Monograph:

Owens, D.H. (1976) Multi-Variable Root-Loci Using the Inverse Transfer Function Matrix. Research Report. ACSE Research Report 49 . Department of Automatic Control and Systems Engineering

MULTIVARIABLE ROOT-LOCI USING THE INVERSE TRANSFER FUNCTION MATRIX

by

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Research Report No. 49.

October 1976.

Abstract

Recently developed analysis of the asymptotic behaviour of the root-loci of linear multivariable systems are reformulated in terms of the inverse system transfer function matrix. The advantage of the approach is the analytical separation of the effects of plant and controller on root-loci which can provide more direct insight into the effect of controller structure and dynamic compensation.

1. Introduction

Recent papers (Kouvaritakis and Shaked 1976, Owens 1976a) have provided theoretical analyses of the asymptotic behaviour of the root-loci of a linear mutlivariable system described by an mxm strictly proper transfer function matrix $G(s)$ when subjected to unity negative feedback with mxm formed path controller transfer function matrix $p\,K(s)$, where p is a real scaler gain parameter and $K(s)$ is proper. The two analyses are quite different in character in that Kouvaritakis and Shaked (1976) derive the asymptotic directions and pivots of the root-locus by spectral expansion of matrices characterizing the composite system $G(s)K(s)$ whereas Owens (1976,a) uses eigen-analysis of the matrix $\{G^{-1}(s) + \rho K(s)\}$ to characterize the asymptotic directions and pivots in terms of geometric relationships in the system input-output space. The use of spectral expansion techniques provides a systematic numerical technique for the calculation of system asymptotes but suffers from the disadvantage that the separate effects of plant and controller on the root-loci are difficult to determine. The geometric approach, as recently presented (Owens 1976a), is partially limited in its application due to the requirement that $\lim_{s\to\infty} s^k G(s)$ should exist for some integer $k \geq 1$ and the finite and nonsingular. It does however, have the advantage that the effect of plant and controller on the root-locus is separated and the controller appears in the relations defining the asymptotes in a quasi-linear manner, hence aiding the assessment and choice of compensator structure and the avoidance of sensitivity problems (Owens, 1976a).
This paper generalizes the results of Owens (1976a) to the analysis of the root-locus structure of strictly proper systems with mxm inverse transfer function matrix

$$G^{-1}(s) \triangleq \sum_{j=0}^{k} s^j A_{k-j} + H(s) \qquad \ldots (1)$$

where H(s) is strictly proper, $k \geq 1$ and $A_o \neq 0$ (in contrast with the requirement $|A_o| \neq 0$ used by Owens (1976a).

2. Characterization of the Asymptotic Root-loci

Consider a unity negative-feedback configuration for the control of an invertible system described by a strictly proper mxm transfer function matrix G(s) with inverse of the form of (1), and forward path controller pK(s) which is proper. Writing, for a transfer functinn matrix L(s),

$$L_\infty^{(j)} \triangleq \lim_{s \to \infty} s^j L(s) \qquad \text{.... (2)}$$

whenever the limit exists, then (Owens 1976a) it is assumed that

$$|K_\infty^{(0)}| \neq 0 \qquad \text{.... (3)}$$

and K(s) is written in the form

$$K(s) = K_\infty^{(0)} + K_1(s) \qquad \text{.... (4)}$$

where $K_1(s)$ is an mxm strictly proper transfer function matrix describing the dynamic elements in the controller structure.

The poles of the closed-loop system are the zero of the closed-loop characteristic polynomial $\rho_c(s) = \rho_o(s)\,|I + pG(s)K(s)|$, where $\rho_o(s)$ is the open-loop characteristic polynomial. If attention is restricted to unbounded poles of $\rho_c(s)$ as the gain p becomes large, then the closed-loop poles are the zeros of the rational polynomial $|G^{-1}(s) + pK(s)|$. Equivalently the unbounded closed-loop poles $(p \to +\infty)$ are the solutions of the eigenvalue problem,

$$\{G^{-1}(s(p)) + p\,K(s(p))\}x(p) = 0 \quad , \quad p > 0 \qquad \text{.... (5)}$$

where, without loss of generality, for any unbounded sequence $\{p_n\}_{n \geq 1}$ it is assumed that the sequence $\{x(pn)\}_{n \geq 1}$ has a non-zero cluster point x_o.

