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PARAMETER MAPPING IN THE STABILITY ANALYSIS OF SYSTEMS WITH MULTIPLE DELAYS

by

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Abstract

The stability of linear systems with multiple delays is considered using parameter plane methods. By mapping contours from the complex variable plane to the parameter plane it is possible to choose control parameters such as loop gain and system delays which will ensure that all closed loop poles have negative real parts.

1. Introduction

The stability of linear control systems with multiple delays is a difficult problem due to the transcendental nature of the closed loop characteristic equations of these systems. Since such systems have an infinite number of closed loop poles, standard techniques due to Routh and Hurwitz are only applicable when delay terms are approximated by series expansions. This approach is hardly satisfactory and in recent years, attempts have been made to utilize the theorems developed by Pontryagin[1] to locate the system's poles.[2] Pontryagin's criterion is not really helpful in the design of control systems and increases in complexity if the system delays do not have a common factor. Eisenberg[3] appears to be the first to have applied the parameter method developed by Mitrović[4] and generalized by Šiljak,[5] to systems with delays. This approach has seen further development[6,7,8] and it is the interpretation of Loo[8] which will be followed here. By dividing the characteristic equation into real and imaginary parts and treating the exponential terms as complex coefficients, it is possible to generate analytical expresssions for various system parameters in terms of complex plane parameters. By then mapping the imaginary axis of the complex plane into the parameter plane, it is possible to locate regions of the parameter plane corresponding to regions of the complex plane and to see the effect of varying system parameters on the location of the system's closed loop poles.

2. A linear feedback system with multiple delays.

Consider the feedback system in Fig. 1 with delays T_1 and T_2. G(s) and H(s) may be ratios of polynomials in the Laplace variable s, or constant. R is a function of s and $\exp(-T_1 s)$

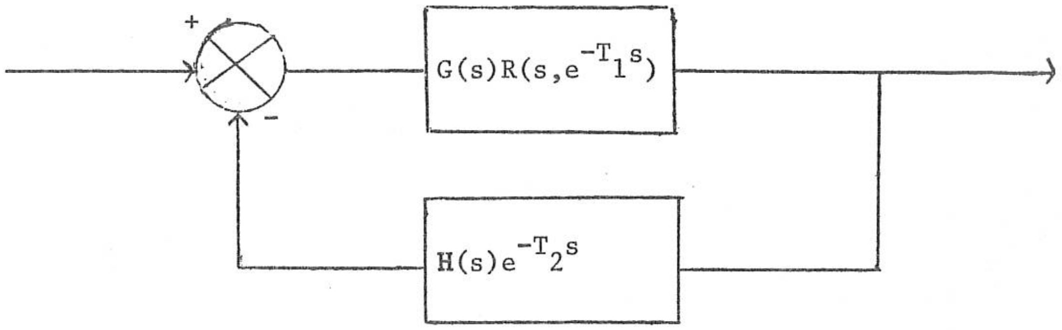


Figure 1

The closed loop characteristic equation of the system in Fig. 1 can be written as

$$F(s) = 1 + G(s)\ H(s).\exp(-T_2 s)\ \circ\ R(s,\ \exp(-T_1 s)) \qquad \ldots\ldots(1)$$

and is clearly dependent on the form of the function R.

3. Characteristic equation with single exponential terms

If we choose $R = \exp\ (-T_1 s)$ and

$$G(s) = K_1 \frac{G_1(s)}{G_2(s)} \qquad \ldots\ldots(2)$$

$$H(s) = K_2 \frac{H_1(s)}{H_2(s)}$$

where G_1, G_2, H_1, H_2 are finite polynomials in s and K_1, K_2 are constant loop gains, equation 1 becomes

$$F(s) = G_2(s)H_2(s) + K_1K_2G_1(s)\ H_1(s)\ \exp(-(T_1+T_2)s) \qquad \ldots\ldots(3)$$

