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Concepts of general system theory in the linear optimal control problem

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Summary

Concepts of generalised flux and thermodynamic forces, and particularly the Onsager-Casimir reciprocal relations characterising the phenomenological equations for general irreversible flow processes, are shown to exist analogously in the formulation of the linear optimal control and estimation problems. Characteristics of the energy function used in the Lagrange and Hamiltonian representation of a system reacting with an environment are discussed, and the condition of zero dissipation in the formulation of the optimal control problem defining the combined solution of the state and adjoint variables is illustrated. Concepts of energy in the general linear system based on quadratic functions are discussed with reference to an electromechanical system, and the solution of the optimal control problem with resulting cross-product weighting terms in the performance criterion is illustrated.

The general electrical network problem is shown to include a Riccati-type differential equation which represents a matrix analogue of the classical operator solution of the defining second-order equations. The scattering matrix of electrical network theory and the associated power relations are also shown to exist analogously in the linear optimal control problem. This can be formulated in terms of 'incident' and 'reflected' variables which possess similar properties to the scattering variables of network theory, and the scattering matrix is shown to be closely related to the matrix solution of the Riccati equation associated with the optimal control problem.

1. Introduction

The phenomenological equations representing a large class of irreversible flow processes involving energy dissipation and mass and energy transport can be based on concepts of 'thermodynamic' forces and fluxes, and entropy change associated with a dissipation function which acts as a potential for the 'thermodynamic' forces. The definition of a state function based on concepts of energy and represented as a scalar product of conjugate variables is also of fundamental importance in general system theory. The existence of a dissipation function will be governed by the form of representation used for the physical system and its interacting environment or connected load source. In the classical Lagrange formulation of the non-conservative system representing, say, the electrical network problem, a dissipation function is defined and the environment is not represented inherently by the system equations. By analogy, the optimal control problem may be considered in terms of a system interacting with an 'environment' or adjoint system. In this case the Hamiltonian formulation combines the properties of both the system and its 'environment' and represents an overall conservative-type system.

The basic laws of physical systems, such as representing the thermodynamics of irreversible processes, are associated with theorems of reciprocity and with the invariance of time reversal[1-4]. A unified analysis of such processes may be formulated in terms of the Lagrangian-type equations of classical mechanics with generalised coordinates and based on the concept of virtual work[5]. The Lagrangian analysis can also be associated with a principle of minimum dissipation. In particular, the general electrical network problem can be formulated in terms of a Lagrange equation and 'thermodynamic' properties. By analogy, the concepts and functional relations for physical systems can also be associated with the optimal control problem and also with the dual problem of optimal filtering. A matrix Riccati differential equation based on a transformation between the state and adjoint variables defines the solution of the linear optimal control problem. A similar matrix Riccati equation can also be shown to exist in the electrical network problem, and represents a matrix analogue of the classical operator solution. Other properties of electrical network theory can be associated with the optimal control problem. Thus the incident and reflected variables in the general theory of scattering applied to the behaviour of an evolving system, and particularly the scattering variables of electrical network theory, can be defined analogously in the optimal control and estimation problems.

The Lagrange-type formulation and particularly the concepts of electrical network theory are of fundamental importance and, by analogy, can provide a physical basis for the more abstract mathematical problems of general system theory. The correspondence of the least-squares problem with the steady-state behaviour of a physical system has been illustrated[6], and the present study extends the analogy to consider the relationships between the formulation of optimal control for the linear dynamic system and certain properties of the general phenomenological equations for irreversible flow processes. Such analogies, based particularly on the electrical network problem, can lead to a greater understanding of the optimal control problem.

2. Energy concepts

Concepts of energy and the definition of a state-function based on the principle of conservation are of fundamental importance in the state-variable representation of the behaviour of physical systems. The state-function may be related to the transfer of active (stored) power between a physical system and its environment, and partial differentiation with respect to the state variables define the dual system variables and also an extremum value which is associated with an equilibrium condition[7]. A state-function for an electromechanical system consisting of an electric motor (system) driving a mechanical load (environment) illustrated in Section 11 can be defined in terms of the active power stored in the magnetic field and transferred to the environment, related to scalar products of the state and input variables as in eqn 127. It represents a conversion of energy related to a subtraction of 'input' and 'output' costs defined on the basis of the first law of thermodynamics. The minimisation of this function associated with equilibrium conditions has been compared with the minimisation of both input and output costs in the optimal regulator problem[8].

In the classical Lagrange formulation of the general non-conservative system which interacts with an environment and dissipates energy, such as representing the particular electromechanical system or the general electrical network of Section 3, a dissipation function is specified for the environment. In the Hamiltonian formulation of the optimal control problem the system interacts with an 'environment' or dual system which is included inherently in the overall representation of the 'closed' system, in a reference frame which includes both state and costate as independent variables. The total output of the system is effectively balanced by the 'output' of the dual system or environment, and the total representation of the combined system is conservative, as illustrated in Section 4. By contrast, in the representation of total system energy for a physical system according to classical mechanics, energy is dissipated and may be converted into a form which is not considered in the analysis, that is, it is not transferred to an environment which is represented in the formulation. For example, in the electrical network, energy is supplied, stored and also dissipated as heat losses. The general dynamic system model will include dissipation and a potential for transferring energy to a load characterised by the level of certain state variables. Thus, in the boiler-turboalternator model, input energy will be transformed to available energy at the boiler stop valve characterised by outlet steam pressure and temperature, with energy loss in the flue gas and with relatively small internal dissipation of fluid energy. The turboalternator will dissipate energy in turbine friction and alternator heat losses, and will transform the available energy in the boiler to mechanical and electrical energy measured by the level of the operating variables such as turbine speed and alternator voltage and current, for transfer to a load source. In the electrical network problem, however, the dissipation to heat energy and the action-reaction phenomena are not included inherently in the overall system model.

Concepts of energy and power in the general dynamic system

$$\dot{x}(t) = A\, x(t) + B\, u(t) \qquad (1)$$

with particular reference to energy exchanges between a system and its controller have been specified in terms of quadratic forms and products of the state and input variables, with appropriate scaling[9]. Thus

$$\text{'stored energy'},\ P_s = (x^t x)/2\ ,\quad \text{'absorbed power'},\ P_a = x^t A x$$
$$\text{'injected power'},\ P_i = x^t B u\ ,\quad \text{'active power'} = P_a + P_i = \dot{P}_s = x^t \dot{x} \qquad (2)$$

In general, the A-matrix will define a non-conservative system. For the free linear conservative system, A is skew-symmetric ($= -A^t$) representing a self-adjoint system[10]. The power forms of eqn 2 based directly on the defining state variable representation do not correspond directly with the power relations developed for the electromechanical system in Section 11. The active power or rate of change of stored energy ($\dot{P}_s$) is similar to the active power delivered to the motor armature (P_m), obtained as the difference between input and dissipated power. However, the 'absorbed power' P_a and 'stored energy' P_s do not contain, for the particular example, the quadratic input weighting and cross-product terms resulting from the voltage-current circuit relations of Section 11. In contrast to the specification of quadratic energy forms in the electromechanical system, the conventional boiler model involving thermal energy transfer and storage will include a change in stored energy for the tube metal related to a linear function of a state variable (temperature) and will not exist as a quadratic form, and also injected power may not be associated with a state variable[26].

