DYADIC APPROXIMATION ABOUT
A GENERAL FREQUENCY POINT

by

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Abstract

Recent results in the dyadic expansion of a general, square transfer function matrix are used to extend the concepts of dyadic approximation about s = 0 to the general case. The results make possible the systematic approximation of linear systems whilst retaining an exact description of system behaviour at a known frequency of interest.

The concept of dyadic transfer function matrix[1] and the ease of feedback control systems design for such plants has been used[1] to motivate the introduction of a technique for approximating a general mxm transfer function matrix $G(s)$. More precisely, if $G^{-1}(o)$ exists and the matrix

$$H_2 = \lim_{s \to o} s^{-1}\{G(o)G^{-1}(s) - I_m\} \qquad \ldots(1)$$

has a complete set of eigenvectors $\{\alpha_j\}_{1 \leqslant j \leqslant m}$ and dual eigenvectors $\{\gamma_j\}_{1 \leqslant j \leqslant m}$ where $\gamma_j^+ \alpha_k = \delta_{jk}$, then the dyadic approximation[1] $G_A(s)$ to $G(s)$ is defined to be

$$G(s) = \sum_{j=1}^{m} \{\gamma_j^+ G(s) G^{-1}(o)\alpha_j\}\alpha_j\gamma_j^+ G(o) \qquad \ldots(2)$$

If $G(s)$ is a dyadic transfer function matrix (DTFM), then $G_A(s) \equiv G(s)$. In more general situations the Taylor series expansions of $G(s)$ and $G_A(s)$ about the origin agree up to the term in s, so that $G_A(s)$ is a low frequency approximation to $G(s)$. More precisely

$$\lim_{s \to o} s^{-1}\{G(s) - G_A(s)\} = 0 \qquad \ldots(3)$$

For the purposes of this letter we will term the above $G_A(s)$ a dyadic approximation (DA) to $G(s)$ about s = o.

A major problem with the above analysis in many practical situations is the possibility of significant errors in the description of intermediate to high frequency behaviour, whereas the design engineer may desire the conceptual and computational benefits of working with a DTFM approximation to G(s) which includes an exact description of G(s) at a known (non-zero) frequency of interest (eg. a resonant frequency, or an intercept frequency[1]). With this in mind the following definition is proposed,

Definition

A DA to G(s) at the point $s = i\omega_1$ is a DTFM $G_A(s,\omega_1)$ satisfying the relation

$$G_A(i\omega_1,\omega_1) = G(i\omega_1) \qquad \ldots(4)$$

It is the purpose of this letter to propose a systematic technique for the construction of $G_A(s,\omega_1)$ such that if G(s) is a DTFM then $G_A(s,\omega_1) \equiv G(s)$. This property is required as, a priori, the best DA to a DTFM G(s) is G(s) itself.

RESULT

If $|G(i\omega_1)| \neq 0$ and $G(-i\omega_1)G^{-1}(i\omega_1)$ has a complete set of eigenvectors $\{\alpha_j(\omega_1)\}_{1\leq j\leq m}$ with dual eigenvectors $\{\gamma_j(\omega_1)\}_{1\leq j\leq m}$ where $\gamma_j^+(\omega_1)\alpha_k(\omega_1) = \delta_{jk}$, then[2,3]

$$G(i\omega_1) = \sum_{j=1}^{m} g_j(\omega_1)\alpha_j(\omega_1)\beta_j^+(\omega_1) \qquad \ldots(5)$$

where $\{g_j(\omega_1)\}_{1\leq j\leq m}$ are non-zero complex scalars and $\{\alpha_j(\omega_1)\beta_j^+(\omega_1)\}_{1\leq j\leq m}$ is a set of linearly independent dyads which is invariant under complex conjugation. The decoupling matrix of G(s) at $s = i\omega_1$ is

$$K_D(\omega_1) = \{\sum_{j=1}^{m} \alpha_j(\omega_1)\beta_j^+(\omega_1)\}^{-1} \qquad \ldots(6)$$

which is well defined[3], real and $\beta_j^+(\omega_1)K_D(\omega_1)\alpha_k(\omega_1) = \delta_{jk}$ or, $1\leq j\leq m$

$$G(i\omega_1)K_D(\omega_1)\alpha_j(\omega_1) = g_j(\omega_1)\alpha_j(\omega_1) \qquad \ldots(7)$$