2.1 Mathematical Preliminaries

It will be seen that the asymptotes of the root-locus depend upon the geometric structure of the coefficient matrice A_o, A_1, ... and the structure of $K_\infty^{(o)}$. For convenience, define the subspaces

$$V_o(\lambda) \triangleq C^m$$

$$V_1(\lambda) \triangleq N(A_o + \lambda K_\infty^{(o)})^\perp$$

$$V_{j+1}(\lambda) \triangleq N(A_j + \lambda K_\infty^{(o)})^\perp \cap N(A_o) \cap \cdots \cap N(A_{j-1})$$

$$j \geq 1$$

.... (6)

for any complex number λ. It is evident that the subspaces $V_1(o), V_2(o), \ldots, V_{\ell-1}(o), V_\ell(\lambda)$ are orthogonal for all $\ell \geq 2$ and for all complex numbers λ. Also, let, for any complex number ,

$$W_o(\lambda) \triangleq C^m \qquad \text{.... (7)}$$

and, for $j \geq o$, $W_{j+1}(\lambda)$ be the subspace of solution vectors x_o of the relations

$$\{\lambda . K_\infty^{(o)} + A_j\}x_o + A_{j-1}x_1 + \cdots\cdots + A_o x_j = 0$$

$$A_{j-1}x_o + \cdots\cdots + A_o x_{j-1} = 0 .$$

$$\vdots$$

$$A_1 x_o + A_o x_1 = 0$$

$$A_o x_o = 0$$

$$x_i \in V_1(o) \oplus \cdots \oplus V_{j+1-i}(o) \quad , \quad 1 \leq i \leq j \qquad \text{.... (8)}$$

Of particular interest to the analysis is the case of $\lambda = 0$, when, from the definitions,

$$W_{j+1}(o) \subset W_j(o) \qquad \text{.... (9)}$$

The following theorem provides the basic means of calculation and characterization of the asymptotic behaviour of the closed-loop system. By a mild abuse of terminology, the statement that a sequence $y_n \to o$ $(n \to \infty)$ in a subspace X will be interpreted in the sense that the orthogonal projection of y_n onto X tends to the zero vector as n tends to infinity.

It follows trivially that $y_n \to o$ $(n \to \infty)$ in $X_1, \ldots, X_r$ implies that $y_n \to o$ $(n \to \infty)$ in $X_1 + \ldots + X_r$.

Theorem 1

Let $\{p_n\}_{n\geq 1}$ be an unbounded sequence of controller gains and suppose that, for some $o \leq q \leq k-1$, the eigenvector $x(p_n)$ of equation (5) takes the form

$$x(p_n) = x_o + \delta_{q,o} \sum_{i=1}^{q} (s(p_n))^{-i} \hat{x}_i + z_q(p_n) \quad \ldots (10)$$

where $\{s(p_n)\}$ is unbounded, $p_n(s(p_n))^{q-k} \to \lambda$ $(n \to \infty)$,

$$\{\lambda K_\infty^{(0)} + A_q\} x_o + A_{q-1}\hat{x}_1 + \cdots + A_o \hat{x}_q = 0$$
$$A_{q-1} x_o + A_{q-2}\hat{x}_1 + \cdots + A_o \hat{x}_{q-1} = 0$$
$$\vdots \qquad \vdots$$
$$A_o x_o = 0$$

$$\hat{x}_i \in V_1(0) \oplus \cdots \oplus V_{q+1-i}(0) \quad , \ 1 \leq i \leq q$$