3. Phenomenological equations for irreversible flow processes

A unified formulation of the empirical laws defining a physical system can be based on the concepts of thermal potential, dissipation, entropy change and generalised thermal force. The basic laws of physical systems also incorporate Onsager's principle of reciprocity and symmetry associated with the thermodynamics of irreversible processes, and will be invariant with respect to time reversal.[11] A fundamental form of variational principle based on similar concepts is also closely related to Onsager's principle of reciprocity[5].

The equations representing conservative dynamical systems, without external magnetic fields and Coriolis forces, are reversible in time such that the state trajectory can be retraced with a reversal of all velocities, and the total system energy and entropy remain constant. With dissipation, the approach to equilibrium is irreversible and results in an increased generation and flow of entropy and a loss of 'availability' of the system energy. Irreversible flow processes, such as representing the conduction of heat, electricity and diffusion, can be described in terms of macroscopic operations using concepts involving fluxes of heat and mechanical energy, and entropy production. Reciprocal relations can then be derived on the basis of microscopic reversibility with flow assumed proportional to a specified potential or state function defined in terms of generalised state coordinates.

The 'thermodynamic' state or displacement of the flow system from equilibrium is specified by a set of macroscopic variables $\alpha = (\alpha_1 \cdot\cdot \alpha_n)$[1-3]. Thermodynamic restoring 'forces', such as temperature gradients and emf's, are then defined in terms of a scalar entropy change $s(\alpha)$ and related linearly to the resulting fluxes $J(=\dot{\alpha})$, which are proportional to the flow of matter, heat or current, by the linear differential matrix equations

$$X = \frac{\partial s}{\partial \alpha} = RJ \ , \qquad J = GX \tag{3}$$

Eqns 3 represent the phenomenological relations for a set of interacting simultaneous irreversible flow processes in terms of conjugate variable vectors X and J, where $R = [R_{ij}]$ and $G = [G_{ij}]$ are nxn mutually reciprocal resistance and conductance matrices respectively. The condition for microscopic reversibility leads to the reciprocal relations $R = R^t$, $G = G^t$. A dissipation function acting as a potential for the thermodynamic forces is then defined in terms of a dissipation matrix R as a quadratic form in the 'fluxes', giving the rate of entropy production for irreversible processes

$$2D(\dot{\alpha}\dot{\alpha}) = \dot{s}(\alpha\dot{\alpha}) = X^t\dot{\alpha} = \dot{\alpha}^t R\dot{\alpha} = X^t GX \ ; \ X = \partial D/\partial\dot{\alpha} \tag{4}$$

Irreversible processes in which the velocity or kinetic energy contribution to entropy change may be significant are defined in terms of the even and odd variables α and $\beta\,(=\dot{\alpha})$ respectively[2,3]. Microscopic reversibility then requires the conditions $\alpha(\tau) = \alpha(-\tau)$ and $\beta(\tau) = -\beta(-\tau)$, and the reciprocal relations introduce opposite signs for the mutual coefficients between the α and β variables. Entropy, as an even function invariant with respect to time reversal, may then be considered as a homogeneous quadratic function with 'potential' and 'kinetic' components,

$$s = -(\alpha^t \Gamma \alpha + \beta^t L \beta)/2 \tag{5}$$

For phenomonological laws of the form

$$R\,\dot{\alpha} = \xi \tag{6}$$

Machlup and Onsager[2] define the thermodynamic forces

$$\xi = \frac{\partial s}{\partial \alpha} + \frac{d}{dt}\left(\frac{\partial s}{\partial \dot{\alpha}}\right) = -\Gamma\alpha - L\dot{\beta} \tag{7}$$

The formulation is thus associated with a second-order homogeneous system

$$L\ddot{\alpha} + R\dot{\alpha} + \Gamma\alpha = 0 \tag{8}$$

and the rate of entropy production

$$\dot{s} = -\alpha^t\Gamma\dot{\alpha} - \beta^t L\dot{\beta} = \xi^t\beta = \dot{\alpha}^t R\dot{\alpha} = 2D(\dot{\alpha}\dot{\alpha}) = \xi^t R^{-1}\xi \tag{9}$$

Now incorporating even and odd conjugate variables defined by

$$\gamma_1 = L^{-1}\Gamma\alpha\,, \qquad \gamma_2 = \beta \tag{10}$$

the system eqns 8 may be represented in the form

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -L^{-1}R \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \tag{11}$$

Eqn 11 illustrates the reciprocal relations $m_{12} = -m_{21}$ with the system equations defined in terms of odd and even quantities.

The form of eqn 7 defining the thermodynamic forces is closely related to the general Lagrangian representation for a non-conservative system with coordinates α referred to an equilibrium position, given by the vector form

$$\frac{\partial \mathcal{L}}{\partial \alpha} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) = \frac{\partial D}{\partial \dot{\alpha}} \tag{12}$$

With forcing v included in the linear dynamical system of eqn 8 the Lagrangian state function is defined

$$\mathcal{L}(\alpha, \dot{\alpha}, v) = v^t\alpha + (\dot{\alpha}^t L\dot{\alpha} - \alpha^t\Gamma\alpha)/2 \tag{13}$$

Thermodynamic forces or affinities may also be defined in terms of a Hamiltonian-type formulation[12], with generalised flux (momentum) components $\phi_1 \ldots \phi_n$ introduced with the vector transformation

$$\phi = \partial\mathcal{L}/\partial\dot{\alpha} = L\dot{\alpha} \tag{14}$$

The scalar Hamiltonian function is then given by

$$H(\alpha, \phi, v) = \phi^t\dot{\alpha} - \mathcal{L} = (\phi^t L^{-1}\phi + \alpha^t\Gamma\alpha)/2 - v^t\alpha \tag{15}$$

Eqn 15 is constant for a conservative system and for the system with forcing it may be equated with the function $- s - v^t\alpha$. Thermodynamic forces may then be associated with the differential

$$dH = -a^t d\alpha - b^t d\phi - \alpha^t dv \tag{16}$$

where $$a = v - \Gamma\alpha = -\partial H/\partial\alpha = \dot{\phi} + R\dot{\alpha} \tag{17}$$

$$b = -L^{-1}\phi = -\partial H/\partial\phi = -\dot{\alpha} \tag{18}$$

The rate of energy dissipation is given by the energy balance (input-rate of change of stored energy)

$$\begin{aligned} 2D &= v^t\dot{\alpha} - \frac{d}{dt}(H + v^t\alpha) = a^t\dot{\alpha} + b^t\dot{\phi} \\ &= \dot{\alpha}^t R\dot{\alpha} = -dH/dt = \dot{s} \end{aligned} \tag{19}$$

Eqns 17, 18 represent a system of 2n first-order Hamiltonian canonical equations for a non-conservative system. The coefficient a is also associated with a steady-state equilibrium condition of the dynamic system obtained as the extremum condition of the Hamiltonian or Lagrangian function with respect to the state α. Eqn 17 can be associated directly with the n-mesh voltage equation for the general RLC network, in terms of mesh current coordinates $\dot{\alpha}_1 .. \dot{\alpha}_n$ and impressed mesh voltages $v_1 .. v_n$, where Γ is a symmetrical reciprocal capacitance matrix and R, L are symmetrical resistance and inductance matrices respectively[12-14].

