairs, A and C are of dimension bxp and bxm respectively, and rank A = p, rank C = m. The branch variables are constrained by Kirchoff's voltage and current laws given by

$$C^T e = 0 \qquad\qquad A^T i = 0 \tag{3}$$

The branch variables e,i and the node-to-datum voltages e' and the currents in the basic meshes i' are also related by

$$e = Ae' \qquad\qquad i = Ci' \tag{4}$$

in which only p branch voltages and m mesh currents are linearly independent. With arbitrary sources E,I the equivalent induced mesh-voltage and nodal-current sources are given by

$$E' = C^T E \qquad\qquad I' = A^T I \tag{5}$$

Eqns 1-5 may now be combined to give the solution for mesh and branch currents

$$i' = (C^T ZC)^{-1} C^T Z(YE - I) \tag{6}$$

$$i = LZ(YE - I) \tag{7}$$

where $$L = C(C^T ZC)^{-1} C^T = Y - YA(A^T YA)^{-1} A^T Y = Y - YMY \tag{8}$$

is the branch-admittance matrix (of driving point and transfer admittances) and

$$M = A(A^T YA)^{-1} A^T = Z - ZC(C^T ZC)^{-1} C^T Z = Z - ZLZ \tag{9}$$

is the branch-impedance matrix. Similarly, the node-to-datum and branch voltages are given by

$$e' = (A^TYA)^{-1}A^TY(ZI - E) \tag{10}$$

$$e = MY(ZI - E) \tag{11}$$

and

$$MY + ZL = I_b \tag{12}$$

The coil variables may then be specified in the form

$$\begin{bmatrix} V \\ J \end{bmatrix} = \begin{bmatrix} ZL & Z-ZLZ \\ L & I_b-LZ \end{bmatrix} \begin{bmatrix} E \\ I \end{bmatrix} = N \begin{bmatrix} E \\ I \end{bmatrix} \tag{13}$$

It is of interest to note that the components of the matrix N possess properties similar to those of the components of the matrix star-product associated with a scattering process.[3,4] It is also significant that a correspondence exists with the components of the matrix differential representation of the general scattering process. It may then be possible to consider conditions of energy dissipation in terms of the inequality properties of a scattering matrix on the basis of this correspondence.

The algebraic relationships established for the network problem may be illustrated by means of Roth's transformation diagram[2] shown in FIG 2.

$$\begin{tikzcd}
i' \arrow[r, "C"] \arrow[d, shift right, "C^TZC"'] & J \arrow[r, "A^T"] \arrow[d, shift right, "Z"'] & I' \arrow[d, shift right, "(A^TYA)^{-1}"'] & \text{— 'through' vars} \\
E' \arrow[u, shift right, "(C^TZC)^{-1}"'] & V \arrow[l, "C^T"] \arrow[u, shift right, "Y"'] & e' \arrow[l, "A"] \arrow[u, shift right, "A^TYA"'] & \text{— 'across' vars}
\end{tikzcd}$$

(mesh) (branch) (node)

FIG 2 Roth's transformation diagram

The components of eqn 13 relating the coil and source variables may also be represented by a transformation or signal-flow-type diagram, as in FIG 3.

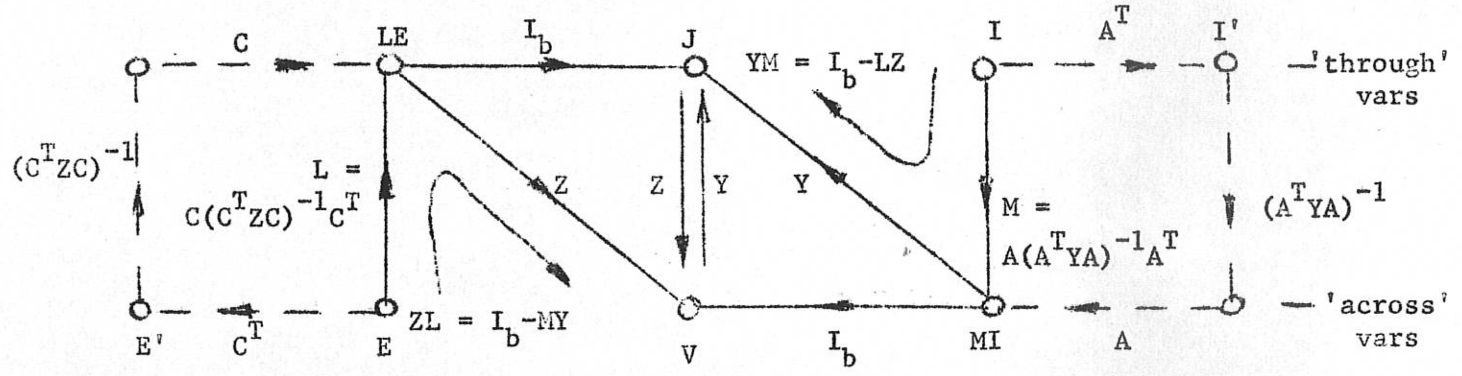

FIG 3 Transformation diagram relating coil and source variables

The transformation diagram of FIG 3 includes the basic characteristics of Roth's diagram, with the impedance- and admittance-type operators directed across the diagram between the conjugate 'through' and 'across' variables, and with the dimensionless or connection-type operators directed horizontally between similar variables. The diagram illustrates particularly the properties of the matrices L and M representing the branch-admittance and branch-impedance matrices respectively, and also the contribution of the internal sources E,I to the coil variables J,V via the transformed variables LE and MI. The open transformations $E \to V$, $I \to J$ appear as dual operators based on the property of eqn 12, and can be identified with the existence of a 'residual'-type component. A coil power function may also be defined in terms of scalar products obtained directly from the transformation diagram. Thus

