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CASCADE CANONICAL FORM FOR LINEAR MULTIVARIABLE SYSTEMS

by

D. H. OWENS, B.Sc.,A.R.C.S.,Ph.D.

Lecturer in the Department of Control
Engineering
University of Sheffield
Mappin Street
Sheffield S1 3JD

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Abstract

A canonical form is derived for systems described by an $m \times \ell$ transfer function matrix $G(s)$ and applied to the calculation of system transmission zeros, the feedback control of multivariable second-order type systems and pole allocation using output feedback.

List of Symbols

A^+	conjugate transpose of the matrix A
dim N	dimension of a linear subspace N
span $\{x_j\}_{1\leq j\leq n}$	subspace generated by linear combinations of the vectors $x_1,\ldots,x_n$
A^T	transpose of the matrix A
$\|A\|$	determinant of the matrix A
$\delta_{j\ell}$	Kronecker delta function
$\{e_j\}_{1\leq j\leq n}$	natural basis in R^n

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1. Introduction

In a recent paper (Owens, 1975) the concept of the classical first order lag has been extended to the multivariable case by defining an mxm multivariable first order system of the form

$$G(s) = \sum_{j=1}^{m} \frac{b_j}{s+b_j} \alpha_j \beta_j^+ \quad , \qquad |G(s)| \neq 0 \qquad \ldots(1)$$

where $\{\alpha_j\beta_j^+\}_{1\leq j\leq m}$ is a set of dyads satisfying the constraint that $\bar{b}_j = b_\ell$ implies $\overline{\alpha_j\beta_j^+} = \alpha_\ell\beta_\ell^+$. Closed-form solutions have been derived (Owens, 1975) for proportional and proportional plus integral unity negative feedback controllers which are direct multivariable generalizations of the equivalent classical controllers. The proposed controllers are capable of producing a feedback system with arbitrarily fast response speeds and small interaction effects.

In classical theory, first order lags can be regarded as fundamental building blocks for the construction of the system dynamic behaviour. For example, if g(s) is a non-zero scalar transfer function with pole set $\{p_j\}_{1\leq j\leq n}$ and zeros $\{z_j\}_{1\leq j\leq m}$ $(m<n)$, then if $k = n-m$,

$$g_\infty^{(k)} \triangleq \lim_{s\to\infty} s^k g(s) \neq 0 \qquad \ldots(2)$$

and,

$$g(s) = \{ \prod_{j=1}^{k} (s-p_j)^{-1}\}\{g_\infty^{(k)} + h(s)\} \qquad \ldots(3)$$

where h(s) is a strictly proper transfer function with pole set $\{p_j\}_{k<j\leq n}$. Quite obviously the decomposition can be extended by application of the same approach to h(s). Such decompositions are useful in the classification of linear systems according to rank and type, the characterization of zeros as due to parallel branches in the system, and in many cases can be used as the basis of a simple model reduction procedure.

Extending the definition (eqn (1)) to include the possibility of open-loop plant integrators, an mxm multivariable first order system G(s) takes the form,

$$G(s) = \sum_{j=1}^{m} (s+b_j)^{-1}\alpha_j\beta_j^{+} \quad , \qquad |G(s)| \not\equiv 0 \qquad \ldots(4)$$

This paper demonstrates how, in quite general situations, such systems are fundamental building blocks for the characterization of the structure of a linear system described by a mxℓ strictly proper transfer function matrix G(s). A canonical form for G(s) analogous to eqn (3) is derived in section 2. In section 3 the canonical form is used to provide a systematic technique for the calculation of system transmission zeros and to provide upper bounds on the number of such zeros. In section 4, the results are used to extend previous analysis (Owens, 1975) of a class of multivariable second order type systems. Finally, in section 5, certain limitations inherent in the idea of pole allocation using output feedback (Fallside and Seraji, 1971 and Ahmari and Vacroux, 1973) are described by a simple application of the canonical form.

2. Cascade Canonical Form

Consider an mxℓ transfer function matrix G(s) of the form

$$G(s) = \sum_{j=1}^{n} (s+b_j)^{-1}\alpha_j\beta_j^{+} \qquad \ldots(5)$$

where $\{\alpha_j\beta_j^{+}\}_{1\leq j\leq n}$ are mxℓ non-zero dyads. Such a transfer function matrix arises in the analysis of ℓ-input, m-output completely controllable and completely observable state space models of state dimension n and possessing a system matrix A having a diagonal canonical form. If A has a non-diagonal Jordan form, an infinitessimal perturbation of its elements will remove this difficulty.

This section considers the decomposition of G(s) into a cascade canonical form analogous to equation (3) using multivariable first order systems of the form defined in equation (4). Result 1 given below is a generalization of the simple classical result that a strictly proper transfer function g(s) can be decomposed as the product of a first order lag and a proper transfer function. If L(s) is a transfer function matrix then, for convenience of notation,

$$L_\infty^{(r)} \triangleq \lim_{s\to\infty} s^r L(s) \qquad \ldots(6)$$

whenever the limit exists.

Result One (see appendix 7.1)

If dim span $\{\alpha_j\}_{1\leq j\leq n} = p_1$ and dim span $\{\beta_j\}_{1\leq j\leq n} = q_1$, then $p_1 \leq m$, $q_1 \leq \ell$ and

$$G(s) = P_1 M_1(s) Q_1 \qquad \ldots(7)$$

where P_1, Q_1 are constant, full rank matrices of dimension $m x p_1$, $q_1 x \ell$ respectively and $M_1(s)$ is a strictly proper $p_1 x q_1$ transfer function matrix of the form

$$M_1(s) = \sum_{j=1}^{n} (s+b_j)^{-1} x_j y_j^+ \qquad \ldots(8)$$

where dim span $\{x_j\}_{1\leq j\leq n} = p_1$, dim span $\{y_j\}_{1\leq j\leq n} = q_1$. Moreover,

$$G(s) = P_1 G_1(s)\{M_{1\infty}^{(1)} + H_1(s)\}Q_1 \qquad \ldots(9)$$

where, by suitable reordering of dyads, $G_1(s)$ is a $p_1 x p_1$ multivariable first order system (in general non-unique) with pole set $\{-b_j\}_{1\leq j\leq p_1}$ and $H_1(s)$ is a $p_1 x q_1$ strictly proper transfer function matrix with pole set $\{-b_j\}_{p_1<j\leq n}$, each pole having unity rank residue.