If

$$G_A(s,\omega_1) = \sum_{j=1}^{m} \{\gamma_j^+(\omega_1)G(s)K_D(\omega_1)\alpha_j(\omega_1)\}\alpha_j(\omega_1)\gamma_j^+(\omega_1)\{K_D(\omega_1)\}^{-1} \quad \ldots(8)$$

then

$$G_A(\bar{s},\omega_1) \equiv \overline{G(s,\omega_1)} \quad \ldots(9)$$

$$G_A(i\omega_1,\omega_1) = G(i\omega_1) \quad \ldots(10)$$

and if G(s) is a DTFM, $G_A(s,\omega_1) \equiv G(s)$.

Proof

Equations (5)-(7) follow from previous results[2,3]. Equation (10) follows directly by substituting equations (6),(7) into (8) and comparing with equation (5). If G(s) is a DTFM of the form

$$G(s) = \sum_{j=1}^{m} g_j(s)\alpha_j\beta_j^+ \quad \ldots(11)$$

then[2], $\alpha_j(\omega_1) = \alpha_j$ and $\beta_j(\omega_1) = \beta_j$, $1 \leqslant j \leqslant m$, for all ω_1. The use of equation (6) and the relations[2,3] $\beta_j^+(\omega_1)K_D(\omega_1)\alpha_k(\omega_1) = \beta_j^+K_D(\omega_1)\alpha_k = \delta_{jk}$ yield the result $G_A(s,\omega_1) \equiv G(s)$. Finally, noting the invariance of the set $\{\alpha_j(\omega_1)\}$ under complex conjugation[2],

$$\overline{G_A(s,\omega_1)} = \sum_{j=1}^{m} \{\gamma_j^+(\omega_1)G(\bar{s})K_D(\omega_1)\alpha_j(\omega_1)\}\alpha_j(\omega_1)\gamma_j^+(\omega_1)\{K_D(\omega_1)\}^{-1}$$

$$= G_A(\bar{s},\omega_1) \quad \ldots(12)$$

which completes the proof of the result.

Property (9) is required to ensure that G_A is physically realizable.

An alternative way of calculating $G_A(s,\omega_1)$ is to write $T(\omega_1) = [\alpha_1(\omega_1),\ldots,\alpha_m(\omega_1)]$ and define[3],

$$H(s,\omega_1) = T^{-1}(\omega_1)G(s)K_D(\omega_1)T(\omega_1) \quad \text{...(13)}$$

which, by eqn (7), is diagonal at $s = i\omega_1$. It is easily shown that

$$G_A(s,i\omega_1) = \sum_{j=1}^{m} H_{jj}(s,\omega_1)\alpha_j(\omega_1)\beta_j^{+}(\omega_1) \quad \text{...(14)}$$

To obtain a physical interpretation of the DA, note from equation (13), that

$$G(s) = \sum_{j=1}^{m}\sum_{k=1}^{m} H_{jk}(s,\omega_1)\alpha_j(\omega_1)\beta_k^{+}(\omega_1) \quad \text{...(15)}$$

Hence, interpreting the $\{\alpha_j(\omega_1)\}$ as output modes characteristic of the frequency ω_1, a comparison of equations (15) and (14) indicates that the DA is obtained by neglecting interaction between the modes at all frequencies.

Finally, it is emphasized that the use of dyadic approximations will only be of use in systems where modal interaction[1] is small in the sense that $H(s,\omega_1)$ is diagonally dominant over the frequency range of interest. However, in those cases where dyadic approximation is a reasonable working tool, the above analysis represents a significant generalization of previous results[1].

References

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