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INTEGRITY OF MULTIVARIABLE FIRST-ORDER TYPE SYSTEMS

by

D. H. Owens, B.Sc.,A.R.C.S.,Ph.D.

Lecturer in the Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield S1 3JD

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Abstract

Recent results on the unity feedback control analysis of multivariable first-order type systems are extended to provide necessary and sufficient conditions for integrity of the closed-loop system.

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1. Introduction

In a recent paper (Owens, 1975) the concept of a multivariable first-order type system has been introduced, and closed-form solutions derived for proportional and proportional plus integral unity negative feedback controllers capable of producing a high performance feedback system with fast response speeds and small interaction effects. This paper is primarily concerned with the important problem of the integrity (Belletrutti and MacFarlane, 1971) of the resulting feedback system. The following failures are considered in this paper,

(1) Sensor failure in loop j

In this situation, failure of measurement equipment in loop j means that the feedback loop returns the erroneous signal $y_j(t) \equiv 0$.

(2) Actuator failure in loop j

In this situation, actuator failure reduces the jth control signal $u_j(t)$ to zero for all time.

Necessary and sufficient conditions are established for the proposed controller to produce a fail-safe system and an approach to the use of compensation networks to achieve integrity outlined.

2. Extension of Previous Results (Owens, 1975)

For the purpose of this paper a multivariable first-order type system is a system described by an mxm transfer function matrix G(s) of the form,

$$G(s) = \sum_{j=1}^{m} (s+b_j)^{-1} \alpha_j \beta_j^+ \quad , \quad |G(s)| \not\equiv 0 \qquad \ldots(1)$$

where $\{\alpha_j \beta_j^+\}_{1 \leq j \leq m}$ is a set of dyads such that $\bar{b}_j = b_\ell$ implies $\overline{\alpha_j \beta_j^+} = \alpha_\ell \beta_\ell^+$. Note that this definition extends the previous (Owens, 1975) to include the possibility of open-loop plant integrators and open-loop unstable systems. The following result extends previous results (Owens, 1975) to this case.

Result One

With the above definitions, $\{\alpha_j\}_{1 \leq j \leq m}$ and $\{\beta_j\}_{1 \leq j \leq m}$ are sets of linearly independent vectors and

$$G_\infty = \lim_{s \to \infty} sG(s) \qquad \ldots(2)$$

exists and is non-singular. A forward path controller

$$K(s) = \{k+c+\frac{kc}{s}\} G_\infty^{-1} - G^{-1}(s)\big|_{s=o} \qquad \ldots(3)$$

produces a closed-loop system with return-difference determinant

$$|T(s)| = (s+k)^m (s+c)^m / s^m \prod_{j=1}^{m} (s+b_j) \qquad \ldots(4)$$

and closed-loop transfer function matrix

$$\{I_m + G(s)K(s)\}^{-1} G(s)K(s) = \frac{k}{s+k} M_1(k,c) + \frac{c}{s+c} M_2(k,c) \qquad \ldots(5)$$

where

$$\lim_{k \to \infty} M_1(k,c) = I_m \quad , \quad \lim_{k \to \infty} M_2(k,c) = 0 \qquad \ldots(6)$$

Proof

The linear independence of $\{\alpha_j\}$ and $\{\beta_j\}$ follows from the condition $|G(s)| \not\equiv 0$ and hence G_∞ is non-singular. Also

$$G(s)G_\infty^{-1} = \sum_{j=1}^{m} \frac{1}{s+b_j} \alpha_j \gamma_j^+ \qquad \ldots(7)$$

where $\gamma_j^+ \alpha_k = \delta_{jk}$, $1 \leq j, k \leq m$, and hence

$$G_\infty G^{-1}(s)\big|_{s=o} = \sum_{j=1}^{m} b_j \alpha_j \gamma_j^+ \quad \text{...(8)}$$