This result has an immediate corollary by applying it to $G^T(s)$,

$$G(s) = P_1\{M_{1\infty}^{(1)} + \hat{H}_1(s)\}\hat{G}_1(s)Q_1 \qquad \ldots(10)$$

where $\hat{G}_1(s)$ is a $q_1 x q_1$ multivariable first order type system with pole set $\{-b_j\}_{1\leqslant j\leqslant q_1}$ and $\hat{H}_1(s)$ is a strictly proper $p_1 x q_1$ transfer function matrix with pole set $\{-b_j\}_{q_1<j\leqslant n}$.

The matrices P_1, Q_1 represent the space spanned by all possible system outputs and the space of effective control action respectively. In most cases $P_1 = I_m$, $Q_1 = I_\ell$ when

$$G(s) = G_1(s)\{G_\infty^{(1)} + H_1(s)\} \qquad \ldots(11)$$

This formula illustrates the decomposition of G(s) into the product of a first order system and a proper transfer function matrix.

The procedure of Result 1 can be applied to $H_1(s)$ to generate a decomposition of the form defined by the following inductive relationships,

$$H_o(s) = G(s) \qquad \ldots(12)$$

$$H_j(s) = P_{j+1}M_{j+1}(s)Q_{j+1}\ , \quad j\geqslant 0 \qquad \ldots(13)$$

$$M_j(s) = G_j(s)\{M_{j\infty}^{(1)} + H_j(s)\}\ , \quad j\geqslant 0 \qquad \ldots(14)$$

where $G_j(s)$, $j\geqslant 1$, is a first order system of dimension $p_j x p_j$. For example,

$$\begin{aligned} G(s) &= H_o(s) = P_1M_1(s)Q_1 \\ &= P_1G_1(s)\{M_{1\infty}^{(1)} + H_1(s)\}Q_1 \\ &= P_1G_1(s)\{M_{1\infty}^{(1)} + P_2M_2(s)Q_2\}Q_1 \\ &= P_1G_1(s)\{M_{1\infty}^{(1)} + P_2G_2(s)\{M_{2\infty}^{(1)} + H_2(s)\}Q_2\}Q_1 \qquad \ldots(15) \end{aligned}$$

The procedure will terminate at the smallest integer ℓ such that $H_\ell(s) \equiv 0$, from which

$$\sum_{j=1}^{\ell} p_j = n \qquad \ldots(16)$$

Hence the system G(s) can be represented by $\ell \leqslant n$ multivariable first order systems of various dimensions linked by proportional transferences P_j, $1 \leqslant j \leqslant \ell$, Q_j, $1 \leqslant j \leqslant \ell$ and $M_{j\infty}^{(1)}$, $1 \leqslant j \leqslant \ell$ as illustrated in Fig.1.

Although the above procedure has an obvious relationship to the classical decomposition of equation (3), the integer k = (number of poles)-(number of zeros) seems to play no part in the multivariable scheme. The following result defines the multivariable equivalent of k and quantifies its effect on the canonical form for $m \geqslant \ell$. The case of $m < \ell$ is treated by taking transposes.

<u>Result Two</u> (see appendix 7.2)

Let $k \geqslant 1$ be the uniquely defined integer such that $G_\infty^{(k)}$ is non-zero and finite. Then, if $m \geqslant \ell$ and rank $G(s) < \ell$ at, at most, a finite number of points of the complex plane,

$$G(s) = \{G_\infty^{(k)} + G_{k+1}(s)\}G_k(s) \;\dots\; G_1(s) \qquad \dots(17)$$

where, by suitable reordering of dyads, $G_j(s)$, $1 \leqslant j \leqslant k$, is an $\ell \times \ell$ multivariable first order system with pole set $-b_i$, $(j-1)\ell+1 \leqslant i \leqslant j\ell$, and $G_{k+1}(s)$ is a strictly proper transfer function matrix with pole set $-b_i$, $k\ell < i \leqslant n$.

If $m=\ell$ then, for any integer $0 \leqslant r \leqslant k$, with the above definitions,

$$G(s) = G_1(s) \;\dots\; G_r(s)\{G_\infty^{(k)} + G_{k+1}(s)\}G_k(s)G_{k-1}(s)\dots G_{r+1}(s) \qquad \dots(18)$$

This result illustrates the direct analogy between the multivariable decomposition and the classical case (eqn(3)) simply by a comparison of equation (18) with equation (3) with r=k. It also indicates that multivariable first order systems are fundamental building blocks for the construction of the dynamic behaviour of G(s) analogous to the first order lags of classical theory. The integer k reduces to the classical case if $m = \ell = 1$. Also, as the dimension of the minimal realization of $G_{k+1}(s)$

is positive, k is bounded by the relation

$$k \min(m,\ell) \leqslant n \qquad \ldots(19)$$

and, with the assumption of Result 2 and $m = \ell$, $km = n$ if, and only if, G(s) can be represented as the product of k multivariable first order systems.

Applications of Results 1 and 2 are illustrated in the following sections. As a final point in this section, it is noted that the cascade canonical form for G(s) has a direct interpretation in terms of a state space canonical form. This is illustrated in the following result.

