Monograph:
Owens, D.H. (1973) A Characterisation of Interaction in Linear Multi-Variable Systems Using a Dyadic Expansion Method. Research Report. ACSE Research Report 15 . Department of Automatic Control and Systems Engineering

A CHARACTERISATION OF INTERACTION IN LINEAR MULTIVARIABLE SYSTEMS USING A DYADIC EXPANSION METHOD

by

D. H. OWENS

Research report No. 15

Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield, S1 3JD.

October 1973

ABSTRACT

In a practical situation, the control engineer may have experience with his system and have some insight into its dynamic behaviour. This paper suggests a theoretical method for making use of this background knowledge in the feedback control design process by manipulating the frequency response information available in the plant transfer function matrix H(s) into a form suitable for physical interpretation. The approach used is that of dyadic expansion of H(s). The technique represents an extension of the dyadic approximation method used in the analysis of nuclear reactor spatial control systems to include a description of interactions in a general case. However, under stated conditions, the representation given in this paper is exact. As the aim of the paper is to enable frequency response data to be interpreted physically, the method described does not generate a complete design technique. Examples indicate however that, if used with physical intuition, the procedure can provide guidelines to practical controller structures for highly interacting systems.

List of Symbols

s	Laplace transform variable
$y(s)$	Nx1 vector of output transforms
$u(s)$	Nx1 vector of control input transforms
$H(s)$	NxN matrix of plant transfer functions
$e(s)$	Nx1 vector of error transforms
$r(s)$	Nx1 vector of reference input transforms
$G(s)$	NxN matrix of control transfer functions
$K(s)$	NxN feedback compensator matrix
A^+	The adjoint or conjugate transpose of the matrix A
$\|\|A\|\|$	Norm of the matrix A
A^T	Transpose of the matrix A
e_j	Nx1 vector with a unit entry in position j and zeros elsewhere
$\|A\|$	Determinant of the matrix A

List of figures

1. General Multivariable Feedback Control System

2. Multivariable System Example

1. INTRODUCTION

The problem of the analysis and design of multivariable feedback control systems has been the subject of a large research effort over the past few decades, and many contributions have been made to a general mathematical theory of linear multivariable feedback control in particular.[1] A method currently receiving a lot of attention is the vector frequency response or transfer function matrix technique[1], which is a generalisation of the frequency response methods introduced by Bode, Nyquist etc... In this formulation the dynamic system under consideration is assumed to have a mathematical description of the form

$$y(s) = H(s)u(s) \qquad (1)$$

A general multivariable control system structure for this system is illustrated in Fig.1. The input/output relationships of such a configuration are as follows

$$\begin{aligned} y(s) &= H(s)u(s) \\ u(s) &= G(s)e(s) \\ e(s) &= r(s) - K(s)y(s) \end{aligned} \qquad (2)$$

or

$$y(s) = \{I + H(s)G(s)K(s)\}^{-1}H(s)G(s)r(s) \qquad (3)$$

The control problem is the choice of cascade and feedback control elements (as represented by the transfer function matrices G(s) and K(s)) such that the overall closed loop system (eqn(3)) satisfies the required stability and transient response specifications.

Results are available[1] describing necessary and sufficient conditions for the stability and integrity of the closed-loop system of Fig.1, and several methods have been suggested for the control design analysis. These techniques are based on ideas of (i) direct structural modification[2] to reduce the problem to the analysis of N single variable systems,

(ii) eigenvalue or characteristic locus analysis[3,4], (iii) loop by loop addition methods[5] and (iv) approximation concepts.[6]

In a practical situation, the control engineer may have direct experience with his system and may have obtained some useful insight into its dynamic behaviour which could be used to guide the control synthesis procedure. However, the methods listed above are not explicitly constructed to make maximum use of this information. The system matrix H(s) contains basic information on the overall dynamical structure and subsystem structure of the plant and also defines the contributions of these subsystems to the outputs and the effect of the control inputs on these subsystems. An essential part of the process of feedback control system analysis is the manipulation of this frequency response information into a form suitable for design work. Hence, if the engineers background information is to be used to help in control synthesis, it seems reasonable that the manipulated form of H(s) should, if possible, be amenable to physical interpretation. An example where such considerations have led to a practical approach to control system synthesis is the dyadic approximation method[7] as applied to the problem of regulating the spatial power distribution in a nuclear reactor. In this case, the individual dyads of the approximation have a direct physical interpretation[7,8,9] in terms of spatial modes of oscillation in the system in the sense that their form is similar to that of the eigensolutions of the underlying partial differential equations. The physical insight provided by this formulation enables the control engineer to identify the source of control difficulties, assess the affect of modelling errors on the final closed-loop design[8] and provides a technique for synthesizing fail-safe feedback control schemes with a performance adequate for nuclear reactor systems.[7,9]

The dyadic approximation method[7] in essence replaces H(s) by an approximation of the form

$$H_o(s) = \sum_{j=1}^{N} h_j(s)\alpha_j\beta_j^+ \qquad (4)$$

where

$$\lim_{s\to o} \frac{H(s) - H_o(s)}{s} = 0 \qquad (5)$$

The $\{\alpha_j\}$ and $\{\beta_j\}$ are sets of linearly independent, frequency independent vectors whose elements are real or exist in complex conjugate pairs, and $\{h_j(s)\}$ is a set of rational scalar transfer functions. Practical experience[8] with nuclear reactor spatial control problems indicates that $\{\alpha_j\}$ corresponds to the dominant eigenmodes of the system, the $\{h_j(s)\}$ are transfer functions describing the dynamics of the dominant eigenmodes, and the $\{\beta_j\}$ describe the effect of the control inputs on each eigenmode. For a more general system, if the error term $H(s)-H_o(s)$ is sufficiently small at all frequencies, then similar physical interpretations can provide direct insight into the essential or dominant interactions in the system.