$$\ldots (11)$$

and $(s(p_n))^j z_q(p_n) \to o$ $(n \to \infty)$ in $V_{q+1-j}(o)$, $1 \leq j \leq q$. If, also

$$R(A_o + \lambda K_\infty^{(0)}) \cap K_\infty^{(0)} W_1(\lambda) = \{0\} \qquad (q = 0)$$

$$\Big\{ R(A_o) + \sum_{j=1}^{q} (A_{q+1-j} + \delta_{j,1} K_\infty^{(0)} \lambda) \Big[\{V_1(0) \oplus \cdots$$
$$\cdots \oplus V_{q+1-j}(0)\}^{\perp} \Big] \Big\} \cap K_\infty^{(0)} W_{q+1}(\lambda) = \{0\} \ , (q > 0) \quad \ldots (12)$$

then the sequence $\{s(p_n)(p_n(s(p_n))^{q-k} - \lambda)\}_{n \geq 1}$ has a finite cluster point β_q which is a solution of relations (11) and, for $q = o$,

$$\{\beta_o K_\infty^{(0)} + A_1 + \lambda K_{1\infty}^{(1)}\} x_o + \{\lambda K_\infty^{(0)} + A_o\} x_1 = 0$$
$$x_1 \in V_1(\lambda) \quad \ldots (13)$$

or, for $q > o$,

$$\{\beta_q K_\infty^{(0)} + A_{q+1} + \lambda K_{1\infty}^{(1)}\}x_0 + \sum_{i=1}^{q}(A_{q+1-i} + \delta_{i,1}\lambda K_\infty^{(0)})(\hat{x}_i + y_i) + A_0 x_{q+1} = 0$$

$$x_{q+1} \in V_1(0)$$

$$y_i \in \{V_1(0) \oplus \cdots \oplus V_{q+1-i}(0)\}^{\perp}, \quad 1 \leq i \leq q$$

... (14)

Moreover, for q = o,

$$x(p_n) = x_0 + s^{-1}x_1 + z_1(p_n)$$

where $s(p_n)z_1(p_n) \to o \ (n \to \infty)$ is $V_1(\lambda)$. Also, for $q > o$, if the sum

$$R(A_0) + \sum_{j=1}^{q}(A_{q+1-j} + \delta_{j,1}K_\infty^{(0)}\lambda)\left[\{V_1(0) \oplus \cdots \oplus V_{q+1-j}(0)\}^{\perp}\right] \quad \ldots (15)$$

is a direct sum, then $y_i \in V_{q+2-i}(\lambda\delta_{i,1})$, $1 \leqslant i \leqslant q$, or, defining

$x_i = \tilde{x}_i + y_i$, $1 \leqslant i \leqslant q$,

$$\{\beta_q K_\infty^{(0)} + A_{q+1} + \lambda K_{1\infty}^{(1)}\}x_0 + \sum_{i=1}^{q}(A_{q+1-i} + \delta_{i,1}\lambda K_\infty^{(0)})x_i + A_0 x_{q+1} = 0$$

$$\{\lambda K_\infty^{(0)} + A_q\}x_0 + A_{q-1}x_1 + \cdots + A_0 x_q = 0$$

$$A_{q-1}x_0 + \cdots + A_0 x_{q-1} = 0$$

$$\vdots \qquad \vdots$$

$$A_0 x_0 = 0$$

$$x_i \in V_1(0) \oplus \cdots \oplus V_{q+1-i}(0) \oplus V_{q+2-i}(\lambda\delta_{i,1}), \quad 1 \leq i \leq q+1$$

... (16)

and

$$x(p_n) = x_0 + \sum_{i=1}^{q+1}(s(p_n))^{-i}x_i + z_{q+1}(p_n)$$

where $(s(p_n))^j z_{q+1}(p_n) \to o \ (n \to \infty)$ in $V_{q+2-j}(\lambda\delta_{j,1})$, $1 \leqslant j \leqslant q+1$.