Result Three (see appendix 7.3)

With the assumptions of Result 2, $m = \ell$ and $r = k$, G(s) has a minimal state space realization of the form

$$\dot{x} = Ax + Bu \quad , \qquad y = Cx \qquad \ldots(20)$$

where

$$A = \begin{bmatrix} A_1 & I_m & 0 & \cdots & & 0 \\ 0 & A_2 & I_m & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & 0 \\ & & & & A_k & C_1 \\ 0 & \cdots & & & 0 & A_{k+1} \end{bmatrix} \qquad \ldots(21)$$

$$B^T = \left[0 \; \ldots \; 0 \;\; (G_\infty^{(k)})^T B_1^{\;T}\right] \qquad \ldots(22)$$

$$C = \left[I_m \; 0 \; \ldots \; 0\right] \qquad \ldots(23)$$

$$A_j = -G_j^{-1}(s)\big|_{s=o} \quad , \qquad 1 \leqslant j \leqslant k \qquad \ldots(24)$$

$$G_{k+1}(s) = C_1(sI_{n-km} - A_{k+1})^{-1}B_1 \qquad \ldots(25)$$

That is, the proposed canonical form decomposes the state space R^n into the direct sum of subspaces X_j, $1 \leq j \leq k+1$, characterized by the diagonal block A_j, $1 \leq j \leq k+1$, respectively. Also dim $X_j = m$, $1 \leq j \leq k$, and dim $X_{k+1} = n-km$. The only subspace contributing directly to the output is X_1 and the only subspaces affected directly by control action are X_k and X_{k+1}. The effective input into the subspace X_j, $1 \leq j \leq k-1$ is the output from X_{j+1} which emphasizes the cascade nature of the canonical form and illustrates the intuitive notion that increasing the value of k increases the overall phase lag in the system and hence has a considerable effect on the behaviour of the system with high gain feedback.

3. System Transmission Zeros

The transmission zeros (Davison and Wang, 1974) of an mxm transfer function matrix G(s) arising from the state space model of equation (20) are the zeros of the zero polynomial

$$P_G(s) \triangleq |sI_n - A| \cdot |G(s)| \qquad \ldots(26)$$

The zeros of an $m \times \ell$ G(s) canbedefined to be the common zeros (with appropriate multiplicity) of all $\min(m,\ell) \times \min(m,\ell)$ transfer function matrices obtained by taking rows ($m \leq \ell$) or columns ($m \geq \ell$) of G(s). For this reason we restrict our attention to the case of $m = \ell$. Also assume that $|G(s)| \not\equiv 0$ to prevent the trivial result $P_G(s) \equiv 0$.

3.1 Physical Source of Zeros

From Result 2, with r = k,

$$G(s) = G_1(s) \ldots G_k(s)\{G_\infty^{(k)} + G_{k+1}(s)\} \qquad \ldots(27)$$

It is shown in appendix 7.4 that

$$P_G(s) = \{\prod_{j=km+1}^{n} (s+b_j)\} \cdot |G_\infty^{(k)} + G_{k+1}(s)| \qquad \ldots(28)$$

which, by comparison with Fig.2, indicates that system zeros are associated solely with the feedforward component of the system structure and hence (Result 3) with a subspace of the state space of dimension n-km.

3.2 Calculation and Number of Transmission Zeros

Equation (28) forms a convenient basis for the calculation of system zeros. To initiate the analysis, the following result identifies situations when G(s) has no zeros.

Result Four (see appendix 7.5)

If $m = \ell$ and $|G(s)| \not\equiv 0$, a sufficient condition for G(s) to have no zeros is

$$n = (k+1)m - r_\infty \qquad \ldots(29)$$

where $$r_\infty \triangleq \text{rank } G_\infty^{(k)} \leqslant m \qquad \ldots(30)$$

This result is easily extended to the non-square case, when eqn (29) becomes

$$n = (k+1) \min (m,\ell) - r_\infty \qquad \ldots(31)$$

Result 4 is particularly useful in assessing the number of zeros as the parameters n, k, m, r_∞ are easily calculated from the state space model and transfer function matrix, without the necessity of evaluating the zero polynomial $P_G(s)$.

Equation (28) is in an unsuitable form for zero calculation. The following result describes an alternative form for $P_G(s)$.

Result Five (see appendix 7.6)

If $m = \ell$, $|G(s)| \not\equiv 0$ and rank $G_\infty^{(k)} = m$, then, using the notation of Result 3,

$$P_G(s) = |G_\infty^{(k)}| \cdot |sI_{n-km} - A_{k+1} + B_1 (G_\infty^{(k)})^{-1} C_1| \qquad \ldots(32)$$

so the system has n-km zeros, each of which is an eigenvalue of the matrix $Z = A_{k+1} - B_1 (G_\infty^{(k)})^{-1} C_1$.

The matrices A_{k+1}, B_1, C_1, are easily obtained from $G_{k+1}(s)$ by either pole-residue analysis or by inspection of a minimal realization of $G_{k+1}(s)$. $G_\infty^{(k)}$ is of small dimension in general and so is easily inverted. The matrix Z is easily constructed and the zeros of the system obtained by calculation of its eigenvalues.

Equation (32) holds only if $|G_\infty^{(k)}| \neq 0$. The case of $r_\infty < m$ is more complex as illustrated by the following result.

Result Six (see appendix 7.7)

If $m=\ell$, $|G(s)| \neq 0$, $r_\infty < m$ and P,Q are non-singular constant matrices satisfying

$$P\ G_\infty^{(k)} Q = \begin{bmatrix} I_{r_\infty} & 0 \\ 0 & 0 \end{bmatrix} \qquad \ldots(33)$$

then, using the notation of Result 3,

$$P_G(s) = |P^{-1}Q^{-1}| \cdot \begin{vmatrix} sI_{n-km} - A_{k+1} + B_1 QPC_1 & T_1 \\ T_2 & 0 \end{vmatrix} \qquad \ldots(34)$$

where T_1 is the $(n-km)x(m-r_\infty)$ matrix consisting of the last $m-r_\infty$ columns of B_1Q and T_2 is the $(m-r_\infty)x(n-km)$ matrix consisting of the last $m-r_\infty$ rows of PC_1. Moreover the system has $\leq n-(k+1)m+r_\infty$ zeros and possesses $n-(k+1)m+r_\infty$ zeros if, and only if, $|T_2T_1| \neq 0$.