It would seem therefore that interaction in linear multivariable systems can usefully be described using dyadic representations. The dyadic approximation described previously is limited in application as only certain practical systems are approximately dyadic. To extend the technique to include a larger class of systems it is necessary to use exact representations. In general an exact dyadic representation of H(s) must take the form

$$H(s) = \sum_{j=1}^{N} h_j(s)\alpha_j(s)\beta_j^+(s) \qquad (6)$$

where now $\{\alpha_j\}$ and $\{\beta_j\}$ are frequency dependent. An example of such an expansion is the eigenvector expansion used in the commutative controller[3] and characteristic locus methods[4].

This paper presents a dyadic representation of H(s) of the form of equation 6. The representation is derived with the objective that the expansion should be amenable to physical interpretation. This objective is incorporated into the analysis in the form of axioms concerning the required properties of $\{\alpha_j(s)\}$ and $\{\beta_j(s)\}$. The technique does not generate a complete control design technique because of the assumption of a human link in the design procedure. However, simple examples indicate that, if used together with existing theory and physical intuition, it can provide insight into system dynamics and provide guidelines to possible feedback controller structures for highly interacting systems.

2. The Dyadic Representation of System Interactions

Consider the problem of manipulating the NxN transfer function matrix H(s) of a general linear system into a form suitable for physical interpretation. As H(s) describes the vector response of the system to vector inputs it is reasonable that, as in the nuclear reactor case, a dyadic expansion of H(s) may be useful for this purpose. However, H(s) has an infinite number of dyadic representations, and hence it is necessary to choose a representation which gives the most insight into the physical interactions in the system. To illustrate this point consider the response of the system at $s = i\omega$, as represented by the complex matrix $H(i\omega)$. Consider arbitrary sets $\{\alpha_j(i\omega)\}$ and $\{\gamma_j(i\omega)\}$ of N complex, linearly independent vectors such that, $1 \leqslant j, k \leqslant N$,

$$\gamma_j^{+}(i\omega)\alpha_k(i\omega) = \delta_{jk} \tag{7}$$

The NxN unit matrix is as follows

$$I = \sum_{j=1}^{N} \alpha_j \gamma_j^{+} \qquad (8)$$

and hence $H(i\omega)$ can be written as

$$H(i\omega) = IH(i\omega) = \sum_{j=1}^{N} \alpha_j \{\gamma_j^{+} H(i\omega)\} \qquad (9)$$

which is a possible dyadic expansion. This expansion is just one of many as the $\{\alpha_j\}$ are arbitrary. From the point of view of physical interpretation, however, most of these expansions will be irrelevant as an arbitrary choice of $\{\alpha_j\}$ is unlikely to bear any relationship to the physics of the problem. It is convenient to write eqn 9 in the more general form (cf eqn 6)

$$H(i\omega) = \sum_{j=1}^{N} h_j(i\omega)\alpha_j(i\omega)\beta_j^{+}(i\omega) \qquad (10)$$

The requirements of physical interpretation used in this paper are:-

(i) The vectors $\{\alpha_j(i\omega)\}$ should reflect the physical structure of the system just as the dyadic approximation method reflects the eigenmode structure of the nuclear reactor spatial control problem.

(ii) The vectors $\{\alpha_j(i\omega)\}$ and $\{\beta_j(i\omega)\}$ should be real vectors or at worst exist in complex conjugate pairs. This requirement is included to aid physical interpretation and is consistent with the fact that system dynamics are generated by real interconnections between subsystems.

Although eigenvector expansions have useful analytical properties[3], the eigenvectors themselves are not in general real and are a sensitive function of parameters which are not fundamental to the dynamics of the system[7]. For example, consider an arbitrary real, non-singular matrix B, then the eigenvectors of H(s) and H(s)B could be significantly different, yet the dynamical structures of the two systems are essentially the same except for a redefinition of control variables, and this can be removed by the use of a cascade controller factor of the form B^{-1}. Therefore, in general, eigenvector expansions will not satisfy the requirements (i) and (ii) above for physical interpretation.

It is the purpose of this section to derive, using geometrical and physical arguments, an appropriate set of vectors $\{\alpha_j(z)\}$ to form a basis for the dyadic representation of H(s) at s = z (which is not necessarily on the imaginary axis). This is done using the idea that if two dynamical systems have indistinguishable dynamic responses at s = z, then, from the practical viewpoint, they can be taken to have identical interaction structures at s = z. The problem is then solved by investigating the properties of a dynamical system described by a dyadic transfer function matrix $H_A(s,z)$ which is indistinguishable from H(s) at s = z. The dyadic structure of $H_A(s,z)$ forms a basis for the dyadic expansion of H(z).

Assume that the interaction present in H(z) can be reconstructed using a dyadic transfer function matrix $H_A(s,z)$ and some other transfer function matrix $\mu(s,z)$ such that, for all s,

$$H_A(s,z) = \sum_{j=1}^{N} h_j(s,z)\alpha_j(z)\beta_j^+(z) \tag{11}$$

$$H(s) = H_A(s,z) + \mu(s,z) \tag{12}$$

and

$$\mu(z,z) = 0 \tag{13}$$

The dyadic matrix $H_A(s,z)$ is said to intersect the system H(s) at s = z. That is, the systems described by $H_A(s,z)$ and H(s) are indistinguishable at s = z and hence, from the practical viewpoint, can be taken to have identical interaction structures at s = z. Bearing in mind that, by assumption, $H_A(s,z)$ represents a dynamical system in its own right which automatically satisfies the two requirements for physical interpretation given above, then the representation

$$H(z) = H_A(z,z) = \sum_{j=1}^{N} h_j(z,z)\alpha_j(z)\beta_j^+(z) \tag{14}$$

is a candidate for the required dyadic expansion of $H(z)$.