The rate eqns 17, 18 may be expressed in the matrix form

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & -I_n \\ I_n & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{20}$$

Affinities a and variables α are even, and affinities b and variables ϕ are odd with respect to time reversal. The components e_{11}, e_{22} for the linear system are symmetrical and represent Onsager coefficients. The Casimir coefficients e_{12}, e_{21} define the reciprocal relations $e_{12} = -e^t_{21}$. The second-order-type system with forcing may also be represented in the 2n-dimensional companion matrix form

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -L^{-1}\Gamma & -L^{-1}R \end{bmatrix} \begin{bmatrix} \alpha \\ \rho \end{bmatrix} + \begin{bmatrix} 0 \\ L^{-1} \end{bmatrix} v \tag{21}$$

4. Reciprocal relations in the linear optimal control and filter problems

Reciprocal relations associated with the general phenomenological equations representing a physical system can also be shown to exist in the formulation of the linear optimal control problem. The problem may be concerned with determining the control vector $u(t)$ in the n-state, m-input linear dynamic system of eqn 1, which minimises the quadratic performance functional

$$J = \int_{t_o}^{T_f} f_o(x(t),\ u(t))dt\ ,\quad f_o = (x^tQx + u^tGu + 2x^tWu)/2 \qquad (22)$$

where Q is an nxn positive semi-definite matrix and G is an nxn positive definite matrix. A scalar product of the state and control variables is included with the nxm matrix W, which may be required to represent say the power functions of an electromechanical system as in Section 11. Similar cross-product weighting also appears in the model following problem[15]. In the maximum principle a Hamiltonian is defined

$$H(p,\ x,\ u) = p_of_o + p^tf \qquad (23)$$

where the adjoint variables p satisfy the linear, homogeneous differential equations

$$\dot{p}_i = -\sum_{j=o}^{n} \frac{\partial f_j}{\partial x_i} p_j\ , \qquad i = 0,\ 1..n \qquad (24)$$

Thus $x = \partial H/\partial p$, $\qquad \dot{p} = -\partial H/\partial x \qquad (25)$

For the linear system with quadratic performance and appropriate boundary conditions

$$H = -f_o + p^t(Ax + Bu) \qquad (26)$$

$$\dot{p} = Qx - A^tp + Wu \qquad (27)$$

The maximum principle requires H to be a maximum along an optimal trajectory, and differentiation of eqn 26 with respect to control u gives

$$u(t) = G^{-1}(B^tp - W^tx) \qquad (28)$$

The optimal Hamiltonian system may then be represented in terms of the state and adjoint variables by the 2n-dimensional differential equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} (A-BG^{-1}W^t) & BG^{-1}B^t \\ (Q-WG^{-1}W^t) & -(A^t-WG^{-1}B^t) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \text{ or } \dot{h}(t) = Mh(t) \qquad (29)$$

The optimally regulated trajectory is then given by the solution of eqn 29 with the two-point boundary conditions, $x(t_o) = 0$, $p(T_f) = 0$. Including a state-adjoint variable relation

$$p(t) = -Px(t) \tag{30}$$

in eqn 29 gives the nonlinear matrix Riccati differential equation

$$\dot{P} = (PB + W)G^{-1}(B^tP + W^t) - PA - A^tP - Q \tag{31}$$

where the nxn symmetrical matrix P is a unique positive definite solution for all positive definite matrices Q. With asymptotic stability ($T_f \to \infty$) a solution of the algebraic matrix Riccati equation is given by[16]

$$P = -U_{21} U_{11}^{-1} \tag{32}$$

where U_{21}, U_{11} represent partitioned eigenvector components associated with the stable modes of the matrix M.

A dynamic programming solution may also be obtained with minimisation of the quadratic performance index

$$J_N = \sum_{i=1}^{N} (u_{i-1}^t G u_{i-1} + x_i^t Q x_i + 2x_{i-1}^t W u_{i-1}) \tag{33}$$

associated with the discrete-time dynamic system

$$x(k+1) = \phi(T)x(k) + \Delta(T) u(k) \tag{34}$$

Minimisation with backward tracing gives the optimal control law

$$u_{N-r} = K_r x_{N-r} \tag{35}$$

where

$$K_r = -(\Delta^t P_{r-1}\Delta + G)^{-1} (\Delta^t P_{r-1}\phi + W^t) \tag{36}$$

$$P_{r-1} = (\phi + \Delta K_{r-1})^t P_{r-2}(\phi + \Delta K_{r-1}) + K_{r-1}^t G K_{r-1} + 2WK_{r-1} + Q \tag{37}$$

$$P_o = Q , \quad r = 1 \tag{38}$$

Substituting the form of eqn 36 into eqn 37, the control algorithm may also be stated by eqns 35 and 36 with

$$P_{r-1} = \phi^t P_{r-2}\phi + (\phi^t P_{r-2}\Delta + W)K_{r-1} + Q \tag{39}$$

The discrete reverse-time control algorithm developed by dynamic programming will reduce, in the limit, to the reverse-time continuous solution. Thus using the first-order approximations, with small sampling interval h,

$$\phi = I + Ah , \quad \Delta = Bh , \quad Q(t) = Q_k h , \quad W(t) = W_k h \tag{40}$$

in eqn 39 gives, in the limit $h \to o$ with $[P(\tau+h)-P(\tau)]/h \to \dot{P}(\tau)$,

$$\dot{P}(\tau) = A^tP + PA - (PB + W)G^{-1}(B^tP + W^t) + Q \tag{41}$$

corresponding with the form of eqn 31 for integration in reverse time with $\tau = T_f - t$. The control law of eqn 35 will reduce similarly to the continuous-time optimal control solution.

The Hamiltonian formulation of the general system problem of Section 3 can be associated with the optimal control problem formulated in terms of the state and adjoint variables x, p with $\mathcal{L} \equiv f_o$. In particular, quantities analogous to the affinities may be identified using eqn 26 to give the differential form

$$dH = -a^tdx - b^tdp + (p^tB - u^tG - x^tW)du \tag{42}$$

where
$$a = Qx - A^tp + Wu = -\partial H/\partial x = \dot{p} \tag{43}$$

$$b = -(Ax + Bu) = -\partial H/\partial p = -\dot{x} \tag{44}$$

With dissipation defined analogously to eqn 19

$$2D = a^t\dot{x} + b^t\dot{p} = 0 \tag{45}$$

Thus the formulation of the optimal control problem incorporates conditions which ensure that the combined optimally controlled system is conservative. Similar properties with reference to the nonquadrature net-phase-shift of Hamiltonian systems have also been discussed[17]. Thus the Hamiltonian system in the optimal control problem is formulated on the basis of equality between energy supplied and stored energy in the overall system, with constant H.