If

$$s = \theta + j\omega \qquad \ldots\ldots(4)$$

then the polynomials $G_2(s)\ H_2(s)$ and $G_1(s)\ H_1(s)$ can be split into real and imaginary parts and written as

$$G_2(\theta,\omega)H_2(\theta,\omega) = R_2(\theta,\omega) + j\ I_2\ (\theta,\omega) \qquad \ldots\ldots(5)$$

$$G_1(\theta,\omega)H_1(\theta,\omega) = R_1(\theta,\omega) + j\ I_1(\theta,\omega) \qquad \ldots\ldots(6)$$

where

$R_2(\theta,\omega)$ and $R_1(\theta,\omega)$ denote the real ; $I_2(\theta,\omega)$ and $I_1(\theta,\omega)$ the imaginary parts of G_2H_2 and G_1H_1 respectively. Thus equation 4 enables us to write

$$F(\theta,\omega) = R_F(\theta,\omega) + j\ I_F(\theta,\omega) \qquad \ldots\ldots (7)$$

where $R_F(\theta,\omega)$ denotes the real and $I_F(\theta,\omega)$ the imaginary part of $F(\theta,\omega)$. Combining equation 3,4,5,6,7 yields.

$$R_F(\theta,\omega) = R_2(\theta,\omega) + K_1K_2\ \exp(-\theta(T_1+T_2)).$$
$$\left[R_1(\theta,\omega)\ .\ \cos(T_1+T_2)\omega + I_1(\theta,\omega)\cdot\sin(T_1+T_2)\omega\right] \qquad \ldots\ldots (8)$$
$$I_F(\theta,\omega) = I_2(\theta,\omega) + K_1K_2\ \exp(-\theta(T_1+T_2)).$$
$$\left[I_1(\theta,\omega)\ .\ \cos(T_1+T_2)\omega - R_1(\theta,\omega)\ \sin(T_1+T_2)\omega\right] \qquad \ldots\ldots (9)$$

Now by writing the polar representations of equations 5 and 6 as

$$G_2(\theta,\omega)\ .\ H_2(\theta,\omega) = r_2(\theta,\omega)\ .\ \exp\left[j\phi_2(\theta,\omega)\right] \qquad \ldots\ldots(10)$$
$$G_1(\theta,\omega)\ .\ H_1(\theta,\omega) = r_1(\theta,\omega)\ .\ \exp\left[j\phi_1(\theta,\omega)\right] \qquad \ldots\ldots(11)$$

equation 3 becomes

$$r_2(\theta,\omega)\ .\ \exp\left[j\phi_2(\theta,\omega)\right]$$
$$= K_1K_2r_1(\theta,\omega)\ \exp\ (-\theta(T_1+T_2)).\ \exp\left[j(\phi_1(\theta,\omega)-\omega(T_1+T_2)+(2n+1)\pi\right]$$
$$\ldots\ldots(12)$$

where $n = o, \pm 1, \pm 2, \ldots\ldots$

By equating magnitude and phase angle of both sides of equation 12 yields

$$K_1K_2 = \frac{r_2(\theta,\omega)}{r_1(\theta,\omega)}\ .\ \exp(\theta(T_1+T_2)) \qquad \ldots\ldots(13)$$

$$T_1+T_2 = \frac{1}{\omega}\ \left[\phi_1(\theta,\omega) - \phi_2(\theta,\omega) + (2n+1)\pi\right] \qquad \ldots\ldots(14)$$
$$n = o,\ \pm 1,\ \pm 2,\ \ldots\ldots\ldots$$

It is now possible to construct the desired $\alpha - \beta$ parameter plane by choosing, say: $\alpha = T_1+T_2$, $\beta = K_1K_2$; $\alpha = T_1$, $\beta = K_1$; $\alpha = T_2$, $\beta = K_1$; $\alpha = T_1, \beta = K_2$; or $\alpha = T_2$, $\beta = K_2$. The appropriate choice of α and β shows how any two of the system parameters T_1, T_2, K_1 and K_2 affect the location of the system's closed loop poles.

By letting $\theta = o$, equations 13 and 14 become functions of ω only and it is possible to map the entire $j\omega$ axis into the parameter plane.

Since the region of interest with respect to pole location is the left half complex plane, rules and graphical methods have been developed[5,9] to determine the corresponding region in the parameter plane. The correspondence is dependent on the sign of the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial R_F(\theta,\omega)}{\partial\alpha} & \dfrac{\partial R_F(\theta,\omega)}{\partial\beta} \\ \\ \dfrac{\partial I_F(\theta,\omega)}{\partial\alpha} & \dfrac{\partial I_F(\theta,\omega)}{\partial\beta} \end{vmatrix} \qquad \ldots\ldots(15)$$

If J is positive/negative, the region of interest in the complex s plane (sometimes referred to as the shaded region) which lies on one side of a contour in the s plane, maps into a region on the same/opposite side of the corresponding contour in the parameter plane.