In order to characterize the vectors $\{\alpha_j(z)\}$ explicitly, note that, as $H(s)$ and $H_A(s,z)$ represent physical systems,

$$H(\bar{s}) = \overline{H(s)} \tag{15}$$

$$H_A(\bar{s},z) = \overline{H_A(s,z)} \tag{16}$$

and hence, from equations (12) and (13),

$$\mu(\bar{z},z) = 0 \tag{17}$$

or

$$H(\bar{z}) = \sum_{j=1}^{N} h_j(\bar{z},z)\alpha_j(z)\beta_j^+(z) \tag{18}$$

i.e. $H_A(s,z)$ intersects $H(s)$ at $s = \bar{z}$.

Assuming that $H(z)$ is non-singular and defining, $1 \leqslant j \leqslant N$,

$$\gamma_j^+(z) = \beta_j^+(z)H^{-1}(z) \tag{19}$$

then, from equation (14), for $1 \leqslant j, k \leqslant N$, we can take

$$\gamma_j^+(z)\alpha_k(z) = \delta_{jk} \tag{20}$$

and, from equation (18),

$$H(\bar{z})H^{-1}(z) = \sum_{j=1}^{N} h_j(\bar{z},z)\alpha_j(z)\gamma_j^+(z) \tag{21}$$

Equations (20) and (21) immediately suggest that the $\{\alpha_j(z)\}$ are defined uniquely to be the eigenvectors of $H(\bar{z})H^{-1}(z)$.

The above results are based entirely on a physical interpretation of the mathematical structure of the problem. A justification of the assertions can be obtained using the following results, (see Appendices 7.1, 7.2 and 7.3).

RESULT 1

If the NxN system transfer function matrix H(s) is of the dyadic form[7]

$$H(s) = \sum_{j=1}^{N} h_j(s)\alpha_j\beta_j^+ \quad \text{and} \quad H^{-1}(o) \text{ exists, where the vectors}$$

$\{\alpha_j\}$ and $\{\beta_j\}$ are frequency independent, then we can take $H_A(s,z) = H(s)$ and the eigenvectors of $H(\bar{z})H^{-1}(z)$ are $\{\alpha_j\}$.

RESULT 2

If $\alpha_j(z)$ is an eigenvector of $H(\bar{z})H^{-1}(z) = A(z)$ with eigenvalue $\lambda_j(z)$, then $\overline{\alpha_j(z)}$ is an eigenvector of A(z) with eigenvalue $(\overline{\lambda_j(z)})^{-1}$.

RESULT 3

$|\lambda_j(z)| = 1$ if, and only if, $\alpha_j(z)$ can be chosen to be real and non-zero.

Results 2 and 3 immediately verify that $\{\alpha_j(z)\}$ satisfy requirement (ii) for physical interpretation. Result 1 states that the $\{\alpha_j(z)\}$, as chosen, ensure that requirement (i) is satisfied at least for dyadic transfer function matrices, and hence may be true for a larger class of systems. This result is very difficult to generalize and may only be consolidated by application to a large number of practical problems.

If $\{\gamma_j(z)\}$ are the dual eigenvectors to $\{\alpha_j(z)\}$ satisfying the equation (7), then the proposed dyadic representation of H(z) takes the form (c.f. eqn 9),

$$H(z) = \sum_{j=1}^{N} \alpha_j(z)\{\gamma_j^+(z)H(z)\} \tag{22}$$

which can be written in the form (cf. equation (10))

$$H(z) = \sum_{j=1}^{N} h_j(z)\alpha_j(z)\beta_j^+(z) \tag{23}$$

if we define, $1 \leqslant j \leqslant N$,

$$h_j(z)\beta_j^+(z) = \gamma_j^+(z)H(z) \tag{24}$$

To show that the vectors $\{\beta_j(z)\}$ can be scaled to satisfy requirement (ii) for physical interpretation, note that the eigenvalue equation for $A(z) = H(\bar{z})H^{-1}(z)$ can be written, using the notation of Results 2 and 3, in the form, $1 \leqslant j \leqslant N$,

$$\gamma_j^+(z)H(\bar{z}) = \lambda_j(z)\gamma_j^+(z)H(z) \tag{25}$$

From Result 2, $\{\alpha_j(z)\}$ are either real or exist in complex conjugate pairs. If $\alpha_j(z)$ is real then $\gamma_j(z)$ is real and, from Result 3, $\lambda_j(z) = e^{2i\theta(z)}$ where $\theta(z)$ is real. Substituting into equation (25) and multiplying both sides by $e^{-i\theta(z)}$ yields the fact that $\gamma_j^+(z)H(z)e^{i\theta(z)}$ is real and hence, defining $h_j(z) = e^{-i\theta(z)}$ and $\beta_j^+(z) = \gamma_j^+(z)H(z)e^{i\theta(z)}$, the result follows for this case.