The rate eqns 43, 44 may also be represented in the form of eqn 20,

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{46}$$

Eqn 46 illustrates the existence of zero dissipation, with analogous Onsager-type coefficients $e_{11} = e_{22} = 0$ and Casimir-type coefficients $e_{21} = -e_{12}^t = I_n$. Thus only Casimir-type reciprocal relations appear in the maximum principle formulation of the linear optimal control problem. Analogous relations also exist between the representation of the free conservative system of Section 3 and the linear optimal control problem based on the correspondence

$$x \equiv \alpha\ , \quad p \equiv \phi\ , \quad Q + A^tP \equiv -\Gamma\ , \quad BG^{-1}B^t - AP^{-1} \equiv L^{-1} \tag{47}$$

with a nonsingular Riccati solution matrix P. This equivalence forms the basis of the analogy between the classical operator solution of the general system and the algebraic matrix Riccati equation as discussed in Section 5.

A similar correspondence may be found with the dual linear optimal filter problem. Thus consider the dynamical system of eqn 1 with outputs y(t) and observed signals z(t) given by

$$y(t) = H(t)\,x(t) \tag{48}$$

$$z(t) = y(t) + v(t) \tag{49}$$

where the functions u(t), v(t) represent independent random white noise processes with zero means and covariance matrices

$$[u(t)u^t(\tau)] = Q(t)\delta(t-\tau)\ , \quad [v(t)v^t(\tau)] = R(t)\delta(t-\tau) \tag{50}$$

where Q(t), R(t) are symmetric, positive definite matrices. The optimal estimate of the state $\hat{x}(t)$ is generated by the Kalman-Bucy filter equations[18]

$$\dot{\hat{x}}(t) = A(t)\,\hat{x}(t) + K(t)\,[z(t) - H(t)\,\hat{x}(t)] \tag{51}$$

$$K(t) = P(t)\,H^t(t)\,R^{-1}(t) \tag{52}$$

where the covariance matrix $P(t) = [x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^t$ is a solution of the matrix differential variance or Riccati-type equation

$$\dot{P} = AP + PA^t - PH^tR^{-1}H\,P + BQB^t \tag{53}$$

The solution of the variance equation can be associated with the Hamiltonian function, defined in terms of state and adjoint variables x, w,

$$H(x,\ w,\ t) = -(x^tBQB^tx - w^tH^tR^{-1}Hw)/2 - w^tA^tx \tag{54}$$

and with the corresponding canonical differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \partial H/\partial w \\ -\partial H/\partial x \end{bmatrix} = \begin{bmatrix} -A^t & H^tR^{-1}H \\ BQB^t & A \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{55}$$

$$w(t) = P(t)\,x(t) \tag{56}$$

The differential of the Hamiltonian function may then be stated

$$dH = a^t dx + b^t dw \tag{57}$$

where

$$a = -Aw - BQB^tx = -\dot{w} \tag{58}$$

$$b = -A^tx + H^tR^{-1}Hw = \dot{x} \tag{59}$$

A dissipation function may also be defined as previously

$$2D = a^t\dot{x} + b^t\dot{w} = 0 \tag{60}$$

The rate equations can then be specified in terms of 'affinities' by the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{61}$$

Eqn 61 represents the dual form of eqn 46 and illustrates the existence of the Casimir-type reciprocal relations in the linear optimal filtering problem. The free conservative system formulation is similarly related to the linear filtering problem with the correspondence

$$x \equiv \alpha \, , \quad w \equiv \emptyset \, , \quad BQB^t + AP \equiv -\Gamma \, , \quad H^tR^{-1}H - A^tP^{-1} \equiv L^{-1} \tag{62}$$

5. The matrix Riccati equation in the electrical network problem

A matrix analogue of the classical operator solution of the general system eqns 21 can be obtained by formulating an algebraic matrix Riccati-type equation, with a solution matrix related to eigenvector components of the form of eqn 32. Thus defining the eigenvector matrix equation for the general system, representing say the electrical network problem,

$$\begin{bmatrix} 0 & I_n \\ -L^{-1}\Gamma & -L^{-1}R \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{63}$$

Equating components and eliminating eigenvalues Λ_1 and Λ_2 then gives algebraic matrix Riccati-type equations of the form

$$P^2 + L^{-1}RP + L^{-1}\Gamma = 0 \tag{64}$$

where $$P = U_{21}U_{11}^{-1} = U_{11}\Lambda_1 U_{11}^{-1} \tag{65}$$

and $$KL^{-1}\Gamma K + KL^{-1}R + I = 0 \tag{66}$$

where $$K = U_{12}U_{22}^{-1} = U_{22}\Lambda_2^{-1}U_{22}^{-1} = P^{-1} \tag{67}$$

Eqns 64 and 66 represent a matrix analogue of the classical operator solution of the defining second-order differential equations. The matrix P introduces a transformation between the dual variables which is a basic characteristic of the Riccati formulation of the optimal control problem, and corresponds to the differential operator transformation of eqn 14.

The algebraic matrix Riccati equation can also be associated with the 2n-dimensional steady-state multiple transmission line equations[19],

$$\begin{bmatrix} dv/dx \\ di/dx \end{bmatrix} = -\begin{bmatrix} 0 & Z(x) \\ Y(x) & 0 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} = M\begin{bmatrix} v \\ i \end{bmatrix} \tag{68}$$

where v, i are the unknown complex voltage and current n-vectors, and Z(x), Y(x) are continuous symmetric series impedance and shunt admittance n-matrices respectively. Including a relation between the dual sets of variables

$$i = P(x)\, v \tag{69}$$

leads to the nonlinear matrix Riccati differential equation

$$P' = P(x)\, Z(x)\, P(x) - Y(x)\,, \qquad (' \equiv d/dx) \tag{70}$$

Sternberg and Kaufman[19] introduce the continuous symmetric nxn matrix solution

$$P(x) = I(x)\, V^{-1}(x) \tag{71}$$

where I(x), V(x) are nxn matrices of mutually conjugate solutions of eqn 68 with

$$I^{*}V - V^{*}I \equiv 0 \tag{72}$$

Similarly, the nxn matrix

$$K(x) = V(x)\, I^{-1}(x) \tag{73}$$

is a continuous symmetric matrix solution of the Riccati-type matrix differential equation

$$K' = KY(x)\, K - Z(x) \tag{74}$$

With I(x), V(x) nonsingular, P(x), K(x) are nonsingular and

$$P^{-1}(x) = K(x) \tag{75}$$

The solutions of eqns 71 and 73 are analogous to the eigenvector solution of the steady-state matrix Riccati differential equation given by

$$P_o = U_{21}\, U_{11}^{-1} \tag{76}$$

where U_{21}, U_{11} are eigenvector components of the matrix M. Thus the eigenvalue problem related to the form of eqn 68 is defined by

$$\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{77}$$

Expanding the components of eqn 77 in the eigenvalue Λ_1 and then eliminating Λ_1 gives the algebraic matrix Riccati equation

$$Y = (U_{21} U_{11}^{-1}) Z (U_{21} U_{11}^{-1}) \tag{78}$$

corresponding to the steady-state form of eqn 70 with $P_o = U_{21} U_{11}^{-1}$. Similarly, partitioning eqn 77 in terms of the eigenvalues Λ_2 leads to the algebraic matrix Riccati equation

$$Z = (U_{12} U_{22}^{-1}) Y (U_{12} U_{22}^{-1}) \tag{79}$$

corresponding to the steady-state form of eqn 74 with $K_o = U_{12} U_{22}^{-1}$. Eqns 78 and 79 are similar to the form of eqns 64 and 66 for the general lossless network ($R = 0$).