Using equations 8 and 9 we can show that for

(i) $\alpha = T_1+T_2$, $\beta = K_1K_2$

$$J = K_1K_2\ \omega\exp(-2\theta(T_1+T_2))r_1^{\ 2}(\theta,\omega) \qquad \ldots\ldots(16)$$

(ii) $\alpha = T_1$, $\beta = K_1$; $\alpha = T_2$, $\beta = K_1$

$$J = K_1K_2^{\ 2}\ \omega\exp(-2\theta(T_1+T_2))r_1^{\ 2}(\theta,\omega) \qquad \ldots\ldots(17)$$

(iii) $\alpha = T_1$, $\beta = K_2$; $\alpha = T_2$, $\beta = K_2$

$$J = K_1^{\ 2}K_2\omega\ \exp(-2\theta\ (T_1+T_2)r_1^{\ 2}(\theta,\omega) \qquad \ldots\ldots(18)$$

Thus for positive values of K_1 and K_2, the sign of the Jacobian is the same as that of ω. Since we are mapping the $j\omega$ axis into the parameter plane, the region to the left of the $j\omega$ axis (the left half complex plane) will map into a region to the left/right of the corresponding contour in the parameter plane, in the direction of increasing ω, for positive/negative values of J.

Finally, since the $j\omega$ axis will map into a different contour for each value of n(n=o, ± 1, ± 2,...), the region of the parameter plane corresponding to the left half complex s plane will be the intersection of the n - regions defined by any n-contours in the parameter plane.

4. Characteristic equation with multiple exponential terms.

If we choose $R = 1/(1-K_1G_3(s)\ \exp\ (-T_1s))$

and

$$G(s) = K_2 \frac{G_1(s)}{G_2(s)} \qquad \ldots\ldots\ (19)$$

$$H(s) = \frac{H_1(s)}{H_2(s)}$$

where G_1, G_2, G_3, H_1, H_2, are finite polynomials or constant and K_1, K_2 are constant loop gains as before, the system in Fig. 1 has a characteristic equation of the form

$$F(s) = G_2(s).H_2(s) - K_1G_2(s)\ H_2(s)G_3(s)\ \exp\ (-T_1s) + K_2G_1(s)H_1(s)\ \exp(-T_2s) \qquad \ldots\ldots\ (20)$$

Letting the complex variable $s = \theta + j\omega$ as before, we can write $G_2(\theta,\omega).H_2(\theta,\omega)$ and $G_1(\theta,\omega)\ .\ H_1(\theta,\omega)$ as in equations 5 and 6.

Define

$$G_2(\theta,\omega).H_2(\theta,\omega).G_3(\theta,\omega) = R_3(\theta,\omega) + j\ I_3(\theta,\omega) \ldots\ldots\ (21)$$

where $R_3(\theta,\omega)$ denotes the real and $I_3(\theta,\omega)$ the imaginary part of $G_2.H_2.G_3$. Then using equations 4,5,6,7 and 21, we can write

$$R_F(\theta,\omega) = R_2(\theta,\omega) - K_1\ \exp(-T_1\theta)\ [R_3(\theta,\omega)\ \cos T_1\omega + I_3(\theta,\omega)\sin T_1\omega] + K_2\ \exp(-T_2\theta)\ [R_1(\theta,\omega)\ \cos T_2\omega + I_1(\theta,\omega)\sin T_2\omega] \qquad \ldots\ldots(22)$$

$$I_F(\theta,\omega) = I_2(\theta,\omega) - K_1\exp(-T_1\theta)\ [I_3(\theta,\omega)\ \cos T_1\omega - R_3(\theta,\omega)\ \sin T_1\omega] + K_2\exp(-T_2\theta)\ [I_1(\theta,\omega)\ \cos T_1\omega - R_1(\theta,\omega)\ \sin T_2\omega] \qquad \ldots\ldots(23)$$