Proof

Dividing the eigenvalue equation by $(s(p_n))^{k-q-1}$ and, for notational convenience, denoting $\acute{s}(p_n)$ by s and x_o by $\tilde{x}_o$, leads to

$$0 = \lim_{n\to\infty} s\Big\{\sum_{j=0}^{k} s^{j+q-k} A_{k-j} + \lambda K_\infty^{(0)} + s^{q-k} H(s)$$
$$+ \{p_n s^{q-k} - \lambda\} K_\infty^{(0)} + p_n s^{q-k} K_1(s)\Big\} x(p_n)$$
$$= \lim_{n\to\infty} s\Big\{\sum_{j=0}^{q}\sum_{i=0}^{q} s^{j-i} A_{q-j}\check{x}_i + \sum_{j=0}^{k-q-1}\sum_{i=0}^{q} s^{j-i+q-k} A_{k-j}\check{x}_i$$
$$+ \sum_{j=0}^{q} s^j A_{q-j} z_q(p_n) + \sum_{j=0}^{k-q-1} s^{j+q-k} A_{k-j} z_q(p_n) + p_n s^{q-k} K_1(s)$$
$$+ \lambda K_\infty^{(0)}\Big\{\sum_{i=0}^{q} s^{-i}\check{x}_i + z_q(p_n)\Big\} + \{p_n s^{q-k} - \lambda\} K_\infty^{(0)} x(p_n)\Big\}$$

... (18)

Using relations (11) leads to

$$0 = \delta_{q,0}\sum_{i=1}^{q} A_{q+1-i}\check{x}_i + A_{q+1}x_0 + \lambda K_\infty^{(0)}\check{x}_1(1-\delta_{q,0}) + \lambda K_{1\infty}^{(1)} x_0$$
$$+ \lim_{n\to\infty} s\Big\{\sum_{j=0}^{q} s^j A_{q-j} z_q + \lambda K_\infty^{(0)} z_q + \{p_n s^{q-k} - \lambda\} K_\infty^{(0)} x(p_n)\Big\}$$

... (19)

If q = o, then

$$0 = A_1 x_0 + \lambda K_{1\infty}^{(1)} x_0 + \lim_{n\to\infty} s\,\Big\{\{A_0 + \lambda K_\infty^{(0)}\} z_0 + \{p_n s^{-k} - \lambda\} K_\infty^{(0)} x(p_n)\Big\} \qquad \text{... (20)}$$

and, using (12), each term has a finite cluster point i.e.

$$\{B_0 K_\infty^{(0)} + A_1 + \lambda K_{1\infty}^{(1)}\} x_0 + \{A_0 + \lambda K_\infty^{(0)}\} x_1 = 0$$

... (21)

for some $x_1 \in N(A_o + \lambda K_\infty^{(o)})^\perp = V_1(\lambda)$ as required. Moreover $x(p_n) = x_o + s^{-1} x_1 + z_1(p_n)$ where $s\, z_1 \to o\ (n \to \infty)$ in $V_1(\lambda)$ and the result is proven for the case of $q = o$.

If $q > o$, write

$$\lim_{n\to\infty} s\left\{\sum_{j=0}^{q} s^j A_{q-j} z_q + \lambda K_\infty^{(o)} z_q + \{p_n s^{q-k} - \lambda\} K_\infty^{(o)} x(p_n)\right\}$$

$$= \lim_{n\to\infty}\left\{ s^{q+1} A_o z_q + \sum_{j=1}^{q} s^j \{A_{q+1-j} + \delta_{j,1}\lambda K_\infty^{(o)}\} z_q + s\{p_n s^{q-k} - \lambda\} K_\infty^{(o)} x(p_n)\right\} \quad \ldots (22)$$

then, using (12), β_q exists and is finite. However, noting that $s^j z_q \to o\ (n \to \infty)$ in $V_{q+1-j}(o)$, $1 \leq j \leq q$, then $s^j z_q \to o\ (n \to \infty)$ in $V_1(o) \oplus \cdots \oplus V_{q+1-j}(o)$ then (22) takes the form