If $\alpha_j(z)$ is complex, then, from result 2, there exists a linearly independent eigenvector $\alpha_k(z) = \overline{\alpha_j(z)}$, and hence $\gamma_k(z) = \overline{\gamma_j(z)}$. Defining $\beta_j(z) = \gamma_j^+(z)H(z)$, $h_j(z) = 1$, $\beta_k(z) = \lambda_k(z)\gamma_k^+(z)H(z)$ and $h_k(z) = (\lambda_k(z))^{-1}$, then, using Result 2 and equation 25

$$\begin{aligned} \overline{\beta_j^+(z)} &= \gamma_k^+(z)H(\bar{z}) \\ &= \lambda_k(z)\gamma_k^+(z)H(z) \\ &= \beta_k^+(z) \end{aligned} \tag{26}$$

as required.

The procedure for the proposed dyadic expansion of $H(s)$ at $s = z$ can be summarised as follows

<u>STEP 1</u>: Compute the matrix $A(z) = H(\bar{z})H^{-1}(z)$. If $H(z)$ is singular, then an arbitrarily small perturbation will ensure non-singularity.

STEP 2: Compute the eigenvectors $\{\alpha_j(z)\}$ of $A(z)$. If $A(z)$ does not have a complete set of eigenvectors then an arbitrarily small perturbation to $H(z)$ will remove this difficulty.

STEP 3: Form the dyadic expansion of equation (22).

STEP 4: By suitable scaling operations, transform the dyadic expansion into the form of equation (23) where $\{\beta_j(z)\}$ are real or exist in complex conjugate pairs.

In the next section, the above procedure is applied to some examples to illustrate how it can reveal system structures which are not immediately apparent by visual inspection of $H(s)$, and to demonstrate how the technique can suggest practical control schemes for highly interacting structures.

3. EXAMPLES

3.1. Example 1

This example is designed to illustrate how the dyadic representation described in the previous section can reveal the physical structure of a plant. This possibility has already been illustrated by result 1 and its nuclear reactor application[9]. The system chosen here is of a form which readers from all fields can readily understand. The procedure of section 2 is applied to the system and it is shown that the dyadic structure revealed corresponds physically with the expected dynamics.

Consider the spring, mass, damper system shown in Fig.2, where the masses are m_1 and m_2, the springs are identical with unity spring constant and the damper has linear characteristics with unit damping constant.

The dynamic equations describing the dynamic responses of the masses to input spring movements are

$$m_1\ddot{y}_1 = -\{y_1 - u_1\} - \{\dot{y}_1 - \dot{y}_2\} \tag{27}$$

$$m_2\ddot{y}_2 = -\{y_2 - u_2\} - \{\dot{y}_2 - \dot{y}_1\} \tag{28}$$

Taking the Laplace transform of these equations with zero initial conditions and defining

$$d_1(s) = m_1s^2 + s + 1 \tag{29}$$

$$d_2(s) = m_2s^2 + s + 1 \tag{30}$$

$$d(s) = d_1(s)d_2(s) - s^2 \tag{31}$$

then the transfer function matrix describing the system takes the form

$$H(s) = \frac{1}{d(s)}\begin{bmatrix} d_2(s) & s \\ s & d_1(s) \end{bmatrix} \tag{32}$$

(a) Consider the case when $m_1 = m_2$ and $z = i\omega$. The eigenvectors of $A(z) = H(-i\omega)H^{-1}(i\omega)$ are linearly independent and given by

$$\alpha_1(i\omega) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha_2(i\omega) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{33}$$

The dual eigenvectors (equation 20) are given by

$$\gamma_1(i\omega) = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \gamma_2(i\omega) = \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{34}$$

Forming the dyadic expansion of equation (22) yields, after some manipulation,

$$H(i\omega) = \frac{0.5}{1-m_1\omega^2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \frac{0.5}{1-m_1\omega^2+2i\omega}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} \tag{35}$$

That is, the interaction effects in the system are characterised by two, real frequency independent dyads and hence H(s) is a dyadic transfer function matrix[7]. Inspection of equation (35) reveals that the first dyad has dynamics with zero damping, and that this behaviour occurs when the two masses are moving in phase and with equal amplitude. This observation is consistent with physical intuition because (see Fig.2), if this condition is satisfied, there is no energy loss in the damping mechanism. The second dyad has a damping constant of two which is characterised by a dynamic response where the two masses move in antiphase with equal amplitude. Again this is in accordance with physical intuition as, if the masses move in this way, the relative velocity is twice the velocity of each mass and hence the damper has effectively twice the damping constant.

Also the expansion (eqn.35) shows that these two natural responses are non-interacting in a feedback situation as inputs of the shape of $\alpha_1(i\omega)$ (respt. $\alpha_2(i\omega)$) produce outputs of the shape $\alpha_1(i\omega)$ (respt. $\alpha_2(i\omega)$). The control analysis of this type of system has been discussed elsewhere[7]. The concept of non-interaction used above, however, will play an important part in the following analysis.

(b) Consider the case when $m_1 \neq m_2$ and $z = i\omega$. Applying the procedure of section 2, the eigenvectors of $A(z) = H(-i\omega)H^{-1}(i\omega)$ are

$$\alpha_1(i\omega) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \alpha_2(i\omega) = \begin{bmatrix} -(1-m_2\omega^2) \\ 1-m_1\omega^2 \end{bmatrix} \tag{36}$$

The dual eigenvectors are (see equation 20)

$$\gamma_1(i\omega) = \frac{1}{2-(m_1+m_2)\omega^2} \begin{bmatrix} 1-m_1\omega^2 \\ 1-m_2\omega^2 \end{bmatrix}$$

$$\gamma_2(i\omega) = \frac{1}{2-(m_1+m_2)\omega^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{37}$$

and hence, from equation 22, after some manipulation,

$$H(i\omega) = \frac{1}{2-(m_1+m_2)\omega^2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$+ \frac{1}{d(i\omega)(2-(m_1+m_2)\omega^2)} \begin{bmatrix} -(1-m_2\omega^2) \\ 1-m_1\omega^2 \end{bmatrix} \begin{bmatrix} -(1-m_2\omega^2), & 1-m_1\omega^2 \end{bmatrix} \tag{38}$$

Again, the dynamic response of the system is characterised by two real dyads, the first of which is frequency independent and identical in form to that in equation (35). The second dyad is frequency dependent.