The matrices P,Q are easily calculated and the matrices A_{k+1}, B_1, C_1 obtained by examination of a minimal realization of $G_{k+1}(s)$. The zeros of G(s) can then be obtained by applying known techniques (Davison and Wang, 1974) to equation (34). The advantage of equation (34) is that the system matrix is $(n-km+m-r_\infty)x(n-km+m-r_\infty)$ compared with previous results which use an $(m+n)x(m+n)$ system matrix.

4. Multivariable Second-order Type Systems

This section extends previous results (Owens, 1975) on the proportional and proportional plus integral control analysis of a class of second-order type multivariable systems to include the possibility of open-loop plant integrators and open-loop unstable systems. For this purpose, a multivariable, minimum-phase, second-order type mxm system $G(s)$ takes the form of equation (5) with $n = m+1$. Also $|G(s)| \not\equiv 0$ and $P_G(s)$ (eqn. (26)) has a zero in the open left-half complex plane. It follows from eqn. (19) that $k = 1$ and, from Result 2, $G(s)$ has a cascade decomposition

$$G(s) = G_1(s)\{G_\infty^{(1)} + (s+b_{m+1})^{-1}xy^+\} \qquad \ldots(35)$$

Also $|G_\infty^{(1)}| \neq 0$, for, from Result 6, if $r_\infty<m$, $G(s)$ has $\leq n-(k+1)m+r_\infty<1$ zeros contrary to assumption. Hence, without loss of generality, take $G_\infty^{(1)} = I_m$.

The following result is a generalization of previous results (Owens, 1975) on the proportional unity negative feedback control of such systems and generalizes these results to include the possibility of open-loop plant integrators and open-loop unstable systems.

Result Seven (see appendix 7.8)

Consider a unity negative feedback control system for the plant of equation (35) and, without loss of generality, take $G_\infty^{(1)} = I_m$. Then,

$$P_G(s) = s + b_{m+1} + y^+x \qquad \ldots(36)$$

so that $p = b_{m+1}+y^+x>0$. The use of the forward path controller

$$K(s) = kI_m - G_1^{-1}(s)|_{s=o} \qquad \ldots(37)$$

leads to a closed-loop system with return-difference determinant $|T(s)|$ of the form

$$|T(s)| \prod_{j=1}^{m+1} (s+b_j) = \{s^2+\{k+b_{m+1}\}s+kp-y^+G_1^{-1}(s)|_{s=o}x\}(s+k)^{m-1} \qquad \ldots(38)$$

The closed-loop system is hence stable for all arbitrarily high gains k satisfying the conditions

$$k+b_{m+1} > 0 \quad , \quad k > p^{-1}y^+G_1^{-1}(s)|_{s=o}x \qquad \ldots(39)$$

and the closed-loop poles are $-k$(m-1 times), $\mu_1(k)$ and $\mu_2(k)$ where

$$\lim_{k\to\infty} k^{-1}\mu_1(k) = -1 \qquad \ldots(40)$$

$$\lim_{k\to\infty} \mu_2(k) = -p \qquad \ldots(41)$$

Moreover, the residue of the closed-loop system at $\mu_2(k)$ tends to zero as $k\to+\infty$ and

$$\lim_{k\to\infty} \lim_{s\to\infty} k^{-1} sG(s)K(s) = I_m \qquad \ldots(42)$$

so that closed-loop interaction effects will be small at high gains, and the response speed arbitrarily fast.

This result is a complete generalization of previous results (Owens, 1975) and makes possible the construction of a feedback controller for the system capable of producing high response speeds, small steady state error and small interaction effects. The above design will also be superior to previous results (Owens, 1975) as interaction effects (eqn.(42)) are smaller.

5. Application to Pole-allocation

This section considers the application of the cascade canonical form discussed in section 2 in the field of pole allocation using output feedback (Fallside and Seraji, 1971 and Ahmari and Vacroux, 1973). The following result identifies situations where dynamic compensation is a design necessity and illustrates the importance of the integer k in the general feedback control problems.

Result Eight (see appendix 7.9)

Let $m=\ell$ and $|G(s)| \not\equiv 0$ and $k\geq 2$. Using the notation of Result 3, if the system is subject to constant output feedback of the form $u = Ky$, producing the closed-loop poles μ_j, $1\leq j\leq n$,

$$\mathrm{tr}(A+BKC) = \sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \{-b_j\} = \mathrm{tr}A \qquad \ldots(43)$$

and, if trA>0, the system cannot be stabilized by constant output feedback.

This result has important implications in the application of pole-allocation techniques. If $k \geqslant 2$ then care must be taken in the choice of desired pole positions eg with constant output feedback, the choice of $\mu_1 < \mathrm{tr}A$ will guarantee the presence of an unstable closed-loop pole. Also, if trA is small, compensation is required to obtain fast response speeds from the feedback system ie dynamic compensation is required to augment the system state and move trA further into the left-half-plane.

6. Conclusions

Using recent results (Owens, 1975) it has been demonstrated that multivariable first order systems are fundamental building blocks for the construction of the dynamic behaviour of linear systems described by an $m \times \ell$ transfer function matrix G(s). A canonical form has been derived which decomposes G(s) into a set of linked multivariable first order systems in an analogous manner to the decomposition of a classical transfer function into first order lags. The canonical form has direct application to the systematic calculation of system transmission zeros, the feedback control of a class of multivariable second-order-type systems and can be used to provide insight into difficulties occurring in pole allocation using output feedback.