A matrix Riccati-type representation may also be obtained for the symmetrical transmission line consisting of identical multi-terminal sections. The steady-state performance of the passive nth section with input and output m-vector voltages and currents $v(n)$, $i(n)$ and $v(n+1)$, $i(n+1)$ may be obtained in the difference equation form[20],

$$\begin{bmatrix} v(n+1) \\ i(n+1) \end{bmatrix} = \begin{bmatrix} C(n) & -Z(n) \\ -Y(n) & C^t(n) \end{bmatrix} \begin{bmatrix} v(n) \\ i(n) \end{bmatrix} \tag{80}$$

Then with an assumed transformation of the form of eqn 69

$$i(n+1) - i(n) = P(n)[v(n+1) - v(n)] + [P(n+1) - P(n)]v(n) \tag{81}$$

and combining with eqn 80 gives

$$P(n+1) - P(n) = P(n)Z(n)P(n) + C^t(n)P(n) - P(n)C(n) - Y(n) \tag{82}$$

In the limit, with a symmetrical coefficient matrix or with $C = 0$, a matrix Riccati differential equation is obtained as in eqn 70.

6. A transition matrix solution of the electrical network problem

A time solution of the electrical network problem can be defined in terms of transition matrix components and illustrates the time invariant properties of a particular lossless network. A time solution of the system eqns 21 will be given by

$$\begin{bmatrix} \alpha(t) \\ \rho(t) \end{bmatrix} = \phi(t-t_o) \begin{bmatrix} \alpha(t_o) \\ \rho(t_o) \end{bmatrix} + \int_{t_o}^{t} \phi(t-\tau) \begin{bmatrix} 0 \\ L^{-1} \end{bmatrix} v(\tau) d\tau \tag{83}$$

where the transition matrix $\emptyset(t-t_o)$ is defined by

$$\emptyset(t) = \sum_{k=o}^{\infty} N^k t^k/k! = \begin{bmatrix} \emptyset_{11}(t) & \emptyset_{12}(t) \\ \emptyset_{21}(t) & \emptyset_{22}(t) \end{bmatrix}, \quad N = \begin{bmatrix} 0 & I_n \\ -L^{-1}\Gamma & -L^{-1}R \end{bmatrix} \tag{84}$$

For the lossless network, component expansions may be obtained in the form

$$\emptyset_{11}(t) = \emptyset_{22}(t) = \sum_{k=o}^{\infty} (-L^{-1}\Gamma)^k (t^2)^k/(2k)! \tag{85}$$

$$\emptyset_{12}(t) = t\sum_{k=o}^{\infty} (-L^{-1}\Gamma)^k t^{2k}/(2k+1)!\ , \quad \emptyset_{21} = -(L^{-1}\Gamma)\emptyset_{12}(t) \tag{86}$$

The series expansions are associated with the matrix $(L^{-1}\Gamma)$ with the diagonal components similar in form to the exponential matrix, and with the components $\emptyset_{12}(t)$, $\emptyset_{21}(t)$ similar to the driving matrix for the discrete linear system. With step voltage inputs $v(kT)$, the discrete-time response of the electrical network will be governed by a difference equation of the form of eqn 34, with a driving matrix given by

$$\Delta(T) = \sum_{k=o}^{\infty} N^k T^{k+1}/(k+1)! = N^{-1}[\emptyset(T) - I_{2n}]\begin{bmatrix} 0 \\ L^{-1} \end{bmatrix}, \quad |N| \neq 0$$

$$= N^{-1}\begin{bmatrix} \emptyset_{12}(T)L^{-1} \\ (\emptyset_{22}(T)-I_n)L^{-1} \end{bmatrix} = \begin{bmatrix} \Gamma^{-1}L(I_n-\emptyset_{22}(T))L^{-1} \\ \emptyset_{12}(T)L^{-1} \end{bmatrix} \tag{37}$$

With diagonal (primitive) impedance matrices the transition matrix components will be diagonal and will determine the transient behaviour of the individual mesh variables. The form of eqns 85-87 illustrates the odd and even nature of the variables α and ρ respectively with respect to time reversal which results essentially from the translation properties of the transition matrix. Thus

$$\emptyset_{11}(-t) = \emptyset_{11}(t), \emptyset_{22}(-t) = \emptyset_{22}(t), \emptyset_{12}(-t) = -\emptyset_{12}(t), \emptyset_{21}(-t) = -\emptyset_{21}(t)$$

The invariance of time reversal in the particular lossless electrical network problem may also be illustrated in the discrete solution, with a sign reversal on the 'velocity' coordinates ρ. Thus

$$\alpha(k+1)(-T) = \emptyset_{11}(T)\alpha(k)(-T) - \emptyset_{12}(T)\rho(k)(-T) + \Gamma^{-1}L(I_n-\emptyset_{22}(T))v(k)(-T) \tag{88}$$

$$\rho(k+1)(-T) = -\emptyset_{21}(T)\alpha(k)(-T) + \emptyset_{22}(T)\rho(k)(-T) - \emptyset_{12}(T)L^{-1}\ v(k)(-T) \tag{89}$$

In the control problem the system eqns 29-31 are invariant with respect to backward time, or time-to-go to final time T_f, defined in terms of the variable $\tau = T_f - t$, with a sign change included on the derivatives. Thus

$$\dot{h}(\tau) = -Mh(\tau) , \qquad p(\tau) = -P(\tau)\, x(\tau) \tag{90}$$

$$\dot{P}(\tau) = P(\tau)A + A^t P(\tau) + Q - (P(\tau)B + W)G^{-1}(B^t P(\tau) + W^t), \; P(\tau_o) = 0 \tag{91}$$

7. The scattering matrix of electrical network theory in the linear optimal control problem

The scattering parameters of an electrical network are related to a transformation between linear combinations of network voltages and currents, and have an important application in the definition of energy constraints and power transfer in passive networks[21,22]. The parameters are analogous to the reflection and transmission coefficients used to describe wave propagation in transmission line theory. Parameters with similar properties can also be established in the linear optimal control problem.