Hence

$$G_\infty K(s) = \sum_{j=1}^{m} \{k+c+\frac{kc}{s} - b_j\}\alpha_j \gamma_j^+ \quad \text{...(9)}$$

so that

$$G(s)K(s) = \sum_{j=1}^{m} \frac{\{(k+c-b_j)s+kc\}}{s(s+b_j)} \alpha_j \gamma_j^+ \quad \text{...(10)}$$

$$T(s) = I_m + G(s)K(s) = \sum_{j=1}^{m} \frac{(s^2+(k+c)s+kc)}{s(s+b_j)} \alpha_j \gamma_j^+ \quad \text{...(11)}$$

$$|T(s)| = (s+k)^m (s+c)^m / s^m \prod_{j=1}^{m} (s+b_j) \quad \text{...(12)}$$

$$T^{-1}(s)G(s)K(s) = \frac{1}{(s+k)(s+c)} \sum_{j=1}^{m} \{(k+c-b_j)s+kc\}\alpha_j \gamma_j^+ \quad \text{...(13)}$$

$$= \frac{k}{s+k} \sum_{j=1}^{m} \frac{(k-b_j)}{(k-c)} \alpha_j \gamma_j^+ + \frac{c}{s+c} \sum_{j=1}^{m} \frac{(b_j-c)}{(k-c)} \alpha_j \gamma_j^+ \quad \text{...(14)}$$

so that

$$M_1(k,c) = k(k-c)^{-1}\{I_m - k^{-1} G_\infty G^{-1}(s)\big|_{s=o}\} \quad \text{...(15)}$$

$$M_2(k,c) = (k-c)^{-1}\{G_\infty G^{-1}(s)\big|_{s=o} - cI_m\} \quad \text{...(16)}$$

and the result follows by letting $k \to \infty$. Q.E.D.

As before (Owens, 1975), the proposed controller (eqn.(3)) replaces the open-loop poles by poles at $-k$, $-c$, each with algebraic multiplicity m. As k increases (equations (5),(6)) the speed of response increases and closed-loop interaction effects and steady state errors become arbitrarily small.

2. Sensor Failure in Loop j

The main result in this section is as follows, where $\{e_j\}_{1\leq j\leq m}$ is the natural basis in R^m.

Result Two

A unit negative feedback control system for the plant G(s) with forward path controller K(s) (eqn.(3)) is stable with respect to sensor failure in loop j if, and only if, c = 0 (ie proportional control only) and

$$p_j = e_j^T G_\infty G^{-1}(s)\big|_{s=o} e_j > 0 \qquad \ldots(17)$$

Proof

Using previous results (Owens, 1973)

$$\frac{\text{closed-loop characteristic polynomial with sensor failure in loop j}}{\text{closed-loop characteristic polynomial without sensor failure in loop j}}$$

$$= e_j^T \{I_m + G(s)K(s)\}^{-1} e_j \qquad \ldots(18)$$

Using equations (11),(8) and the fact that $\gamma_\ell^+ \alpha_k = \delta_{\ell k}$

$$T^{-1}(s) = \sum_{\ell=1}^{m} \frac{s(s+b_\ell)}{(s+k)(s+c)} \alpha_\ell \gamma_\ell^+$$

$$= \frac{s}{(s+k)(s+c)} \{sI_m + G_\infty G^{-1}(s)\big|_{s=o}\} \qquad \ldots(19)$$

and hence

$$e_j^T T^{-1}(s) e_j = \frac{s(s+p_j)}{(s+k)(s+c)} \qquad \ldots(20)$$

If c>0, then the failed system has a pole at the origin ie the system is not asymptotically stable in the presence of sensor failure in any loop. The case of proportional control (c=o) yields in a similar manner

$$e_j^T T^{-1}(s) e_j = (s+p_j)/(s+k) \qquad \ldots(21)$$

The result follows by noting that stability requires $p_j>0$. Q.E.D.

A surprising observation is that the closed-loop poles in the presence of sensor failure in loop j are $-k$ (algebraic multiplicity $m-1$) and $-p_j$ (which does not depend upon the gain parameter k). In practical terms this means that integrity can be tested at low gains. In fact the following result demonstrates that sensor-integrity can be deduced quite easily from the closed-loop step responses, a fact that could be of significance in applications.

Result Three

A unity negative feedback control configuration for the system G(s), with forward path proportional controller K(s) (equation (3) with c=o) is stable with respect to sensor failure in loop j if, and only if, the steady state error in output j in response to a unit step demand in output j is strictly positive.

Proof

From equations (5),(15)

$$\{I_m+G(s)K(s)\}^{-1}G(s)K(s) = \frac{k}{s+k}\{I_m-k^{-1}G_\infty G^{-1}(s)|_{s=o}\} \qquad \ldots(22)$$

The result follows by noting that $k>0$, observing that the diagonal terms are $k(s+k)^{-1}\{1-k^{-1}p_j\}$ and invo ing equation (17). Q.E.D.