7. Appendices

7.1 Proof of Result 1

Let $\{z_j\}_{1\leq j\leq p_1}$ be a basis for span $\{\alpha_j\}$ and $\{y_j\}_{1\leq j\leq q_1}$ be a basis for span $\{\beta_j\}$ and define $P_1 = [z_1 \ldots z_{p_1}]$, $Q_1 = [y_1, \ldots, y_{q_1}]^+$. By definition $q_1 \leq \ell$ and $p_1 \leq m$ and, $1 \leq j \leq n$,

$$\alpha_j = \sum_{k=1}^{p_1} \eta_{jk} z_k \quad , \quad \beta_j = \sum_{k=1}^{q_1} \varepsilon_{jk} y_k \qquad \ldots(44)$$

or, if $x_j = [\eta_{j1} \ldots \eta_{jp_1}]^T$, $y_j = [\varepsilon_{j1} \ldots \varepsilon_{jq_1}]^T$ then $\alpha_j = P_1 x_j$ and $\beta_j^+ = y_j^+ Q_1$ ie using equations (5),(8)

$$G(s) = \sum_{j=1}^{n} (s+b_j)^{-1} \alpha_j \beta_j^+ = \sum_{j=1}^{n} (s+b_j)^{-1} P_1 x_j y_j^+ Q_1$$

$$= P_1 M_1(s) Q_1 \qquad \ldots(45)$$

and, by construction, dim span $\{x_j\}_{1\leq j\leq n} = p_1$ and dim span $\{y_j\}_{1\leq j\leq n} = q_1$. Hence, by suitable reordering, suppose that $x_1, \ldots, x_{p_1}$ are linearly independent and, without loss of generality, take $x_i^+ x_j = \delta_{ij}$, then

$$M_1(s) = \sum_{j=1}^{n} \frac{1}{s+b_j} x_j y_j^+ = \sum_{j=1}^{p_1} \frac{1}{s+b_j} x_j y_j^+ + \sum_{j=p_1+1}^{n} \frac{1}{s+b_j} x_j y_j^+$$

$$= \{\sum_{j=1}^{p_1} \frac{1}{s+b_j} x_j x_j^+\}\{\sum_{j=1}^{p_1} x_j y_j^+\} + \sum_{j=p_1+1}^{n} \frac{1}{s+b_j} x_j y_j^+ \qquad \ldots(46)$$

Defining the $p_1 x p_1$ multivariable first order system

$$G_1(s) = \sum_{j=1}^{p_1} (s+b_j)^{-1} x_j x_j^+ \qquad \ldots(47)$$

then

$$G_{1\infty}^{(1)} = I_{p_1} \qquad \ldots(48)$$

and

$$M_1(s) = G_1(s)\{\sum_{j=1}^{p_1} x_j y_j^+ + \{\sum_{j=1}^{p_1} (s+b_j)x_j x_j^+\}\{\sum_{j=p_1+1}^{n} (s+b_j)^{-1} x_j y_j^+\}\}$$

$$= G_1(s)\ \{N + H_1(s)\} \qquad \ldots(49)$$

where, noting $\sum_{j=1}^{p_1} x_j x_j^+ = I_{p_1}$, N is a constant $p_1 x q_1$ matrix of the form,

$$N = \sum_{j=1}^{p_1} x_j y_j^+ + \sum_{j=p_1+1}^{n} x_j y_j^+ = M_{1\infty}^{(1)} \qquad \ldots(50)$$

and $H_1(s)$ is a $p_1 x q_1$ strictly proper transfer function matrix of the form

$$H_1(s) = \sum_{j=p_1+1}^{n} \frac{1}{s+b_j} \hat{x}_j y_j^+ \qquad \ldots(51)$$

where, $p_1 < j \leqslant n$,

$$\hat{x}_j = G_1^{-1}(s)\big|_{s=-b_j} x_j \qquad \ldots(52)$$

which proves the result.

7.2 Proof of Result 2

As rank $G(s) < \ell$ at, at most, a finite number of points of the complex plane, it follows (equation (5)) that dim span $\{\beta_j\} = \ell$ and hence $q_1 = \ell$ and we can take $Q_1 = I_\ell$. Using result 1,

$$G(s) = P_1\{M_{1\infty}^{(1)} + H_1(s)\}G_1(s) \qquad \ldots(53)$$

where $G_1(s)$ is an $\ell x \ell$ first order type system with pole set $\{-b_j\}_{1\leqslant j\leqslant \ell}$ and $H_1(s)$ is a strictly proper transfer function matrix with poles $-b_j$, $\ell < j \leqslant n$, each having unity rank residue. If k = 1, then $M_{1\infty}^{(1)} \neq 0$ and the result follows by application of equation (48) to obtain the identity $P_1 M_{1\infty}^{(1)} = G_\infty^{(1)}$. If k>1, then $M_{1\infty}^{(1)} = 0$ and rank $H_1(s) < \ell$ at, at most, a finite number of points of the complex plane so that $Q_2 = I_\ell$ and

$$G(s) = P_1 P_2\{M_{2\infty}^{(1)} + H_2(s)\}G_2(s)G_1(s) \qquad \ldots(54)$$

where $G_2(s)$ is a $\ell x \ell$ first order system with pole set $-b_j$, $\ell+1 \leqslant j \leqslant 2\ell$ and $H_2(s)$ is a strictly proper transfer function matrix with pole set $-b_j$, $2\ell < j \leqslant n$. If $k=2$ then $M_{2\infty}^{(1)} \neq 0$ and the result follows by application of (48) to $G_1(s)$ and $G_2(s)$ to obtain the identity $P_1P_2M_{2\infty}^{(1)} = G_\infty^{(2)}$. In general, for $1 \leqslant j \leqslant k$,

$$G(s) = P_1P_2..P_j\{M_{j\infty}^{(1)} + H_j(s)\}G_j(s)...G_2(s)G_1(s) \qquad ...(55)$$

where $G_r(s)$, $1 \leqslant r \leqslant j$, is an $\ell x \ell$ multivariable first order type system with pole set $-b_i$, $(r-1)\ell+1 \leqslant i \leqslant r\ell$, $G_{r\infty}^{(1)} = I_\ell$, $1 \leqslant r \leqslant j$, and $H_j(s)$ is a strictly proper transfer function matrix with pole set $-b_i$, $j\ell < i \leqslant n$, each having unity rank residue. Noting that $G_\infty^{(j)} = P_1...P_jM_{j\infty}^{(1)} = 0$, $j<k$, the procedure terminates at $j=k$ to provide the required result.