Now by writing $G_2.H_2.G_3$ in polar form as

$$G_2(\theta,\omega).H_2(\theta,\omega).G_3(\theta,\omega) = r_3(\theta,\omega).\ \exp\ [j\phi_3(\theta,\omega)] \qquad \ldots\ldots(24)$$

and combining with equations 10 and 11 enables equation 20 to be written as

$$r_2(\theta,\omega)\ \exp\left[j\phi_2(\theta,\omega)\right] = K_2 r_1(\theta,\omega)\ \exp(-T_2\theta).\exp\left[j\phi_1(\theta,\omega) - \omega T_2 + (2n+1)\pi\right]$$
$$+ K_1 r_3(\theta,\omega)\ \exp(-T_1\theta).\exp\left[j\phi_3(\theta,\omega) - \omega T_1\right]$$
$$n = o,\ \pm 1,\ \pm 2$$
$$\ldots\ldots (25)$$

Equating magnitude and phase angle as before yields

$$K_2 = \frac{\left[r_2(\theta,\omega) - K_1 r_3(\theta,\omega)\ \exp(-T_1\theta)\right]\ \exp(T_2\theta)}{r_1(\theta,\omega)} \qquad \ldots\ldots (26)$$

$$K_1 = \frac{\left[r_2(\theta,\omega) - K_2 r_1(\theta,\omega)\ \exp(-T_2\theta)\right]\ \exp(T_1\theta)}{r_3(\theta,\omega)} \qquad \ldots\ldots (27)$$

$$T_1 + T_2 = \frac{1}{\omega}\left[\phi_1(\theta,\omega) + \phi_3(\theta,\omega) - \phi_2(\theta,\omega) + (2n+1)\pi\right] \qquad \ldots\ldots (28)$$

By choosing the system parameters to be defined in the parameter plane,the Jacobian, defined in equation 15 can be calculated for appropriate α and β. Using equations 22 and 23 it is easy to show that for

(i) $\alpha = T_1$, $\beta = K_1$

$$J = K_1\omega\exp(-2\theta T_1)\ \left[R_3^{\ 2}(\theta,\omega) + I_3^{\ 2}(\theta,\omega)\ (\sin^2 T_1\omega - \cos^2 T_1\omega)\right.$$
$$\left. + 2R_3(\theta,\omega)I_3(\theta,\omega)\ \sin T_1\omega\ \cos T_1\omega\right] \qquad \ldots\ldots (29)$$

(ii) $\alpha = T_2$, $\beta = K_2$

$$J = K_2\ \omega\exp(-2\theta T_2)\ \left[r_1^{\ 2}(\theta,\omega)\right] \qquad \ldots\ldots (30)$$

5. Examples

(a) Referring to Figure 1 let

$$G(s) = \frac{K_1(s+10)}{(s+2)(s+1)} \qquad \ldots\ldots (31)$$

$$H(s) = \frac{K_2}{s+5} \qquad \ldots\ldots (32)$$

Then with $s = \theta + j\omega$, equations 10 and 11 yield

$$r_1(\theta,\omega) = \left[(\theta+10)^2 + \omega^2\right]^{\frac{1}{2}} \qquad \ldots\ldots (33)$$

$$\phi_1(\theta,\omega) = \arctan\left(\frac{\omega}{\theta+10}\right) \qquad \ldots\ldots (34)$$

$$r_2(\theta,\omega) = \left[(\theta^3-3\theta\omega^2+8\theta^2-8\omega^2+17\theta+10)^2 + (3\theta^2\omega-\omega^3+16\theta\omega+17\omega)^2\right]^{\frac{1}{2}} \quad \ldots\ldots (35)$$

$$\phi_2(\theta,\omega) = \arctan\left[\frac{3\theta^2\omega-\omega^3 + 16\theta\omega+17\omega}{\theta^3-3\theta\omega^2+8\theta^2-8\omega^2+17\theta+10}\right] \quad \ldots\ldots (36)$$

