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FEEDBACK STABILITY OF OPEN-LOOP UNSTABLE SYSTEMS: CONTRACTION-MAPPING APPROACH

by

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Abstract

Recent results obtained by Freeman on the stability of linear multivariable feedback systems using the concept of contraction mapping are extended to include the case of open-loop unstable plants. A consequence of the results is a form of Rosenbrock's stability criterion for open-loop unstable minimum phase systems.

In a recent paper[1] Freeman has used the contraction mapping theorem from functional analysis to develop a stability criterion for linear multivariable feedback systems. Although the results represent only sufficient conditions for the feedback stability of a given plant, the results allow a consideration of certain distributed systems[1] and provide an elegant proof of a form of Rosenbrock's stability theorem[2] for open-loop stable systems. In this letter we consider the extension of the results presented by Freeman[1] to include the possibility of open-loop unstable systems and demonstrate that previous extensions of Rosenbrock's work[2,3] are an immediate consequence.

Using the notation of Rosenbrock[2] we consider a multivariable unity feedback system with mxm forward path transfer function matrix Q(s). If y(s), r(s) are the m-vectors of output and reference input transforms respectively, the dynamic equations of the feedback system are written in the form

$$y(s) = Q(s)r(s) - Q(s)y(s) + z_o(s) \qquad (1)$$

where $z_o(s)$ is a term due to the inclusion of non-zero initial conditions as discussed by Freeman[1]. The problem of finding a condition for stability is placed in the context of devising conditions under which the transformation $r(s) \rightarrow y(s)$ defined by equation (1) is a mapping of the Banach space Y of holomorphic vector functions of s in a relatively compact domain Ω in the complex plane into itself[1]. The main result is as follows:

<u>RESULT 1</u>

If (i) $|Q(s)| \not\equiv 0$ and $Q^{-1}(s)$ $(= \hat{Q}(s))$ has elements which are holomorphic in Ω.

(ii) $\hat{Q}(s)z_o(s) \in Y$

(iii) $r(s) \in Y$

(iv) A(s) is an mxm transfer function matrix such that I+A has a bounded inverse on Y

then a sufficient condition for the existence of a unique output transform $y(s) \in Y$ is that

$$M = \|(I+A)^{-1}(\hat{Q}-A)\|_m < 1 \quad (2)$$

where $\|\cdot\|_m$ is the operator norm induced by the norm on Y[1].

Proof

From eqn (1) and assumption (i)

$$y(s) = r(s) + \hat{Q}z_o(s) - \hat{Q}y(s) \quad (3)$$

Adding A(s)y(s) to both ends of this equation

$$y(s) = (I+A)^{-1}r(s) + (I+A)^{-1}\hat{Q}z_o(s) - (I+A)^{-1}(\hat{Q}-A)y(s) \quad (4)$$

which can be written in the operator form

$$y(s) = W_A(s)y(s) \quad (5)$$

where, due to the assumptions of the theorem, W_A maps Y into itself. Note that, for any $y^1(s)$ and $y^2(s)$ in Y,

$$\|W_A(s)y^1(s) - W_A(s)y^2(s)\|_m = \|(I+A)^{-1}(\hat{Q}-A)\{y^1(s) - y^2(s)\}\|_m$$

$$\leq \|(I+A)^{-1}(\hat{Q}-A)\|_m \cdot \|y^1(s) - y^2(s)\|_m \quad (6)$$

and the result follows from the definition of a contraction and the contraction mapping theorem[1].

If assumption (ii) is strengthened by demanding that it be true for all possible initial conditions, the above result has a direct interpretation in terms of the stability of the feedback configuration[1]. Note however that assumptions (i) and (ii) make possible the analysis of a large class of open loop unstable systems providing that $\hat{Q}(s)$ is holomorphic in Ω (i.e. Q(s) is 'minimum phase') and that cancellations of zeros of $\hat{Q}(s)$ (i.e. poles of Q(s)) ensure that the product is holomorphic in Ω.

Although very little restriction has been placed on A(s), an interesting case arises when A(s) is taken to be diagonal of the form

$$(A(s))_{ii} = (\hat{Q}(s))_{ii} \tag{7}$$

when[1] Result 1 requires that (equation 2)

$$M = \max_{i} \sup_{s\in\partial\Omega} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{(\hat{Q}(s))_{ij}}{1+(\hat{Q}(s))_{ii}} < 1 \tag{8}$$

and that the single-variable unity feedback systems (assumption (iv))

$$\frac{(\hat{Q}(s))_{ii}^{-1}}{1 + (\hat{Q}(s))_{ii}^{-1}} \quad , \quad 1 \leqslant i \leqslant m \tag{9}$$

are stable. Let Q(s) be the transfer function matrix of the completely controllable and completely observable state space model S

$$\begin{aligned} \dot{x}(t) &= Fx(t) + Ge(t) \\ y(t) &= Hx(t) \end{aligned} \tag{10}$$

from which, if x_o is the system state at time t = 0,

$$z_o(s) = H(sI-F)^{-1}x_o \tag{11}$$

If S is stable and $\hat{Q}$ maps Y into Y then $\hat{Q}(s)z_o(s)$ is holomorphic on Ω (see (ii) of Result 1) and hence relations (8) and (9) are simply a form of Rosenbrock's stability criterion for open loop stable systems[2]. Note however that we do not explicitly require that $\hat{Q}(s)$ be diagonally dominant on $\partial\Omega$. If S is unstable, then, from Result 1, if equations (8) and (9) hold and $\hat{Q}$ maps Y into Y then the feedback configuration is stable from the origin $x_o = 0$ for any input with transform r(s) holomorphic in Ω. As S is controllable and observable then the feedback system is controllable and observable. It follows directly, using a contradiction argument, that the feedback system is hence stable for all initial conditions x_o. Again, this is simply a form of Rosenbrock's stability criterion for open-loop unstable systems[3] without the normal requirement that $\hat{Q}(s)$ be diagonally dominant on $\partial\Omega$.

In summary, previous results obtained by Freeman on the application of the contraction mapping theorem to the stability of open-loop stable feedback configurations have been extended to include cases when the open-loop system is minimum phase and possibly unstable. A natural consequence of the approach is a simple derivation of a form of Rosenbrock's stability theorem[2,3] which requires only that the matrix $I+\hat{Q}(s)$ be diagonally dominant on $\partial\Omega$.

Finally, Freeman[1] has pointed out that sharper criterion may be obtained by working in Banach spaces with other norms. Alternatively, for the norm used above, it should be noted[6] that the feedback system is stable under similarity transformation $T^{-1}Q(s)T$ whereas the norm of $(I+T^{-1}AT)^{-1}(T^{-1}\hat{Q}T-T^{-1}AT)$ may differ from the norm of $(I+A)^{-1}(\hat{Q}-A)$. Hence, transformation of the problem to an alternative basis (using, say, techniques of dyadic approximation[4,5]) may yield a contraction even though a contraction was not obtained in the original basis.

<u>References</u>

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