Finally, if $m=\ell$, equation (18) follows by applying the above procedure to $G^T(s)$ r times to obtain

$$G(s) = G_1(s)...G_r(s)\{G_\infty^{(r)} + L(s)\} \qquad ...(56)$$

If $r<k$ then $G_\infty^{(r)} = 0$ and, $|G(s)| \not\equiv 0$ implies $|L(s)| \not\equiv 0$, and the result follows by application of the above analysis k-r times.

7.5 Proof of Result 3

Using the notation of Result 2

$$G(s) = G_1(s) \;....\; G_k(s)\{G_\infty^{(k)} + G_{k+1}(s)\} \qquad ...(57)$$

The minimal realization of $G_{k+1}(s)$ has dimension n-km and takes the form

$$\dot{x}_{k+1}(t) = A_{k+1}x_{k+1}(t) + B_1u(t) \qquad ...(58)$$

$$z(t) = C_1x_{k+1}(t) \qquad ...(59)$$

Also, by examination of equation (47), $1 \leqslant j \leqslant k$,

$$G_j(s) = \{sI_m - A_j\}^{-1} \qquad ...(60)$$

where $$A_j = -G_j^{-1}(s)|_{s=o} \qquad ...(61)$$

These systems have minimal realizations of dimension m, of the form

$$\dot{x}_j = A_j x_j + x_{j+1} \quad , \qquad 1 \leqslant j \leqslant k-1 \qquad \ldots(62)$$

$$\dot{x}_k = A_k x_k + G_\infty^{(k)} u + z \qquad \ldots(63)$$

The result follows by combining equations (58),(59),(62),(63) in a single state space form with state vector $x(t) = \{x_1^T(t),..,x_k^T(t),x_{k+1}^T(t)\}^T$.

7.4 Evaluation of the Zero Polynomial

From equation (27),(26)

$$P_G(s) = \{ \prod_{j=1}^{n} (s+b_j)\} \cdot \{ \prod_{j=1}^{k} |G_j(s)|\} \cdot |G_\infty^{(k)} + G_{k+1}(s)| \qquad \ldots(64)$$

Using equation (48), $G_{j\infty}^{(1)} = I_m$ and hence, $1 \leqslant j \leqslant k$,

$$|G_j(s)| = \prod_{\ell=(j-1)m+1}^{j\ell} (s+b_\ell)^{-1} \qquad \ldots(65)$$

Hence,

$$P_G(s) = \{ \prod_{j=km+1}^{n} (s+b_j)\} \cdot |G_\infty^{(k)} + G_{k+1}(s)| \qquad \ldots(66)$$

as required.

7.5 Proof of Result 4

Using equation (28), we need only look at $|G_\infty^{(k)} + G_{k+1}(s)|$. From the assumption of the theorem

$$G_\infty^{(k)} = \sum_{j=1}^{r_\infty} u_j v_j^+ \qquad \ldots(67)$$

where $\{u_j\}_{1 \leqslant j \leqslant r_\infty}$ $\{v_j\}_{1 \leqslant j \leqslant r_\infty}$ are sets of linearly indepedent vectors. Also, by suitable reordering of poles

$$G_{k+1}(s) = \sum_{j=r_\infty+1}^{m} (s+b_j)^{-1} u_j v_j^+ \qquad \ldots(68)$$

where $\{u_j\}_{j>r_\infty}$, $\{v_j\}_{j>r_\infty}$ are sets of linearly independent vectors. If $N = \sum_{j=1}^{m} u_j v_j^+$ then

$$|G_\infty^{(k)} + G_{k+1}(s)| = |N| \prod_{j=r_\infty+1}^{m} (s+b_j)^{-1} \qquad \ldots(69)$$

and hence $P_G(s) = |N|$, which proves the result as $|G(s)| \not\equiv 0$ implies $P_G(s) \not\equiv 0$.

7.6 Proof of Result 5

$$P_G(s) = \{\prod_{j=km+1}^{n} (s+b_j)\} |G_\infty^{(k)} + G_{k+1}(s)|$$

$$= \{\prod_{j=km+1}^{n} (s+b_j)\}|G_\infty^{(k)}|\cdot|I_m+(G_\infty^{(k)})^{-1}C_1(sI_{n-km}-A_{k+1})^{-1}B_1|$$

$$= \{\prod_{j=km+1}^{n} (s+b_j)\}|G_\infty^{(k)}|\cdot|I_{n-km}+B_1(G_\infty^{k})^{-1}C_1(sI_{n-km}-A_{k+1})^{-1}|$$

$$= \{\prod_{j=km+1}^{n} (s+b_j)\}\frac{|G_\infty^{(k)}|\cdot|sI_{n-km}-A_{k+1}+B_1(G_\infty^{k})^{-1}C_1|}{|sI_{n-km}-A_{k+1}|} \qquad \ldots(70)$$

and the result follows by noting that (Result 2), $\prod_{j=km+1}^{n} (s+b_j) = |sI_{n-km}-A_{k+1}|$.

7.7 Proof of Result 6

$$|P|\cdot|G_\infty^{(k)}+G_{k+1}(s)|\cdot|Q| = |PG_\infty^{(k)}Q+PC_1(sI_{n-km}-A_{k+1})^{-1}B_1Q|$$

$$= \left|I_m+[PC_1,e_{r_\infty+1},\ldots,e_m]\begin{bmatrix}(sI_{n-km}-A_{k+1})^{-1} & 0\\ 0 & -I_{m-r_\infty}\end{bmatrix}\begin{bmatrix}B_1Q\\ e_{r_\infty+1}^T\\ \vdots\\ e_m^T\end{bmatrix}\right|$$

$$= \left|I_{n-km+m-r_\infty}+\begin{bmatrix}B_1Q\\ e_{r_\infty+1}^T\\ \vdots\\ e_m^T\end{bmatrix}[PC_1,e_{r_\infty+1},\ldots,e_m]\begin{bmatrix}(sI_{n-km}-A_{k+1})^{-1} & 0\\ 0 & -I_{m-r_\infty}\end{bmatrix}\right|$$