Substituting equations 33,34,35,36 into equations 13 and 14 yields

$$K_1 = \left[\frac{(\theta^3-3\theta\omega^2+8\theta^2-8\omega^2+17\theta+10)^2+(3\theta\omega^2-\omega^3+16\theta\omega+17\omega)^2}{K_2\left[(\theta+10)^2+\omega^2\right]^{\frac{1}{2}}}\right]^{\frac{1}{2}} \quad \ldots\ldots (37)$$

$$T_1 = \frac{1}{\omega}\left[\arctan\left(\frac{\omega}{\theta+10}\right) - \arctan\left(\frac{3\theta^2\omega-\omega^3+16\theta+17\omega}{\theta^3-3\theta\omega^2+8\theta^2-8\omega^2+17\theta+10}\right)+(2n+1)\pi\right]-T_2 \quad \ldots\ldots (38)$$

$$n = o, \pm 1, \pm 2, \ldots$$

Now let $\theta = o$ and consider the T_1-K_1 plane (i.e. $\alpha = T_1$, $\beta = K_1$) by letting

$$T_2 = 0.5 \qquad K_2 = 0.5 \quad \ldots\ldots (39)$$

Therefore equations (37) and (38) become

$$K_1 = \frac{\left[(-8\omega^2+10)^2+(-\omega^3+17\omega)^2\right]^{\frac{1}{2}}}{0.5.\left[\omega^2+100\right]^{\frac{1}{2}}} \quad \ldots\ldots (40)$$

$$T_1 = \frac{1}{\omega}\left[\arctan\left(\frac{\omega}{10}\right) - \arctan\left(\frac{17\omega-\omega^3}{10-8\omega^2}\right)+(2n+1)\pi\right]-0.5 \quad \ldots\ldots (41)$$

From equation 17,

$$J = K_1\omega \quad \ldots\ldots (42)$$

Figure 2 shows contours in the T_1-K_1 plane for various values of n. As ω increases along each contour, we know from equation 42 that the region corresponding to the left half complex plane will lie to the left/right of the contour for positive/negative ω. The crosses along each contour indicate the desired region and the intersection of each region bounded by a contour dependent on n, will be that region bounded by the contour for n = o, -1.

Figure 3 shows the relevent contour in the parameter plane and determines suitable values of K_1 and T_1 which ensure that all the systems closed loop poles lie in the left half complex plane.

(b) Referring to Figure 1 let

$$G(s) = K_2 \qquad \text{..... (43)}$$

$$H(s) = 1 \qquad \text{..... (44)}$$

Then letting $s = \theta+j\omega$, equations 10, 11 and 24 yield

$$r_1 = r_2 = r_3 = 1 \qquad \text{..... (45)}$$

$$\phi_1 = \phi_2 = \phi_3 = o \qquad \text{..... (46)}$$

Substituting equations 45 and 46 into equations 26 and 28 yields

$$K_2 = (1 - K_1 \exp(-T_1\theta)) \exp (T_2\theta) \qquad \text{..... (47)}$$

$$T_2 = \frac{(2n+1)\pi}{\omega} - T_1 \qquad \text{..... (48)}$$

$$n = o, \pm 1, \pm 2, \ldots.$$

Now let $\theta = o$ and consider $T_2 - K_2$ plane with

$$K_1 = 0.5$$

$$T_1 = \text{constant}$$

From equation 30 ,

$$J = K_2\omega \qquad \text{..... (49)}$$

Figure 4 shows the one contour for all n, T_1 and T_2 in the T_2-K_2 plane. Clearly, since J takes the sign of ω, the region of stability (denoted by crosses) can be achieved for all T_1,T_2 and if