The step by step design procedure (Owens, 1975) is easily extended to cope with an investigation of sensor-integrity, by either, calculating $G_\infty G^{-1}(s)|_{s=o}$ and checking that the diagonal terms are strictly positive, or by calculation of the open-loop step responses and the use of Result three. Unfortunately, it is not possible to guarantee that integrity will be present in all loops. The following result illustrates that, in certain cases of practical interest, stable, minimum-phase compensation networks can be used to provide the desired characteristics.

Result Four

If the system is open-loop stable (ie $b_\ell > 0$, $1 \leq \ell \leq m$), then a forward path controller of the form $K(s) = K_1(s)K_2(s)$ with

$$K_1(s) = G_\infty^{-1} \sum_{\ell=1}^{m} \frac{(s+b_\ell)}{(s+k_\ell)} \alpha_\ell \gamma_\ell^+ \qquad \ldots(23)$$

$$K_2(s) = k(GK_1)_\infty^{-1} - (G(s)K_1(s))^{-1}\big|_{s=o}\ , \quad k>0 \qquad \ldots(24)$$

can, by suitable choice of stable poles $\{-k_\ell\}_{1\leq\ell\leq m}$, produce a closed-loop system which is asymptotically stable and stable with respect to sensor failure in any loop.

Proof

By equations (7),(23)

$$G(s)K_1(s) = \sum_{\ell=1}^{m} (s+k_\ell)^{-1}\alpha_\ell\gamma_\ell^+ \qquad \ldots(25)$$

so that, $G(s)K_1(s)$ is a multivariable first-order system with pole set $\{-k_\ell\}_{1\leq\ell\leq m}$, and $K_2(s)$ is simply the proposed controller (Result one) for such a system. As $k>o$ and $G(s)$ is open-loop asymptotically stable, the closed-loop system is asymptotically stable. For the resulting system

$$p_j = e_j^T(GK_1)_\infty(G(s)K_1(s))^{-1}\big|_{s=o}\ e_j$$

$$= \sum_{\ell=1}^{m} k_\ell\{e_j^T\alpha_\ell\}\{\gamma_\ell^+ e_j\} \quad , \quad 1\leq j\leq m \qquad \ldots(26)$$

using $(GK_1)_\infty = I_m$ and equation (25). The proof of the result now depends upon the existence of a set of stable poles $\{-k_\ell\}_{1\leq\ell\leq m}$ such that $p_j>0$, $1\leq j\leq m$. Noting that $I_m = \sum_{\ell=1}^{m} \alpha_\ell\gamma_\ell^+$ then, for any strictly positive real number q

$$\sum_{\ell=1}^{m} q\{e_j^T \alpha_\ell\}\{\gamma_\ell^+ e_j\} = q > 0 \qquad \ldots(27)$$

and the result follows by choosing $k_1 = k_2 = \ldots = k_m = q$. Q.E.D.

The construction used in the proof of Result 4 may produce practical difficulties by requiring an excessive amount of phase advance in the compensator. In these situations, an examination of the individual terms in the summations of equation (26) may make possible a compromise solution for $\{k_\ell\}_{1\leq\ell\leq m}$.

3. Actuator Failure in Loop j

The following results are direct parallels to those of section 2.

Result Five

A unit negative feedback control system for the plant G(s) with forward path controller K(s) (eqn (3)) is stable with respect to actuator failure in loop j if, and only if, c = 0 (ie proportional control) and

$$q_j = e_j^T G^{-1}(s)|_{s=o}\, G_\infty e_j > 0 \qquad \ldots(28)$$

Proof

It is easily shown that c = 0 is required. Using previous results (Owens, 1973)

$$\frac{\text{closed-loop characteristic polynomial with actuator failure in loop j}}{\text{closed-loop characteristic polynomial without actuator failure in loop j}}$$

$$= e_j^T\{I + K(s)G(s)\}^{-1} e_j \qquad \ldots(29)$$

If $\beta_\ell^+ \Psi_k = \delta_{\ell k}$, $1\leq\ell$, $k\leq m$, then

$$G_\infty^{-1} G(s) = \sum_{\ell=1}^{m} (s+b_\ell)^{-1} \Psi_\ell \beta_\ell^+ \qquad \ldots(30)$$