$$= \{ \prod_{j=km+1}^{n} (s+b_j)^{-1} \} \begin{vmatrix} sI_{n-km}-A_{k+1}+B_1QPC_1 & T_1 \\ T_2 & 0 \end{vmatrix} \quad \ldots(71)$$

where we have used the identity $\begin{vmatrix} e_{r_\infty+1}^T \\ \vdots \\ e_m \end{vmatrix} |e_{r_\infty+1},\ldots,e_m| = I_{m-r_\infty}$. The first part of the result follows by comparison of equation (71) with equation (28). By inspection G(s) has <n-km zeros. Also, as $|G(s)| \not\equiv 0$ then $P_G(s) \not\equiv 0$, and noting that if

$$L(s) = T_2(sI_{n-km}-A_{k+1}+B_1QPC_1)^{-1}T_1 \quad \ldots(72)$$

then

$$P_G(s) = |P^{-1}Q^{-1}|\cdot|L(s)|(-1)^{m-r_\infty}|sI_{n-km}-A_{k+1}+B_1QPC_1|$$

$$= P_L(s)\ (-1)^{m-r_\infty}\ |P^{-1}Q^{-1}| \quad \ldots(73)$$

then $|L(s)| \not\equiv 0$ and the zeros of G(s) are the zeros of L(s) and vice versa. It follows directly that G(s) has $\leq (n-km)-(m-r_\infty) = n-(k+1)m+r_\infty$ zeros and has exactly $n-(k+1)m+r_\infty$ zeros (Result 2) if, and only if, $L_\infty^{(1)} = T_2T_1$ is non-singular.

7.8 Proof of Result 7

As $G_{1\infty}^{(1)} = I_m$ then

$$G_1(s) = \sum_{j=1}^{m} \frac{1}{s+b_j}\alpha_j\gamma_j^+ \quad \ldots(74)$$

where $\gamma_j^+\alpha_i = \delta_{ij}$. From result 5, noting that $A_{k+1} = -b_{m+1}$, $B_1 = y^+$, $C_1 = x$

$$P_G(s) = s + b_{m+1} + y^+x \quad \ldots(75)$$

Also

$$K(s) = \sum_{j=1}^{m} \{k-b_j\}\alpha_j\gamma_j^+ \quad \ldots(76)$$

so that

$$G_1(s)K(s) = \sum_{j=1}^{m} \frac{(k-b_j)}{(s+b_j)} \alpha_j \gamma_j^+ \qquad \ldots(77)$$

$$T_A(s) = I_m + G_1(s)K(s) = \sum_{j=1}^{m} \frac{(s+k)}{(s+b_j)} \alpha_j \gamma_j^+ \qquad \ldots(78)$$

$$T_A^{-1}(s) = \frac{1}{s+k} \sum_{j=1}^{m} (s+b_j)\alpha_j \gamma_j^+ \qquad \ldots(79)$$

and hence,

$$|T(s)| = |I_m + G(s)K(s)| = |I_m + G_1(s)\{I_m + (s+b_{m+1})^{-1} x_j y_j^+\}K(s)|$$

$$= |T_A(s)| \cdot |I_m + T_A^{-1}(s) G_1(s)(s+b_{m+1})^{-1} x_j y_j^+ K(s)|$$

$$= \frac{(s+k)^m}{\prod_{j=1}^{m}(s+b_j)} \{1 + y_j^+ K(s) T_A^{-1}(s) G_1(s) x_j \frac{1}{s+b_{m+1}}\} \qquad \ldots(80)$$

But

$$K(s)T_A^{-1}(s)G_1(s) = \{\sum_{j=1}^{m}(k-b_j)\alpha_j\gamma_j^+\}\{\sum_{j=1}^{m}\frac{(s+b_j)}{(s+k)}\alpha_j\gamma_j^+\}\{\sum_{j=1}^{m}(s+b_j)^{-1}\alpha_j\gamma_j^+\}$$

$$= \sum_{j=1}^{m} \frac{(k-b_j)}{(s+k)} \alpha_j \gamma_j^+$$

$$= \frac{1}{(s+k)} \{kI_m - G_1^{-1}(s)|_{s=o}\} \qquad \ldots(81)$$

from which

$$|T(s)| = \frac{(s+k)^{m-1}}{\prod_{j=1}^{m+1}(s+b_j)} \{s^2 + (k+b_{m+1})s + kb_{m+1} + y^+xk - y^+G^{-1}(s)|_{s=o}x\} \qquad \ldots(82)$$

Equation (38) follows from the substitution $p = b_{m+1} + y^+x$. Equations (40), (41) follows by Taylor series expansion of the quadratic term in the numerator of $|T(s)|$. Finally

$$\lim_{s\to\infty} k^{-1}sG(s)K(s) = k^{-1}G_{1\infty}^{(1)}\{kI_m - G^{-1}(s)|_{s=o}\}\to I_m \quad (k\to\infty) \qquad \ldots(83)$$

7.9 Proof of Result 8

For simplicity we prove the result for k = 2. Using result 3, the feedback system is described by the state-space model

$$\dot{x} = \{A+BKC\}x = \begin{bmatrix} A_1 & I_m & 0 \\ G_\infty^{(2)}K & A_2 & C_1 \\ B_1K & 0 & A_3 \end{bmatrix} x \qquad \ldots(84)$$

so that the trace remains unaltered by constant output feedback. Hence, if trA>0, then tr(A+BKC) = trA>0 so the feedback system has, at least, one right-half-plane pole.