$$\lim_{n\to\infty} s\left\{\sum_{j=0}^{q} s^j A_{q-j} z_q + \lambda K_\infty^{(o)} z_q + \{p_n s^{q-k} - \lambda\} K_\infty^{(o)} x(p_n)\right\}$$

$$= A_o x_{q+1} + \sum_{j=1}^{q}\{A_{q+1-j} + \delta_{j,1}\lambda K_\infty^{(o)}\} y_j + \beta_q K_\infty^{(o)} x_o$$

$$x_{q+1} \in N(A_o)^\perp = V_1(o)$$

$$y_i \in \{V_1(o) \oplus \cdots \oplus V_{q+1-i}(o)\}^\perp, \quad 1 \leq i \leq q \quad \ldots (23)$$

Combining (23) with (19) yields equation (14), as required.

Finally, if (15) is a direct sum, then each term in (22) has a finite cluster point. Suppose that $s^{q+1} A_o z_q \to A_o x_{q+1}\ (n \to \infty)$ for $x_{q+1} \in N(A_o)^\perp = V_1(o)$, then $z_q = s^{-q-1} x_{q+1} + \psi_{q+1}$ where $s^{q+1}\psi_{q+1} \to o(n \to \infty)$ in $V_1(o)$. Also suppose that $s^q A_1 z_q \to A_1 y_q$ for some $y_q \in N(A_1)^\perp$ then $s^q A_1 \psi_{q+1} \to A_1 y_q\ (n \to \infty)$ and $y_q \in N(A_o)$ i.e. $y_q \in N(A_1)^\perp \cap N(A_o) = V_2(o)$, and $z_q = s^{-q} y_q + s^{-(q+1)} x_{q+1} + \psi_q$ where $s^{q+1}\psi_q \to o\ (n \to \infty)$ in $V_1(o)$ and $s^q \psi_q \to o\ (n \to \infty)$ in $N(A_1)^\perp \supset N(A_1)^\perp \cap N(A_o) = V_2(o)$. Continuing in an inductive manner with the notation

$$s^j \{A_{q+1-j} + \delta_{j,1} \lambda K_\infty^{(o)}\} z_q \to \{A_{q+1-j} + \delta_{j,1} \lambda K_\infty^{(o)}\} y_j \quad (n \to \infty)$$

then, without loss of generality, we can take $y_j \in N(A_{q+1-j} + \delta_{j,1} \lambda K_\infty^{(o)})^\perp \cap N(A_o) \cap \ldots \cap N(A_{q-j}) = V_{q+2-j}(\lambda\delta_{j,1})$ and $z_q = s^{-1} y_1 + \ldots + s^{-q} y_q + z_{q+1}(p_n)$ where $s^j z_{q+1} \to o$ $(n \to \infty)$ in $V_{q+2-j}(\lambda\delta_{j,1})$ s required. Writing $x_i = \tilde{x}_i + y_i$, $1 \leqslant i \leqslant q$, then $x_i \in V_1(o) \oplus \ldots \oplus V_{q+1-i}(o) \oplus V_{q+2-i}(\delta_{j,1}\lambda)$, $1 \leqslant i \leqslant q + 1$, and relations (16) follow from (11), (14) and the properties of the subspaces $V_j(\lambda)$. Q.E.D.

2.2 Calculation of the Asymptotic Directions

The theorem of the preceding section provides a direct computational method for the calculation of the asymptotic directions of the root-locus (Kouvaritakis and Shaked, 1976). Let $\{p_n\}_{n \geqslant o}$ be unbounded, $\{s(p_n)\}_{n \geqslant o}$ be a corresponding sequence of unbounded closed-loop poles and denote by λ_j any finite cluster point of the sequence $\{p_n(s(p_n))^{-j}\}_{n \geqslant o}$. If $\lambda_j \neq 0$, then equivalently, the root-locus has a j^{th} order infinite zero (Kouvaritakis and Shaked, 1976) of the asymptotic form $s(p) = \lambda^{-1/j} p^{1/j}$ where $p^{1/j}$ is the positive real j^{th} root of p and $\lambda^{-1/j}$ is any j^{th} root of λ^{-1}.