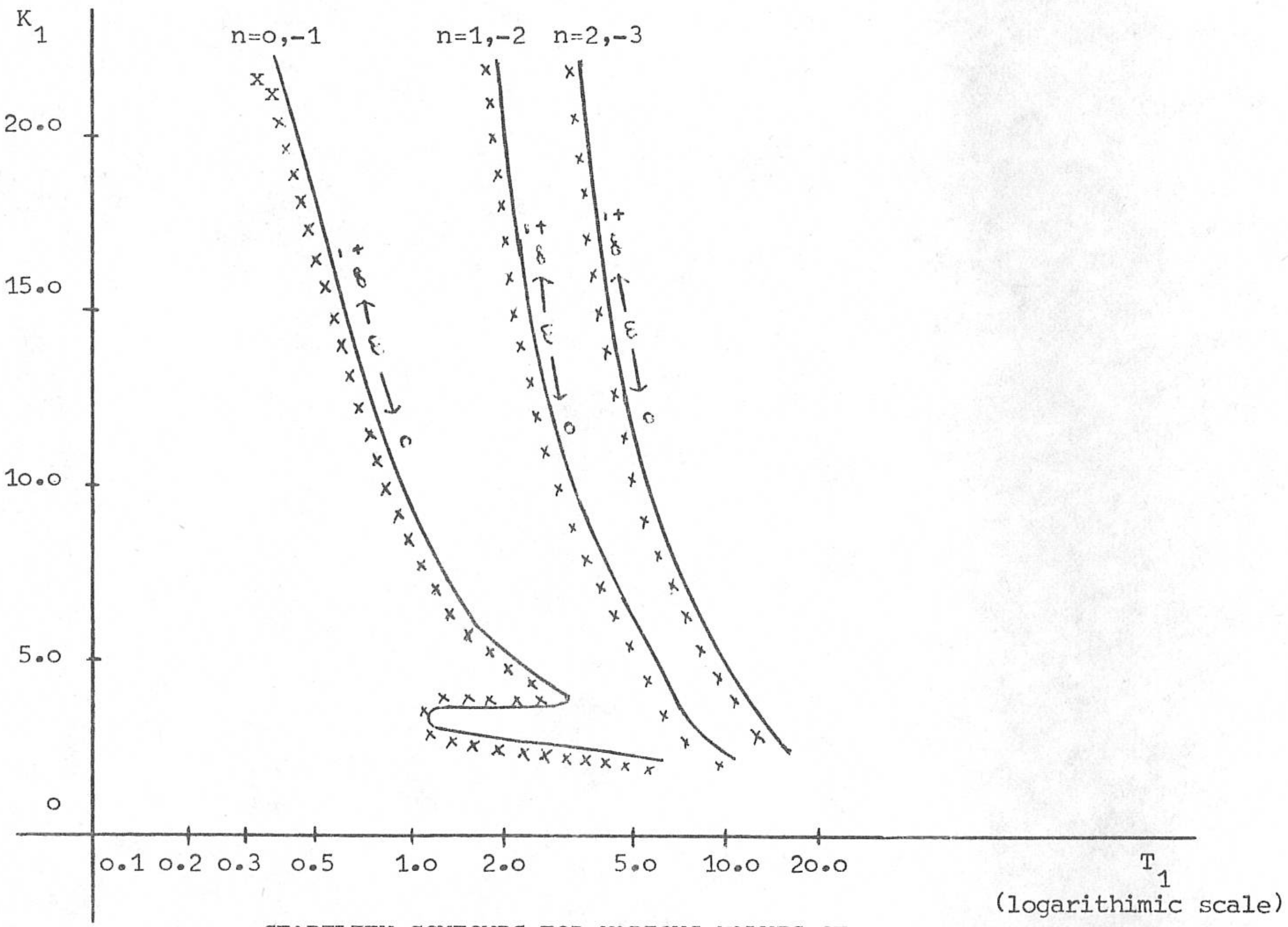
$$K_2 < 1 - K_1 \qquad \text{..... (50)}$$

This problem occurs in the rolling of metal strip and has been analysed by the Nyquist criterion[10] and analytical methods[11] with some difficulty. The parameter plane method reduces the stability analysis of the problem to the simple analysis of Figure 4.