The problem considered in the next few paragraphs is whether or not the information contained in equation (38) does in fact correspond physically to the expected dynamic behaviour, and whether or not the analysis can indicate possible feedback controller structures (see Fig. 1).

At low frequencies, the dyads in equation (38) approximate to the forms

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{39}$$

and

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{40}$$

which are identical to those of the equal mass problem (see equation 35). Physically, equation (39) suggests that, at low frequencies, the vector sum of the inputs to the plant produces more or less identical movements of the masses. Equation (40) states that, under the same conditions, the vector difference between the input produces more or less the same equal movements of the masses but with a difference in sign. These behaviours are as expected physically, for, at low frequencies, the difference between the masses becomes of secondary importance as the system dynamics is dominated by the stiffness of the springs. Under these conditions, low frequency vector inputs to the system of equal amplitude and equal (respt. opposite) phase are expected to produce movements of the masses of equal amplitude and equal (respt. opposite) phase.

The above observations suggest that an integral feedback controller factor of the form,

$$G_1(s) = \frac{k_1}{s}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \frac{k_2}{s}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} \qquad (41)$$

where k_1, k_2 are unspecified gain constants, is a reasonable structural form for the control of the low frequency behaviour of the system. This controller factor separates the low frequency part of the input into components of $\alpha_1(o)$ and $\alpha_2(o)$, operates upon these components independently with the integrator k_1/s and k_2/s respectively, and then projects the results into control vectors which produce outputs proportional to $\alpha_1(o)$ and $\alpha_2(o)$ respectively. Physically the vectors $\alpha_1(o), \alpha_2(o)$ are then approximately non-interacting in a feedback situation at low frequency in thesense that an input to $H(s)G(s)$ of form $\alpha_1(o)$ (respt. $\alpha_2(o)$) produces an output of the approximate form $\alpha_1(o)$ (respt. $\alpha_2(o)$ modified by an overall gain and phase factor. This fact enables the controller gains k_1, k_2 to be chosen independently to provide the required low frequency response in the modes $\alpha_1(o), \alpha_2(o)$.

The high frequency response of the system is described (see equation 38) by the dyadic forms

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \; , \; 1] \qquad (42)$$

and

$$\begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} [m_2 \; , \; -m_1] \qquad (43)$$

At high frequencies, the system dynamics is dominated by the inertia represented by the masses m_1 and m_2. To illustrate how the dyads of equations (42) and (43) are consistent with this physical fact, consider the case when m_1 is much greater than m_2. It can be expected that the energy required to excite high frequency oscillations of m_1 of a certain amplitude is much greater than the energy required to excite m_2 to that amplitude. The dyad of equation (42) represents the case of equal amplitude oscillation. Any input which excites this dyad only must take the form $F\{m_1, m_2\}^T$ where F is a scalar. Inspection of this form reveals that the energy input to each mass must be proportional to that mass and hence the energy input to m_1 is proportionately larger than that to m_2.

Using similar arguments as for the low frequency analysis, the high frequency dynamics of the system can be controlled using a proportional feedback controller factor $G_2(s)$ which decouples the high frequency dyads of equations 42 and 43 e.g.

$$G_2(s) \;=\; k_3 \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} [m_1, m_2] + k_4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1, \; 1] \qquad (44)$$

The vectors $\alpha_1(i\omega)$ and $\alpha_2(i\omega)$ are then approximately non-interacting in a feedback situation at high frequency and hence k_3, k_4 can be chosen independently to provide the required high frequency response in these modes.

From the point of view of feedback control, equations (41) and (42) could be combined to produce an overall feedback controller of the proportional plus integral (PI) form (see Fig. 1)

$$G(s) = G_1(s) + G_2(s) \tag{45}$$

This controller structure contains only four unspecified constants k_j, $1 \leqslant j \leqslant 4$, as compared with the eight constants which would be required to specify an arbitrary PI controller. Hence, at the very minimum, the dyadic representation has provided a physical basis for the reduction of the number of unknown parameters in the multivariable control problem. The final control design analysis could now be undertaken using known design techniques[1], choosing the unknown gains to satisfy the desired performance criteria.

The above analysis can be extended to incorporate the possibility of proportional plus integral plus differential (PID) control. In this case, the high frequency controller of equation (44) is converted to differential form by multiplication by s. It is then necessary to investigate the intermediate frequency dynamics using the representation of equation (38) in order to determine a structure for the proportional controller term.

At intermediate frequencies, an unusual feature of the dyadic representation is the presence of an additional pole pair on the imaginary axis at a frequency of

$$\omega' = \sqrt{\frac{2}{m_1+m_2}} \tag{46}$$

Mathematically, this pole arises when $H(-i\omega)H^{-1}(i\omega)$ only has a Jordan form[10] and the eigenvectors $\alpha_1(i\omega)$ and $\alpha_2(i\omega)$ become colinear. The properties of this pole are entirely consistent with the form of $\alpha_1(i\omega)$, for, if $y_1 = y_2$ for all time, then, adding equations (27) and (28)

$$(m_1 + m_2)\ddot{y}_1 = -2y_1 + u_1 + u_2 \tag{47}$$

That is, a response of this form is excited by the vector sum of the inputs and has a characteristic frequency of ω' with zero damping. To investigate the problem of what this phenomenum indicates physically, and how it influences the feedback dynamics of the system, it is necessary to consider the concept of dyadic or modal decoupling:-

DEFINITION

An NxN system is said to permit modal decoupling at $s = z$ if, and only if, there exists a set of linearly independent vectors $\{x_j(z)\}$ and a real non-singular matrix $B(z)$ such that $\{x_j(z)\}$ are either real or exist in complex conjugate pairs and are the eigenvectors of $H(z)B(z)$. If this is the case then the identity $\overline{(H(z)B(z))}\,(H(z)B(z))^{-1} = \bar{H}(z)H^{-1}(z)$ implies that the $\{x_j(z)\}$ can be identified with the $\{\alpha_j(z)\}$.