$$G^{-1}(s)|_{s=o} G_\infty = \sum_{\ell=1}^{m} b_\ell \Psi_\ell \beta_\ell^+ \qquad \ldots(31)$$

so that

$$K(s)G(s) = \sum_{\ell=1}^{m} \frac{(k-b_\ell)}{(s+b_\ell)} \Psi_\ell \beta_\ell^+ \qquad \ldots(32)$$

and

$$\{I_m + K(s)G(s)\}^{-1} = \sum_{\ell=1}^{m} \frac{(s+b_\ell)}{(s+k)} \Psi_\ell \beta_\ell^+$$

$$= \frac{1}{(s+k)} \{sI_m + G^{-1}(s)|_{s=o} G_\infty\} \qquad \ldots(33)$$

The result follows by substitution into equation (29) and comparison with equation (28). Q.E.D.

Integrity can be deduced simply from G(s) by calculation of $G^{-1}(s)|_{s=o}$ and G_∞ and examination of the diagonal terms of the product. Result three has no useful parallel in the case of actuator failure. However, result four can be extended as follows,

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Result Six

If the system is open-loop stable (ie $b_\ell > 0$, $1 \leq \ell \leq m$), then a forward path controller as given in Result 4 can, by suitable choice of stable poles $\{-k_\ell\}_{1\leq\ell\leq m}$, produce a closed-loop system which is asymptotically stable and stable with respect to actuator failure in any loop.

Proof

From equations (23)-(25),

$$K_2(s) = kI_m - \sum_{\ell=1}^{m} k_\ell \alpha_\ell \gamma_\ell^+ \qquad \ldots(34)$$

and hence, noting that $G_\infty^{-1} = \sum_{\ell=1}^{m} \Psi_\ell \gamma_\ell^+$,

$$K_1(s)K_2(s)G(s) = \{\sum_{\ell=1}^{m} \frac{(s+b_\ell)}{(s+k_\ell)} \Psi_\ell \gamma_\ell^+\}\{\sum_{\ell=1}^{m} (k-k_\ell)\alpha_\ell \gamma_\ell^+\}\{\sum_{\ell=1}^{m} (s+b_\ell)^{-1} \alpha_\ell \beta_\ell^+\}$$

$$= \sum_{j=1}^{m} \frac{(k-k_\ell)}{(s+k_\ell)} \Psi_\ell \beta_\ell^+ \qquad \ldots(35)$$

and hence,

$$(s+k)e_j^T\{I_m + K(s)G(s)\}^{-1}e_j = s + e_j^T \sum_{\ell=1}^{m} k_\ell \Psi_\ell \beta_\ell^+ e_j \qquad \ldots(36)$$

The proof of the result now rests on the existence of a stable pole set $\{-k_\ell\}_{1\leqslant\ell\leqslant m}$ such that $q_j = e_j^T \sum_{\ell=1}^{m} k_\ell \Psi_\ell \beta_\ell^+ e_j > 0$, $1\leqslant j\leqslant m$. This follows in a similar manner to the proof of Result 4. Q.E.D.

4. Illustrative Example

To illustrate the results, consider the multivariable first-order type system

$$G(s) = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+5 & s+1 \\ 2 & s+1 \end{bmatrix}$$

$$= \frac{1}{(s+3)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \frac{1}{(s+1)} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \ldots(37)$$

so that

$$\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \ldots(38)$$

$$\gamma_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \ldots(39)$$

Also

$$G_\infty = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad G(o) = \frac{1}{3} \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} \qquad \ldots(40)$$

so that

$$G_\infty G^{-1}(o) = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix} \quad , \quad G^{-1}(o)G_\infty = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \qquad \ldots(41)$$

ie., using Results 2 and 5, $p_1 = -1$, $p_2 = 5$, $q_1 = 1$, $q_2 = 3$ and the closed-loop system with a controller of the form proposed will be unstable if a transducer fails in loop one.