For the n-port passive network $\underline{N}$ of FIG. 1, a square scattering matrix S is defined by the transformation

$$v_r = Sv_i \qquad \text{or} \qquad (v-i)/2 = S(v+i)/2 \tag{92}$$

where v_r, v_i represent vectors of normalised 'reflected' and 'incidence' voltages or scattering variables at the network ports.

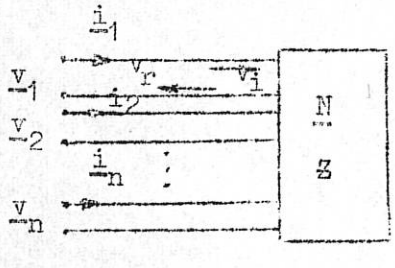

FIG. 1. n-port network

The normalised voltage and current variables v, i are related to actual voltage and current vectors $\underline{v}$, $\underline{i}$ by

$$v = R_o^{-\frac{1}{2}}\, \underline{v} , \qquad i = R_o^{\frac{1}{2}}\, \underline{i} , \qquad \underline{v} = \underline{Z}\, \underline{i} \tag{93}$$

Combining eqns 92 and 93 gives the scattering matrix related to the normalised impedance matrix $Z(= R_o^{-\frac{1}{2}} Z R_o^{-\frac{1}{2}})$,

$$S = (Z + I_n)^{-1}(Z - I_n) = (Z - I_n)(Z + I_n)^{-1} \quad (94)$$

$$\text{and} \quad Z = (I_n - S)^{-1}(I_n + S) = (I_n + S)(I_n - S)^{-1} \quad (95)$$

The total network power is defined by

$$P_T = P_d + jQ = \underline{v}^{*t}\underline{i} = (\underline{v}^{*t}\underline{i} + \underline{i}^{*t}\underline{v})/2 + (\underline{v}^{*t}\underline{i} - \underline{i}^{*t}\underline{v})/2 \quad (96)$$

with real dissipated power

$$P_d = P_i - P_r = v_i^{*t} v_i - v_r^{*t} v_r = v_i^{*t} F\, v_i, \; F = I_n - S^{*t} S \quad (97)$$

and reactive power

$$jQ = v_r^{*t} v_i - v_i^{*t} v_r = v_i^{*t} L\, v_i, \quad L = S^{*t} - S \quad (98)$$

where F is a nonsingular positive hermitian dissipation matrix, and L is a skew-hermitian matrix. For the reactive lossless network $P_d = 0$, $S^{-1} = S^{*t}$, and represents a unitary matrix. For maximum delivered power, $S = 0$, $P_d = v_i^{*t} v_i$, $Z = I_n$, $Z = R_o$, where P_d is equal to the sum of the available powers at the network ports with the network matched to the source impedance R_o. The coefficients of S thus measure the deviation of the circuit impedance or load from the normalising number R_o or from matched maximum power transfer conditions.

The linear optimal control problem may also be formulated in terms of a transformation matrix S which is analogous to the scattering operator of electrical network theory. Thus we may consider defining new 'state' (incident) and 'adjoint' (reflected) variables related by a transformation or 'scattering' matrix S,

$$x_i = (x - p)/2 = (I+P)x/2, \; x_r = (x+p)/2 = (I-P)x/2, \; x_r = Sx_i \quad (99)$$

Then

$$S = (I-P)(I+P)^{-1} \quad (100)$$

$$P = (I+S)^{-1}(I-S) \quad (101)$$

Thus the P-matrix of the Riccati solution in the optimal control problem may be considered to possess properties analogous to the scattering matrix of electrical network theory. Similarly, the assumed transformation or symmetrical 'scattering' matrix in the control problem corresponds with the normalised network impedance matrix Z. Also with P defined in terms of the eigenvector matrix components of eqn 32,

$$S = (U_{11} + U_{21})(U_{11} - U_{21})^{-1} \qquad (102)$$

Other scattering variables may be defined using combinations of the state and control variables together with an interrelating transformation matrix. By analogy with the network problem a 'power' function may also be defined by the product

$$P_1 = x^t p = -(x_i^t x_i - x_r^t x_r) = x_i^t(S^t S - I)x_i = x_i^t F x_i \qquad (103)$$

and 'injected power'

$$P_i = x^t Bu = x_i^t(S^t BG^{-1}B^t S - BG^{-1}B^t)x_i \qquad (104)$$

The free system is then associated with the condition $S = I$. A concept of maximum 'delivered power' may be considered with the condition $S = 0$, $P = I$, with the sign convention determined by the form of eqn 99. Thus it may be possible to relate the coefficients of S to a deviation from matched or maximum 'power transfer' conditions. From the algebraic matrix Riccati equation such a condition would define the relation

$$Q = BG^{-1}B^t - (A + A^t) \qquad (105)$$

and is associated with an incidence variable $x_i = x$ and zero reflected variable x_r. The solution of the Riccati equation for the electromechanical system of Section 11 using the Q matrix of eqn 105 is shown to reduce correspondingly to the unit matrix. However, in general, the solution of the algebraic matrix Riccati equation for a positive-definite matrix P requires an assumed positive-definite Q matrix. Thus Q must be constrained to be a positive-definite form, which may not always exist in eqn 105. For example, such a condition is not ensured for the system represented in companion matrix form with a single non-zero B-matrix element, as illustrated in Section 11. In application to the optimal control of an electric arc furnace model[27], a Q-matrix defined by eqn 105 has been found, however, to give relatively good stable response with a controlled system matrix of the form $(A - BG^{-1}B^t) = -(A^t + Q)$.

If the variables x_i are constrained by the condition $x_i^t x_i = 1$ then the power function of eqn 103 will be bounded by the inequalities $\lambda_{min}(F) \leqslant P \leqslant \lambda_{max}(F)$. This follows, as in the electrical network problem[23,24], using the transformation $F = T^t \Lambda T$, where T is a real orthogonal matrix and Λ is a diagonal matrix of eigenvalues. Then

$$P_1 = (Tx_i)^t \Lambda (Tx_i) = \sum_{j=1}^{n} (Tx_i)_j^2 \, \lambda_j(F) \tag{106}$$

The 'injected power' function of eqn 104 will possess similar properties.

The optimal control problem may also be formulated in terms of the 'incident' and 'reflected' variables of eqn 99. Thus from eqn 29 with $W = 0$,

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_r \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (A-A^t-BG^{-1}B^t-Q) & (A+A^t+BG^{-1}B^t-Q) \\ (A+A^t-BG^{-1}B^t+Q) & (A-A^t+BG^{-1}B^t+Q) \end{bmatrix} \begin{bmatrix} x_i \\ x_r \end{bmatrix} \tag{107}$$

This representation retains the same form as eqn 29 with symmetrical off-diagonal components. 'Affinities' may also be identified in the 'scattering' formulation with

$$dH = -a^t dx_i - b^t dx_r + [(x_r - x_i)^t B - u^t G]du \tag{108}$$

where

$$a = (Q + A + A^t)x_i + (Q + A - A^t)x_r + Bu = -\partial H/\partial x_i = 2\dot{x}_r \tag{109}$$

$$b = (Q - A + A^t)x_i + (Q - A - A^t)x_r - Bu = -\partial H/\partial x_r = -2\dot{x}_i \tag{110}$$

Then
$$2D = a^t \dot{x}_i + b^t \dot{x}_r = 0 \tag{111}$$

Eqn 111 illustrates the conservative-type conditions in the optimal control problem defined in terms of the 'scattering' variables. The overall system may also be associated with a matrix Riccati differential equation. Thus combining the 'scattering' matrix with eqn 107 gives

$$2\dot{S} = -SM_{12}S - SM_{11} + M_{22}S + M_{21} \tag{112}$$

where M_{ij} represent the matrix components of eqn 107. Also

$$2\dot{S}x_i = a + Sb \tag{113}$$

It is interesting to note that the Riccati differential eqn 112 may now be decomposed directly into components associated with skew-symmetrical and symmetrical matrices.