The above concept has a direct physical meaning for proportional feedback control systems. If $H(s)$ permits modal decoupling at $s = z$, then the use of a proportional forward path controller $B(z)$ implies that, if the eigenvalue equation for $H(z)B(z)$ is written in the form, $1 \leq j \leq N$,

$$H(z)B(z)x_j(z) = a_j(z)x_j(z) \tag{48}$$

then

$$\{I + H(z)B(z)\}^{-1}H(z)B(z)x_j(z) = \frac{a_j(z)}{1+a_j(z)}\,x_j(z) \tag{49}$$

i.e. an input to the feedback system at $s = z$ of shape $x_j(z)$ produces an output of the same form but modified by a scalar factor. Physically, this situation corresponds to zero energy transfer between modes in the feedback system.

It follows from the definition that, at frequencies where $\bar{H}(s)H^{-1}(s)$ only has a Jordan form, $H(s)$ does not permit modal decoupling. For the purpose of this example, it follows that $H(s)$ does not permit modal decoupling at $s = i\omega'$. However, it is possible to choose a controller factor B such that $\alpha_1(i\omega')$ is an eigenvector of $H(i\omega')B$, by noting that

$$H(i\omega')\begin{bmatrix}-1\\1\end{bmatrix} = \frac{m_2 - m_1}{(m_1+m_2)d(i\omega')}\begin{bmatrix}1\\1\end{bmatrix} \tag{50}$$

and setting

$$B = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \tag{51}$$

It follows that, in any basis of real vectors containing $\alpha_1(i\omega')$, $H(i\omega')B$ takes a triangular form (it cannot take a diagonal form by the above argument). That is, there is an inevitable transfer of energy into the mode $\alpha_1(i\omega) = (1,1)^T$ at $\omega = \omega'$ and hence this form of response will play a unique part in the dynamic behaviour of the feedback system in the sense that all real inputs to the system will produce a component of output of this form at that frequency.

The above analysis can be used to infer a reasonable practical controller structure for $H(s)$ by defining a basis set containing $\alpha_1(i\omega')$ by the similarity transformation

$$S = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} \tag{52}$$

and setting the proportional controller factor to be of the form $BG_3(s)$ where

$$G_3(s) = k_5\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + k_6\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + k_7\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} \tag{53}$$

As $\alpha_1(i\omega')$ is an eigenvector of $H(i\omega')B$, then the first dyad influences only $\alpha_1(i\omega')$. As $S^{-1}H(i\omega')BS$ takes a triangular form, the second dyad controls only the second member of the basis set but does interact with $\alpha_1(i\omega')$. The third dyad is chosen to compensate for this interactive effect.

The final PID controller structure derived for the system is (see Fig. 1)

$$G(s) = G_1(s) + sG_2(s) + BG_3(s) \tag{54}$$

which contains seven unspecified gain parameters, as compared with the twelve parameters required to construct a general PID controller.

In summary, the above example illustrates that the dyadic representation method of section 2 is a description which corresponds physically with the expected dynamics of the system. The formulation reveals an unexpected energy transfer mechanism in the system and gives some insight into a possible feedback controller structure. The final control design analysis can be undertaken using known techniques[1] (e.g. the characteristic locus method) using the seven unspecified gains as parameters to satisfy the required performance objectives.

3.2. Example 2

The previous example illustrates how the dyadic representation method of section two provides information on system dynamics which can be used to set up a feedback controller structure for the system. This next example illustrates how the method can indicate a means of synthesizing controllers for a quite different system structure.

Consider the system

$$H(s) = \frac{1}{d(s)} \begin{bmatrix} 3+5s & 1+3s+2s^2+4s^3 \\ 1+2s-s^2-s^3 & 1+2s+2s^2+3s^3 \end{bmatrix} \tag{55}$$

where d(s) is a polynomial in s of degree greater than three. The eigenvectors of $H(-i\omega)H^{-1}(i\omega)$ are given by

$$\alpha_1(i\omega) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \alpha_2(i\omega) = \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix} \tag{56}$$

and hence

$$\gamma_1(i\omega) = \frac{1}{1-2\omega^2}\begin{bmatrix}-\omega^2\\ 1\end{bmatrix} \quad , \quad \gamma_2(i\omega) = \frac{1}{1-2\omega^2}\begin{bmatrix}1\\ -2\end{bmatrix} \tag{57}$$