$$\begin{aligned} P &= V^TJ = (MI + ZLE)^T (YMI + LE) \\ &= I^TMI + E^TLE \end{aligned} \tag{14}$$

The solution of the electrical network problem given by eqn 13 includes properties of the constrained matrix inverse.[1] The matrix M of eqn 9 represents the constrained inverse $Y^+_{\mathcal{E}}$ of Y with respect to the subspace $\mathcal{E}$ to which all branch voltages (e) satisfying Kirchoff's voltage law of eqn 3 are constrained. It represents a transformation into $\mathcal{E}$ and MY is the identity in $\mathcal{E}$. The admittance matrix L similarly represents the constrained inverse $Z^+_{\mathcal{J}}$ with respect to the subspace $\mathcal{J}$ of all branch currents (i) which satisfy Kirchoff's current law of eqn 3. Kirchoff's laws define the orthogonal complementary voltage and current subspaces $\mathcal{E}, \mathcal{J}$ of the b-dimensional vector space U, and the vectors e,i are orthogonal with $e^T i = 0$. According to a theorem of Bott and Duffin,[1] using the notation of FIG 1, the equation

$$i - Ye = h \quad (= YE - I) \qquad e \in \mathcal{E},\ i \in \mathcal{J} \tag{15}$$

where h is an arbitrary vector of U, has a unique solution given by

$$e = -Mh = MI - MYE \tag{16}$$

$$i = (I - YM)h = LZh = LE - LZI \tag{17}$$

Eqns 16, 17 then correspond directly with the branch variable components of eqn 13. Properties of the constrained inverse matrix have also been associated with a system of connected elastic shafts.[1]

The solutions of eqn 13 also include properties of the generalised inverse of a singular matrix. Thus the equations

$$\begin{aligned} V &= MI + (I_b - MY)E \\ J &= LE + (I_b - LZ)I \end{aligned} \tag{19}$$

correspond directly with the solution of the problem

$$y = Ax \tag{20}$$

where A is a matrix of order mxn and rank r(<n), defined by[5]

$$x = A^{\dagger}y + (I_n - A^{\dagger}A)z \tag{21}$$

where $A^{\dagger}$ is the generalised inverse matrix of order nxm satisfying $AA^{\dagger}A = A$ and z is an arbitrary n-vector. The correspondence of eqns 18,19 with eqn 21 suggests that the coil voltage V includes a 'main' component MI obtained as a transformation of the branch current source I by the branch-impedance matrix M which may be compared with the generalised inverse matrix $A^{\dagger}$. A component $(I_b - MY)E$ also exists as a 'residual'-type contribution resulting from the arbitrary voltage source E, which compares with the component $(I_n - A^{\dagger}A)z$ in the general solution of eqn 21. A similar correspondence may be established with the component contributions to the coil current J in eqn 19. The matrices M,L are significant in many linear system problems, and possess properties similar to those of the generalised inverse matrix with

$$LZL = L \qquad MYM = M \tag{22}$$

and the matrices $(I_b - ZL)$ and $(I_b - MY)$ are symmetric and idempotent with

$$MYMY = MY \qquad ZLZL = ZL \tag{23}$$

and $ZLMY = MYZL = ML = 0$ (24)

The residual vector in the minimum-norm solution of eqn 20 is given by[6]

$$y - Ax = (I_m - AA^{\dagger})y \tag{25}$$

which by comparison with eqn 18 corresponds to

$$(I_b - YM)I = LZI \tag{26}$$

which can be identified with a 'residual' component of the branch current of eqn 7. Also in the generalised inverse problem, the sum of the squared residuals is a minimum given by

$$P = y^T(I_m - AA^{\dagger})y \tag{27}$$

and a similar form exists in the network problem with

$$P = I^T LZI \quad \text{or} \quad P = E^T MYE \tag{28}$$

A direct correspondence has been shown to exist between the general electrical network problem and the generalised-inverse matrix problem, the general solution of which will fit within the framework of the transformations associated with a linear graph. Other aspects of this correspondence have been discussed previously,[7-10] including particularly the relationship between the constrained and the generalised inverse matrix.[7] The solution of the electrical network problem has also been illustrated by means of a modified form of transformation diagram which highlights, particularly, the decomposition of the solutions for coil voltage and current into 'main' and 'residual'-type components and also the significance of the properties of the constrained inverse matrix. It would appear appropriate to associate a linear graph with the constrained or generalised inverse matrix problem, and the construction of an appropriate network and corresponding transformation diagram may have application in developing possible forms of solution.

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