The general theory of scattering is concerned with the behaviour of an evolving system as time t tends to $-\infty$ compared with its asymptotic behaviour as t tends to $+\infty$. In this respect similar basic concepts may exist with the transformation between the state and adjoint variables in the optimal control problem. Also, in the dynamic theory of scattering applied to the wave equation a scattering operator is defined in terms of forward and reverse wave operators ($S = W_+^{-1}W_-$)[25] which may be identified with the solution of the Riccati equation in the optimal control problem related to 'forward' and 'backward' transition-matrix components[16], thus

$$P(\tau) = -\phi_{21}(\tau)\phi_{11}(\tau)^{-1} \tag{114}$$

8. Conclusions

The basic laws of electrical network theory and properties of the linear optimal control and filtering problems have been shown to be closely associated with the general phenomenological equations of irreversible physical processes. Similar reciprocal relations exist, and the matrix Riccati equation in the optimal control problem associated with a transformation between dual sets of variables has also been shown to exist in the electrical network problem as a matrix analogue of the classical operator solution. The scattering matrix of electrical network theory which is of importance in defining power transfer in passive networks has also been established by analogy in the linear optimal control problem using 'incidence' and 'reflected' variables which possess similar properties to the scattering variables of network theory. Such concepts and relations originating in electrical network and general system theory are of fundamental importance and can provide greater physical insight and understanding of the optimal control problem.

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11. Appendix

11.1 Power relations in an electromechanical system

FIG. 2 represents a separately excited dc motor connected to a variable voltage supply and driving an inertia load with damping.

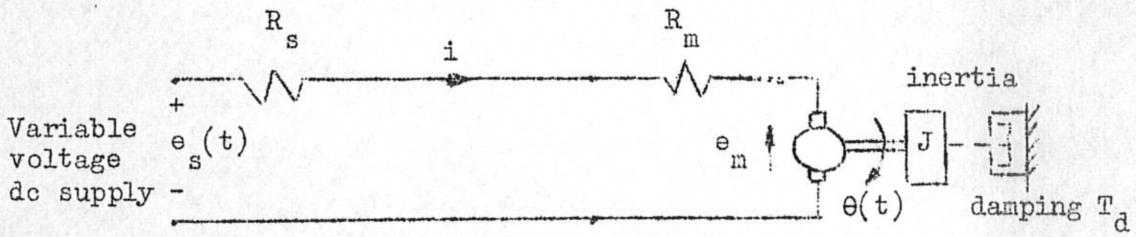

FIG. 2. Electromechanical system with damping and inertia loading

System equations: $e_s(t) = (R_s + R_m)i + e_m$, neglecting field inductance (115)

$$e_m = k\,\omega\,, \quad k \text{ constant} \tag{116}$$

$$T = k\,i = J\,\ddot{\theta} + T_d\,\dot{\theta}\,,\ T \text{ is developed motor torque} \tag{117}$$

State variable representation:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} e_s\,, \qquad \dot{x} = Ax + Bu \tag{118}$$

where $a_{22} = -[T_d + k^2/(R_s + R_m)]/J$, $b_2 = k/J(R_s + R_m)$

Input power to system P_i, to motor armature P_m as active power and total dissipated power P_d referred to units of the state equations are represented in the form

$$P = u^t G u + x^t \bar{A} x + x^t \bar{B} u\,, \quad u = e_s\,, \quad x = (\theta,\omega)^t \tag{119}$$

Thus $P_i = e_s i/J = (e_s^2 - e_s k\,\omega)/J(R_s + R_m)$ (120)

$$G_i = 1/J(R_s + R_m)\,, \quad \bar{A}_i = 0\,, \quad \bar{B}_i = \begin{bmatrix} 0 \\ -k/J(R_s+R_m) \end{bmatrix} = -B \tag{121}$$

$$P_d = [T_d\omega^2 + (e_s - e_m)^2/(R_s + R_m)]/J \tag{122}$$

$$G_d = G_i\,, \quad \bar{A}_d = \begin{bmatrix} 0 & 0 \\ 0 & (T_d + k^2/(R_s+R_m))/J \end{bmatrix}, \quad \bar{B}_d = 2\,\bar{B}_i \tag{123}$$

$$P_m = (e_m i - T_d \omega^2)/J = P_i - P_d \tag{124}$$

$$G_m = 0 , \quad \bar{A}_m = -\bar{A}_d , \quad \bar{B}_m = -\bar{B}_i = B \tag{125}$$

Also, motor input and dissipated power

$$P_t = vi/J , \quad P_{dm} = (T_d \omega^2 + i^2 R_m)/J \tag{126}$$

A state function associated with equilibrium conditions based on the theorem of least power may also be defined in terms of power transferred to the environment. Thus

$$P = \tfrac{1}{2} x^t \bar{A} x + x^t B u , \quad \bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix}$$

$$= \tfrac{1}{2} \omega^2 a_{22} + b_2 e_s \omega \tag{127}$$

Then the condition $\partial P/\partial \omega = 0$ defines the steady-state system equations. The state function is associated with power transferred to the mechanical load (environment) as the difference of equivalent injected and dissipated power (with $R = -a_{22}$), and corresponds to the form of active power P_m. The state function includes no weighting of the system input cost with a term $u^t G u$ as would be required in the optimal control problem, say for determining the supply voltage $e_s(t)$ for control of motor shaft position. In this case a performance criterion associated with an output state and dissipated power may be considered in the form

$$J = \int_0^{T_f} (\gamma \theta^2 + \lambda P_d) dt \tag{128}$$

Dissipation and entropy-type functions may also be defined for the electromechanical system, as in Section 3. Thus with the correspondence $L \equiv I$, $R \equiv -a_{22}$, $\Gamma = 0$ and with state variables (θ, ω) and input $b_2 e_s$, the entropy function of eqn 5 which acts as a potential for the thermodynamic forces is

$$s = -(\theta^2 \Gamma + \omega^2 L)/2 + b_2 e_s \theta = -\omega^2/2 + b_2 e_s \theta \tag{129}$$

with $\bar{A}_s = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, $\quad \bar{B}_s = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}$

The form of s is similar to the state function with a unit dissipation factor and with coupling of the input and state variable θ. The equivalent Onsager 'thermodynamic' force with forcing is

$$\zeta = R\,\dot{\theta} = \frac{\partial s}{\partial \theta} + \frac{d}{dt}\left(\frac{\partial s}{\partial \omega}\right) = -\dot{\omega} + b_2 e_s \qquad (130)$$

and the rate of entropy production

$$\dot{s} = \zeta^t \beta = (b_2 e_s - \dot{\omega})\omega = R\omega^2 \quad : \quad \bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & -a_{22} \end{bmatrix} \qquad (131)$$

Thus the rate of entropy production is included as a component of the dissipated power P_d.