In a similar manner to example 1, after some manipulation, the dyadic representation of $H(i\omega)$ takes the form

$$H(i\omega) = \frac{\{1+2i\omega\}}{d(i\omega)}\begin{bmatrix}2\\ 1\end{bmatrix}[1,\ 1-\omega^2] + \frac{\{1+i\omega\}}{d(i\omega)}\begin{bmatrix}1\\ \omega^2\end{bmatrix}[1,\ -1] \tag{58}$$

i.e. the interaction effects in the system are characterized by two frequency dependent dyads. In a physical problem these dyads may reflect underlying properties of the system (e.g. spatial modes in distributed parameter systems[8]). In a feedback situation these dyads are modally interacting[7] as

$$[1,\ 1-\omega^2]\begin{bmatrix}1\\ \omega^2\end{bmatrix} = 1 + \omega^2 - \omega^4 \neq 0$$

$$[1,\ -1]\begin{bmatrix}2\\ 1\end{bmatrix} = 1 \neq 0 \tag{59}$$

i.e. the output from one dyad will excite the other if a feedback loop is connected around the system. It may be desirable on physical grounds to retain these interactions, but, if necessary, a practical control design procedure can be obtained by removing the interactive effect of $\alpha_1(i\omega)$ on $\alpha_2(i\omega)$. This is achieved by choosing a non-singular controller factor G_1 such that

$$[1\ \ -1]\ G_1 = [1\ \ -2] \tag{60}$$

from which it is noted that $\alpha_1(i\omega) = (2,\ 1)^T$ is an eigenvector of $H(i\omega)G_1$ for all frequency. Moreover, in any basis set containing $\alpha_1(i\omega)$, $H(i\omega)G_1$ is of triangular form. For example, let

$$G_1 = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \tag{61}$$

and consider the basis set defined by the similarity transformation

$$S = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \tag{62}$$

then

$$S^{-1}H(s)G_1S = \frac{1}{d(s)}\begin{bmatrix} 2+4s+s^2+2s^3 & -1-s+s^2+2s^3 \\ 0 & 1+s+2s^2+2s^3 \end{bmatrix} \tag{63}$$

It is well known[1] that the stability of a feedback configuration is governed by the relationship,

$$\frac{\text{closed-loop characteristic polynomial}}{\text{open-loop characteristic polynomial}} = |I + H(s)G(s)| \tag{64}$$

Hence letting $G(s) = G_1G_2(s)$ where $S^{-1}G_2(s)S$ is of upper triangular form, and defining $\{F_j(s)\}$ to be the product of the corresponding diagonal elements of $S^{-1}H(s)G_1S$ and $S^{-1}G_2(s)S$, then

$$|I + H(s)G(s)| = |I + S^{-1}H(s)G_1SS^{-1}G_2(s)S| = \prod_{j=1}^{2} (1 + F_j(s)) \tag{65}$$

This relationship can be used to investigate closed-loop stability.[1]

Although the stability of the closed-loop system depends only on the diagonal elements of $S^{-1}H(s)G_1S$ and $S^{-1}G_2(s)S$, the overall transient response will depend on both the diagonal and off diagonal terms. Insight can be obtained into the closed-loop response by examining the error response,

$$E(s) = \{I + H(s)G(s)\}^{-1} \tag{66}$$

where, if

$$E_d(s) = \{I + S^{-1}H(s)G_1SS^{-1}G_2(s)S\}^{-1} \tag{67}$$

then

$$E(s) = SE_d(s)S^{-1} \tag{68}$$

The stability and transient performance can be examined for the triangular system of equation (67). The real system response can then be generated using equation (68).

The dyadic representation method has provided a feedback control structure of a practical and easily realisable form, without losing a great deal of design freedom. For example, the controller factor G_1 is not unique, nor is the similarity transformation S, and the controller $G_2(s)$ is unspecified except for the upper triangular form of $S^{-1}G_2(s)S$. The actual choice of parameters will depend upon the precise physical nature of the system and other performance objectives.

4. Conclusions

The paper presents a theoretical method of manipulating the frequency response information available in the plant transfer function matrix into a form amenable to physical interpretation, with the objective of aiding the control engineer in his attempts to understand the physical interaction structure of linear multivariable systems. In this way, any previous experience with a system and insight into its dynamic behaviour may be incorporated into the control synthesis procedure to give useful insight into the sort of controller interactions required to satisfy the performance objectives.

The technique chosen is that of dyadic expansion of H(s). The objective of physical interpretation is incorporated into the analysis in the form of axioms concerning the required property of such a dyadic expansion, and the structure of the expansion is identified in terms of the properties of the matrix $H(\bar{s})H^{-1}(s)$.

The suggested procedure is illustrated by two examples which indicate that the technique reveals properties of H(s) which have a direct relationship to the physical properties of the underlying state-space model of the system.

By interpreting the dyadic expansion in terms of energy transfer mechanisms, the methods are shown to indicate possible controller structures for the systems and reveal conditions under which these transfer mechanisms cannot be decoupled.

The technique presented does not alone generate a complete design technique, as, by its very nature, it assumes a human link in the design procedure. However, the examples indicate that, if used together with existing theory and physical intuition, the procedure can provide guidelines to aid in the choice of synthesis procedure.

5. References

1. A. G. J. MACFARLANE: A Survey of Some Recent Results in Linear Multivariable Feedback Theory, Automatica, Vol.8, pp.455-492, 1972.
2. H. H. ROSENBROCK: Design of multivariable control systems using the inverse Nyquist array, Proc. I.E.E., Vol.116, No.11, November 1969, pp.1929-1936.
3. A. G. J. MACFARLANE: Commutative Controller: A new technique for the design of multivariable control systems, Electronic Letters, 6, 121-123 (1970).
4. J. BELLETRUTTI and A. G. J. MACFARLANE: Characteristic Loci techniques in multivariable control system design, Proc. I.E.E. 118, 1291-1297 (1971).
5. D. Q. MAYNE: The Design of Linear Multivariable Systems, Automatica, Vol.9, pp.201-207, 1973.
6. C. F. CHEN: Model Reduction of Multivariable Control Systems by means of Matrix Continued Fractions, Fifth World Congress of the IFAC, Pams, 12-16th June, 1972.
7. D. H. OWENS: A Dyadic approximation method for multivariable control systems analysis with a nuclear reactor application, Proc. I.E.E., Vol.120, No.7, July 1973, pp.801-809.