6. Conclusion

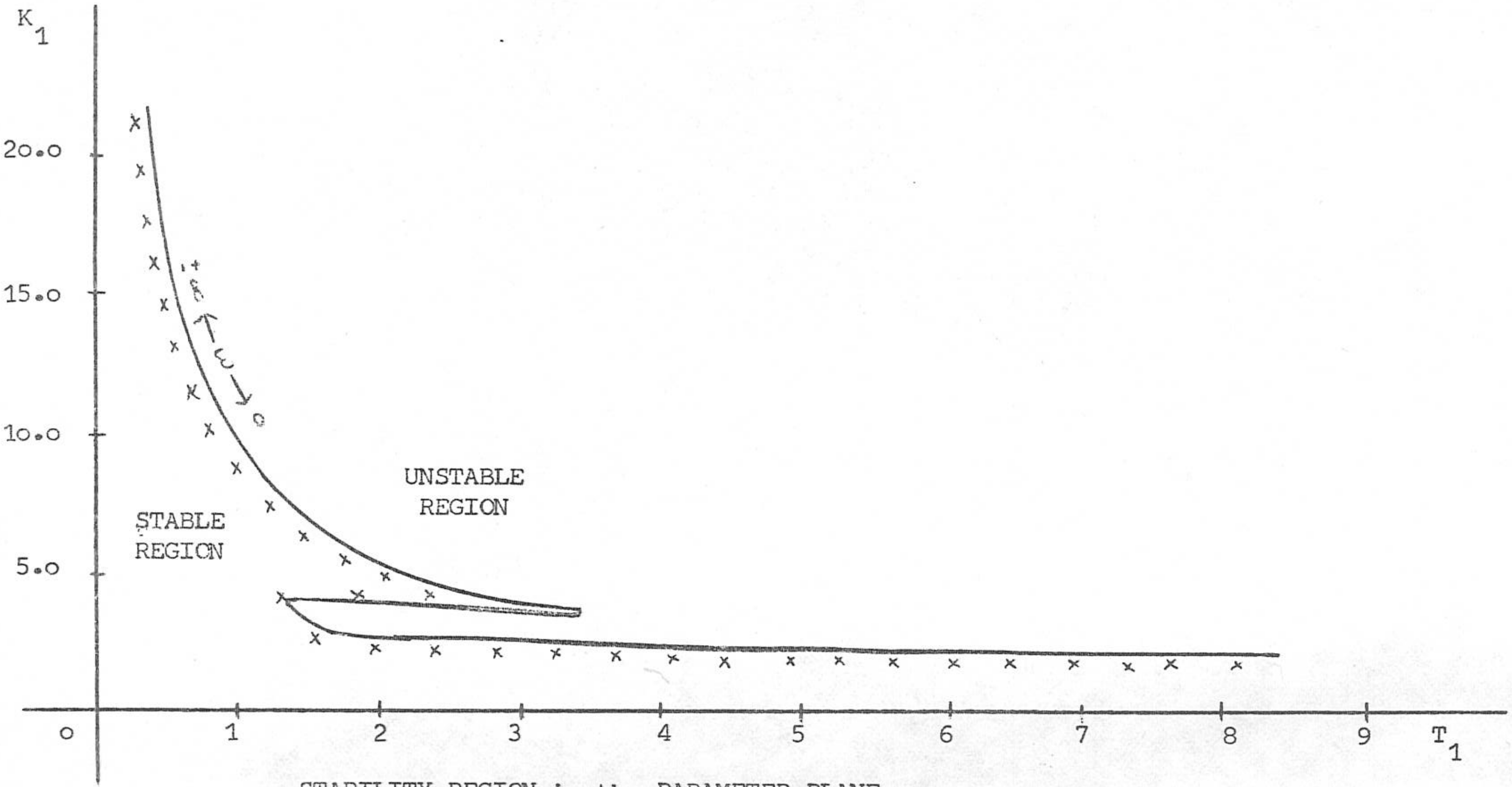
The parameter plane method has been used in the stability analysis of two different linear feedbacks systems with multiple delays. The approach could clearly be applied to any similar system with variable control parameters. It is hoped to extend this work to non linear systems and to provide a user orientated computer package to assist in the design of such control systems.