By analogy with the electrical network problem of Section 3

the Lagrangian function

$$\mathcal{L} = \omega^2/2 + b_2 e_s \theta = s + \omega^2 \qquad (132)$$

and the Hamiltonian

$$H(\theta, \omega, e_s) = \omega\dot{\theta} - \mathcal{L} = \omega^2/2 - b_2 e_s \theta = -s \qquad (133)$$

Then

$$dH = -b_2 e_s d\theta + \omega d\omega - b_2 \theta de_s \qquad (134)$$

$$= -a\,d\theta - b\,d\omega - b_2 \theta de_s \qquad (135)$$

where affinities

$$a = b_2 e_s = \frac{-\partial H}{\partial \theta} = \dot{\omega} + R\,\dot{\theta} \qquad (136)$$

$$b = -\omega = \frac{-\partial H}{\partial \omega} = -\dot{\theta} \qquad (137)$$

Then rate of energy dissipation

$$2D = a\,\dot{\theta} + b\,\dot{\omega} = b_2 e_s \dot{\theta} - \omega\dot{\omega} = -a_{22}\omega^2 = \dot{s} \qquad (138)$$

Optimal control of the electromechanical system for min J based on the form of eqn 128 with infinite-time settling may be obtained by direct solution of the steady-state matrix Riccati equation obtained from eqn 31. The particular form of the matrices A, B and W $= \begin{bmatrix} 0 \\ w_2 \end{bmatrix}$ leads to a solution of the P-matrix components given by the equations

$$b_2^2\, p_{12}^2 = q_1 G \tag{139}$$

$$p_{12} b_2 (b_2 p_{22} + w_2) - G\, p_{11} - G\, p_{12} a_{22} = 0 \tag{140}$$

$$(p_{22} b_2 + w_2)^2 - 2G(p_{12} + a_{22} p_{22}) - G\, q_2 = 0 \tag{141}$$

where the coefficients G, q_1, q_2 and w_2 are identified with the components of eqns 119-126. The control input is then given by

$$e_s = -G^{-1}(B^t P + W^t)x = -G^{-1}(b_2 p_{12},\ b_2 p_{22} + w_2)x \tag{142}$$

With the single-input system, component p_{11} is not required. The optimal control laws and resulting dissipated power functions obtained for various performance criteria related to dissipated and input energy with weighting on output motor position are illustrated in Table 1. The coefficient α represents the ratio of mechanical to electrical power dissipation, and also defines the ratio of voltage drop to back emf $((e_s - e_m)/e_m)$ in the steady-state equations. The coefficient β is associated with the measurement of position and is predominant in the coefficient ψ.

In terms of the scattering variables of Section 7 a maximum 'power' function condition is defined using the state weighting matrix of eqn 105. For the electromechanical system this reduces to the form

$$Q = \begin{bmatrix} 0 & -1 \\ -1 & (G^{-1} b_2^2 - 2a_{22}) \end{bmatrix}, \quad x^t Q x = \omega^2 (G^{-1} b_2^2 - 2a_{22}) - 2\theta\omega \tag{143}$$

Solution of eqns 139-141 then gives the condition $p_{11} = p_{22} = 1$, $p_{12} = 0$. However, for the particular system representation, Q is not positive definite with zero weighting on position θ, and thus cannot be used to define the position control problem. The form of Q associated with a companion A-matrix will only permit control of the single variable corresponding to the nonzero element of matrix B. In this case Q introduces positive weighting only on motor speed ω with the element q_{22}, similar to the quadratic power functions, and with $Q = \begin{bmatrix} 0 & 0 \\ 0 & q_{22} \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. For control of position θ a term q_{11} may be introduced, which will lead to control of the form given in Table 1 based on calculation of the elements p_{12} and p_{11}. A weighting of the variable θ may also be introduced by including a cross-product matrix W with a unit P-matrix solution of eqn 31.

Then

$$Q = (B+W)G^{-1}(B^t+W^t) - (A+A^t) \tag{144}$$

and $$u = -G^{-1}(B^t + W^t)\,x \tag{145}$$

For the electromechanical system this gives

$$Q = \begin{bmatrix} G^{-1}w_1^2 & , G^{-1}w_1(b_2+w_2) - 1 \\ G^{-1}w_1(b_2+w_2) - 1 & , G^{-1}(b_2+w_2)^2 - 2a_{22} \end{bmatrix} \tag{146}$$

Then $$u = -G^{-1}[w_1, b_2+w_2]x = -(w_1\bar{R}J/\lambda)\theta - k\omega(1+w_2/b_2)/\lambda, \quad G = \lambda/(\bar{R}J) \tag{147}$$

and the controlled system matrix

$$A_c = A - BG^{-1}(B^t+W^t) = \begin{bmatrix} 0 & 1 \\ -b_2w_1G^{-1} & , a_{22}-b_2(b_2+w_2)G^{-1} \end{bmatrix} \tag{148}.$$

Eigenvalue locations and dissipated power functions may then be determined based, say, on sensitivity to the cross-product weighting coefficients w_1 and w_2.

$$e_s = -\pi\theta + k\omega(1+\alpha-\psi) \; : \; P_d = [\alpha + (\frac{-\pi\theta}{k\omega} + \alpha - \psi)^2]\omega^2/((1+\alpha)\tau)$$

J	π	ψ^2	β
$\int_0^\infty (\lambda P_d+\gamma\theta^2)dt$, $\int_0^\infty (\lambda P_i+\gamma\theta^2)dt$	$k[(1+\alpha)\tau\gamma/\lambda]^{\frac{1}{2}}$ $(=\pi_1)$	$\alpha^2+\alpha+\beta_1$	$2[(1+\alpha)\tau]^{3/2}(\gamma/\lambda)^{\frac{1}{2}}$
$\int_0^\infty (\lambda P_{dn}+\gamma\theta^2)dt$, $\int_0^\infty (\lambda P_t+\gamma\theta^2)dt$	$\pi_1(\bar{R}/R_n)^{\frac{1}{2}}$	$\alpha^2+\alpha\frac{\bar{R}}{R_n}+\beta_2$	$\beta_1(\bar{R}/R_n)^{\frac{1}{2}}$

TABLE 1. Optimal control law and dissipated power function coefficients for various performance criteria

$$\alpha = RT_d/k^2, \; \tau = -1/a_{22} = (J/T_d)\alpha/(1+\alpha), \; \bar{R} = R_s + R_n$$