8. D. H. OWENS: A Multivariable Control Analysis of Distributed Parameter Nuclear Reactors, Ph.D. Thesis, Imperial College of Science and Technology, London University, April, 1973.

9. D. H. OWENS: Multivariable control system design concepts in failure analysis of a class of nuclear reactor spatial control systems, Proc. I.E.E., Vol.120, No.1, Jan. 1973, pp.119-125.

10. R. BELLMAN: Introduction to Matrix Analysis, McGraw-Hill, 1970.

6. Appendices

6.1 Proof of Result 1

Consider a dyadic transfer function matrix of the form[7]

$$H(s) = \sum_{j=1}^{N} h_j(s)\alpha_j\beta_j^+ \quad \text{and} \quad H^{-1}(o) \text{ exists} \tag{69}$$

where $\{\alpha_j\}$ and $\{\beta_j\}$ are sets of linearly independent, frequency independent vectors. Choosing $\mu(s,z) = 0$ and $H_A(s,z) = H(s)$, equations 11 to 13 are automatically satisfied, which proves the first point of the result.

Consider now the matrix $H(\bar{z})H^{-1}(z)$. It can be shown in an analogous way to Result 2 of Ref. 7 that

$$h_j(z)\beta_j^+H^{-1}(z)\alpha_k = \delta_{jk} \tag{70}$$

and hence, $1 \leqslant k \leqslant N$,

$$H(\bar{z})H^{-1}(z)\alpha_k = \sum_{j=1}^{N} h_j(\bar{z})\alpha_j(\beta_j^+H^{-1}(z)\alpha_k) \tag{71}$$

which, in view of equation (70), can be written in the form

$$H(\bar{z})H^{-1}(z)\alpha_k = \frac{h_k(\bar{z})}{h_k(z)}\alpha_k \tag{72}$$

which proves the result.

6.2 Proof of Result 2

By definition

$$A(z)\alpha_j(z) = \lambda_j(z)\alpha_j(z) \tag{73}$$

and hence, using equation (15) and the definition of A(z),

$$A(\bar{z})A(z)\alpha_j(z) = \alpha_j(z) = \lambda_j(z)A(\bar{z})\alpha_j(z) \tag{74}$$

Taking complex conjugates

$$A(z)\overline{\alpha_j(z)} = \left\{\overline{\frac{1}{\lambda_j(z)}}\right\} \overline{\alpha_j(z)} \tag{75}$$

which proves the result.

6.3 Proof of Result 3

If $\alpha_j(z)$ can be chosen to be real then, from Result 2, $\alpha_j(z)$ is an eigenvector of A(z) corresponding to two eigenvalues of $\lambda_j(z)$ and $\overline{\lambda_j(z)}^{-1}$. The eigenvalues must be equal, i.e., $|\lambda_j(z)|^2 = 1$.

Conversely, if $|\lambda_j(z)| = 1$, then, from Result 2, $\alpha_j(z)$ and $\overline{\alpha_j(z)}$ are eigenvectors of A(z) corresponding to the same eigenvalue. Therefore, either $\alpha_j(z)$ and $\overline{\alpha_j(z)}$ are linearly dependent and hence $\alpha_j(z)$ can be made real, or $\alpha_j(z)$ and $\overline{\alpha_j(z)}$ are linearly independent and hence the real and imaginary parts of $\alpha_j(z)$ are real linearly independent eigenvectors of A(z) spanning the same subspace as $\alpha_j(z)$ and $\overline{\alpha_j(z)}$.

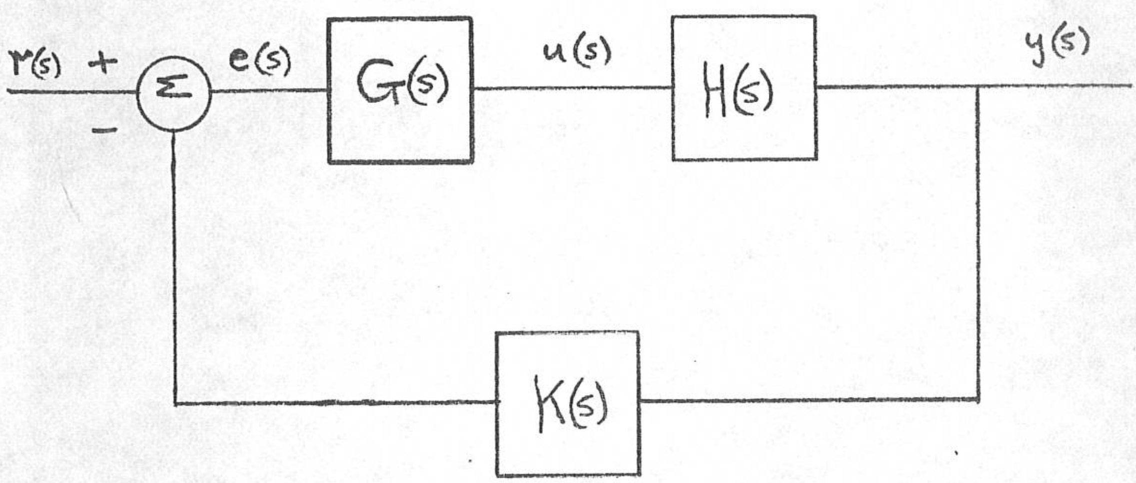

Fig. 1.

Fig. 2