FIGURE 2



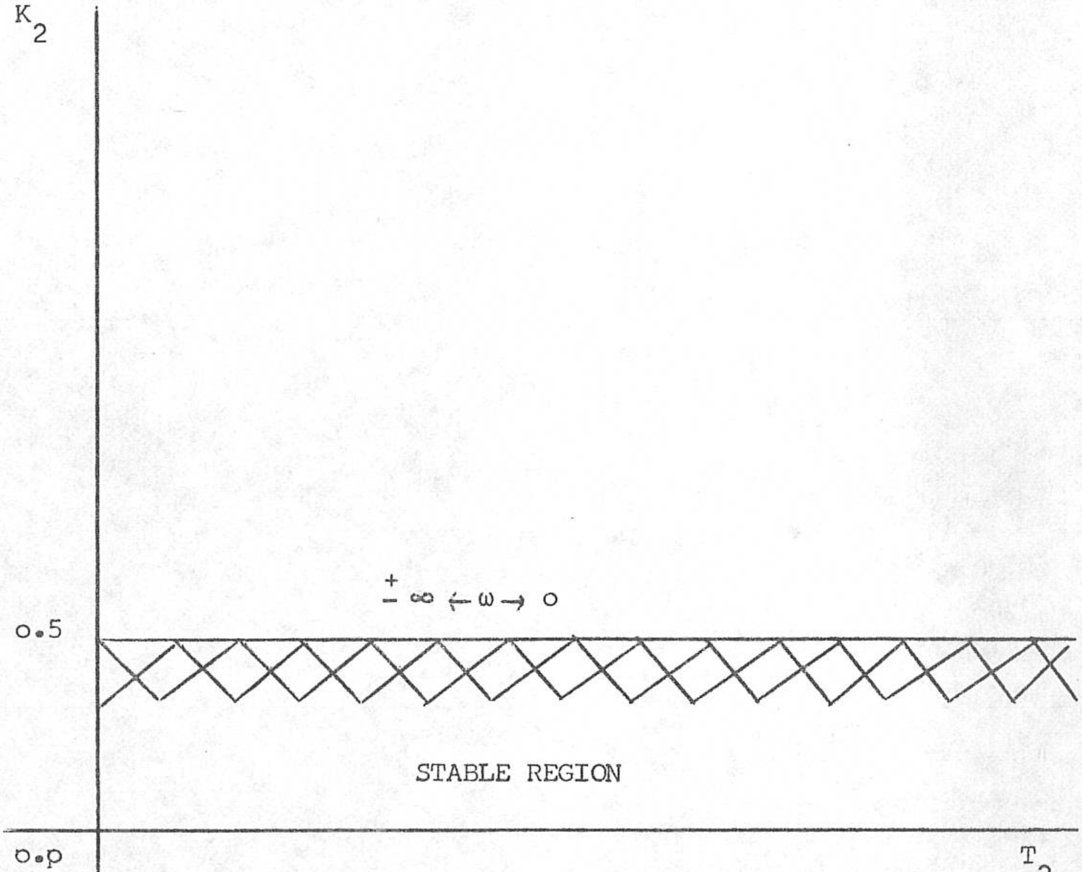
STABILITY CONTOURS FOR VARIOUS VALUES OF n

FIGURE 3



STABILITY REGION in the PARAMETER PLANE

FIGURE 4


K_2
$\pm\infty \leftarrow \omega \rightarrow 0$
0.5
STABLE REGION
0.0
T_2

BIBLIOGRAPHY

1. PONTRYAGIN, L.S.: 'On the zeros of some elementary transcendental functions', Amer. Math. Soc., Translations, (2), 1955,1,pp.95-110.

2. HO, K.W., and CHAN, W.K.: 'Stability of third-order control systems with time lag using Pontryagin's method for zero determination', Proc. I.E.E., 1969,116,(12), pp.2063-2068.

3. EISENBERG, L.: 'Stability of linear systems with transport lag', I.E.E.E. Trans., 1966 AC-11, pp. 247-254.

4. MITROVIĆ, D.: 'Graphical analysis and synthesis of feedback control systems, I - Theory and Analysis,' A.I.E.E. Trans.(Applic. and Industry), 1959, A.I.-77, pp.476-487.

5. SILJAK, D.D.,: 'Analysis and synthesis of feedback control systems in parameter plane, I - Linear continuous systems,' A.I.E.E. Trans. (Applic. and Industry, 1964, AI-75, pp. 449-458.

6. KARMARKAR, J.S. and SILJAK, D.D.: 'Stability analysis of systems with time delays,' Proc. I.E.E., 1970, 117(7), pp. 1421-1424.

7. KARMARKAR, J.S.,: 'Graphical stability criterion for linear systems,' ibid., 1970, 117,(7), pp. 1430-1433. get these

8. LOO, S.G.: 'Stability of linear stationary systems with time delay,' Int. J. Contr., 1969, 9, pp. 103-109.

9. SILJAK, D.D.: 'Generalization of the parameter plane method,' I.E.E.E. Trans. Auto. Control, 1966, AC-11, No. 1, pp. 63-70.

10. EDWARDS, J.B.,: 'Stability problems in the control of multipass processes,' Proc.I.E.E., Vol.121, No. 11, 1974, pp. 1425-1432.

11. GREENBERG, J.M., and EDWARDS, J.B., 'Stability of systems with multiple transportation lags ' I.E.E.E., Trans. on Circuits and Systems, (submitted for publication).