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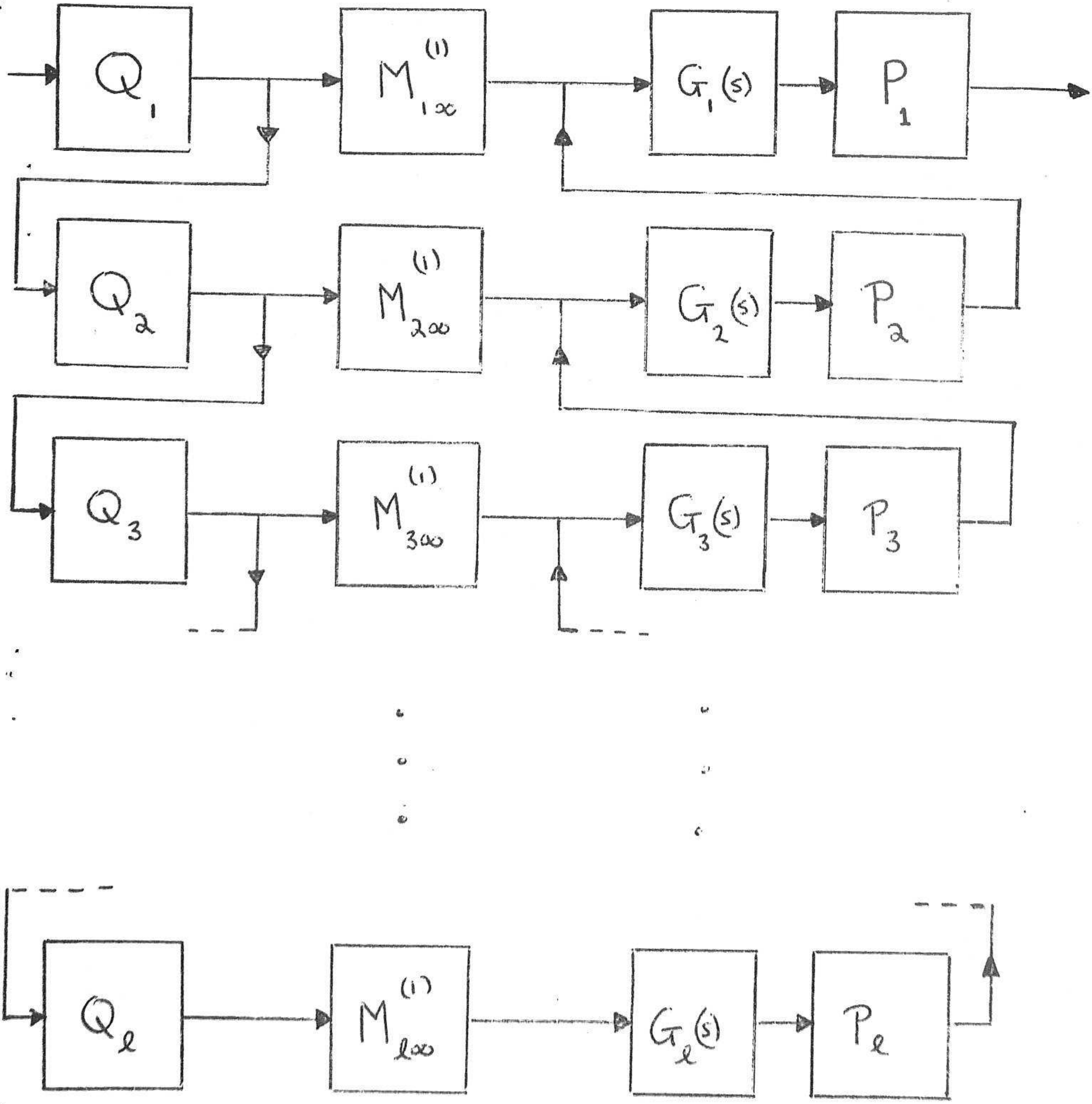
Q_1
$M^{(1)}_{1\infty}$
$G_1(s)$
P_1
Q_2
$M^{(1)}_{2\infty}$
$G_2(s)$
P_2
Q_3
$M^{(1)}_{3\infty}$
$G_3(s)$
P_3
Q_ℓ
$M^{(1)}_{\ell\infty}$
$G_\ell(s)$
P_ℓ

Fig. 1

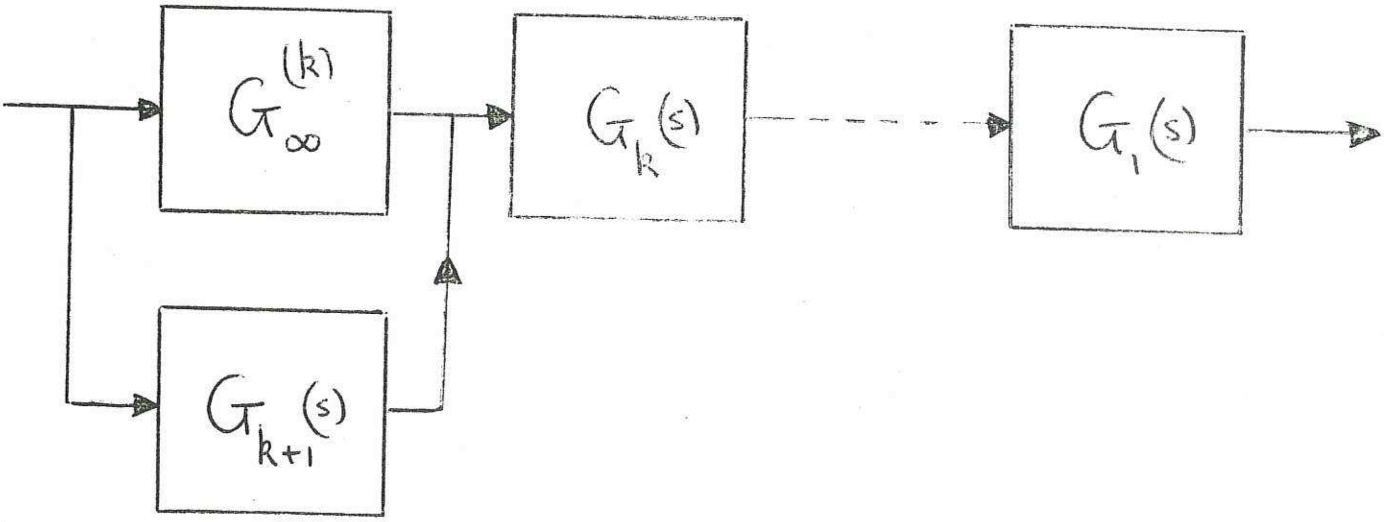

Fig. 2.