To offset this difficulty, try a controller of the form $K(s) = K_1(s)K_2(s)$ where (result 4)

$$K_1(s) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left\{ \frac{(s+3)}{(s+k_1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix} + \frac{(s+1)}{(s+k_2)} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \right\} \qquad \ldots(42)$$

$$K_2(s) = k\, I_2 - (G(s)K_1(s))^{-1}\big|_{s=o} \qquad \ldots(43)$$

then

$$G(s)K_1(s) = \frac{1}{s+k_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix} + \frac{1}{s+k_2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \ldots(44)$$

which is a multivariable first order type system. From equations (36) and (26), the resulting closed-loop system is totally failure stable if, and only if,

$$e_j^T \sum_{\ell=1}^{2} k_\ell \alpha_\ell \gamma_\ell^+ e_j > 0 \qquad j = 1,2 \qquad \ldots(45)$$

$$e_j^T \sum_{\ell=1}^{2} k_\ell \Psi_\ell \beta_\ell^+ e_j > 0 \qquad j = 1,2 \qquad \ldots(46)$$

ie

$$-k_1 + 2k_2 > 0$$
$$2k_1 - k_2 > 0$$
$$k_2 > 0$$
$$k_1 > 0 \qquad \ldots(47)$$

which is satisfied if k_1, k_2 are positive and

$$\frac{1}{2} < \frac{k_1}{k_2} < 2 \qquad \ldots(48)$$

To simplify the compensation networks it is reasonable to set either $k_1 = 3$ or $k_2 = 1$. Choosing $k_1 = 3$, eqn (48) reduces to

$$1.5 < k_2 < 6 \qquad \ldots(49)$$

which represents the range of k_2 required to guarantee both sensor and actuator integrity in every loop. For convenience, choose $k_2 = 4$.

The design procedure can now be completed by the choice of an appropriate value of k. In fact, from section 2,

$$\{I_2+G(s)K(s)\}^{-1}G(s)K(s) = \frac{k}{s+k}\{I_2-k^{-1}(GK_1)_\infty(G(s)K_1(s))^{-1}|_{s=o}\}$$

$$= \frac{k}{s+k}\ \{I_2 - k^{-1}\begin{bmatrix}5 & -2\\ 1 & 2\end{bmatrix}\} \qquad \ldots(50)$$

so that, for example, any choice of k greater than 20 will ensure a fast response with interaction effects less than 10%. The steady state error in channel one however will be of the order of 25%. This can be reduced by increasing the value of k.

5. Effect of Failures on Closed-loop Response

It is of interest to investigate the effect on the closed-loop transient performance of a failure in any loop. Consider the case of sensor failure in loop j, and assume that the closed-loop system is stable in the presence of this failure. The closed-loop transfer function matrix with proportional control and failure in loop j is, using equations (5),(21),

$$\{I_m + G(s)K(s)(I_m-e_je_j^T)\}^{-1}G(s)K(s)$$

$$= \{I_m - T^{-1}(s)G(s)K(s)e_je_j^T\}^{-1}T^{-1}G(s)K(s)$$

$$= \{I_m + \frac{kM_1(k,o)}{(s+p_j)}\ e_je_j^T\}\ \frac{k}{s+k}\ M_1(k,o)$$

$$= \frac{k}{s+k}\hat{M}_1(k) + \frac{1}{s+pj}\hat{M}_2(k) \qquad \ldots(51)$$

where

$$\hat{M}_1(k) = \{I_m - \frac{k}{k-p_j} M_1(k,o)e_j e_j^T\} M_1(k,o) \qquad \ldots(52)$$

$$\hat{M}_2(k) = \frac{k^2}{k-p_j} M\ (k,o)e_j e_j^T M_1(k,o) \qquad \ldots(53)$$

An examination of these relationships and the use of eqn (15) indicates that, for high gains, a unit step demand in output $\ell \neq j$ will, in general, produce a high quality performance from all loops $k \neq j$ and a large transient deviation in output j. A unit step demand in output j will produce, in general, large transient deviations in all channels.

6. Conclusions

Previous results (Owens, 1975) have been extended to include the possibility of open-loop plant integrators and open-loop unstable systems. The proposed controllers produces a closed-loop system capable of producing arbitrarily fast response speeds, small closed-loop interaction effects and small steady-state errors. Closed-loop integrity with respect to sensor or control actuator failure is easily deduced from the transfer function matrix G(s) and a systematic approach is outlined to the synthesis of compensation networks producing a system stable with respect to any single sensor or actuator failure. Important observations are that the integrity of the system is independent of controller gain so that integrity can be tested experimentally using a low gain controller. In the case of sensor failure, integrity can be tested by an examination of the steady state errors in response to appropriate unit step demands in output.

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