Dividing the eigenvalue equation by $(s(p_n))^k$ and letting $n \to \infty$ yields

$$\lim_{n \to \infty} \{A_o + p_n(s(p_n))^{-k} K_\infty^{(o)}\} x(p_n) = 0 \qquad \ldots (24)$$

from which, as $K_\infty^{(o)}$ is non-singular and $x_o \neq 0$, $\{p_n(s(p_n))^{-k}\}_{n>o}$ is bounded and has cluster points λ_k which are the solutions of the relation

$$\{A_o + \lambda_k K_\infty^{(o)}\} x_o = 0 \qquad \ldots (25)$$

This equation can be used to calculate all the k^{th} order infinte zeros. If A_o is nonsingular then previous results (Owens 1976a) can be used to calculate both the asymptotic directions and pivots. If, however A_o is singular, then (25) always has a solution $\lambda_k = 0$ with $A_o x_o = 0$ and $x(p_n) = x_o + z_o(p_n)$. Comparing these observations with the conditions of theorem 1 (with q=o and λ=o) yeilds the result that, subject to (12) and (15), $\lambda_{k-1} = \beta_o$ exists and is finite and a solution of the relations (equation (13))

$$\{\lambda_{k-1} K_\infty^{(o)} + A_1\} x_o + A_o x_1 = 0 \quad , x_1 \in V_1(o)$$
$$A_o x_o = 0 \quad , x_o \neq 0 \qquad \ldots (26)$$

If (26) possesses a solution with $\lambda_{k-1} = o$, then it is easily verified that the conditions of theorem 1 are again satisfied so that, subject to (12) and (15) $\lambda_{k-2} = \beta_1$ exists and is a solution of the relations

$$\{\lambda_{k-2} K_\infty^{(o)} + A_2\} x_o + A_1 x_1 + A_o x_2 = 0$$
$$A_1 x_o + A_o x_1 = 0 \quad , x_1 \in V_1(o) \oplus V_2(o)$$
$$A_o x_o = 0 \quad , x_2 \in V_1(o) \quad , x_o \neq 0$$

$$\ldots (27)$$

In general, subject to the validity of (12) and (15) for $\lambda = o$, the closed-loop system possesses $(k-q)^{th}$ order infinite zeros, generated by the solutions of the relations

$$\{\lambda_{k-q} K_\infty^{(o)} + A_q\} x_o + A_{q-1} x_1 + \cdots + A_o x_q = 0$$
$$A_{q-1} x_o + \cdots + A_o x_{q-1} = 0$$
$$\vdots \qquad \vdots$$
$$A_o x_o = 0 \quad , \quad x_o \neq 0$$

$$x_i \in V_1(o) \oplus \cdots \oplus V_{q+1-i}(o) \quad , 1 \leq i \leq q. \qquad \ldots (28)$$

In most practical applications only the cases of $q = o$, 1 are of interest (i.e. the case of only k^{th} and $(k-1)^{th}$ order infinite zeros) where the relations (28) simplify to equations (25) and (26) respectively.

2.3 Calculation of the Pivots

Given a solution $\lambda_{k-q} \neq 0$ of (28) and application of Theorem 1 with $\lambda = \lambda_{k-q}$ indicates that, under the stated conditions (12), the sequence $\{s(p_n)(p_n(s(p_n))^{q-k} - \lambda_{k-q})\}_{n>o}$ has a finite cluster point β_q. Writing $s(p_n) = \lambda^{-1/k-q}\, p_n^{1/k-q} + \mu(p_n)$ where $p_n^{-1/(k-q)}\, \mu(p_n) \to o \quad (n \to \infty)$, then it is easily shown that

$$\beta_q = \lim_{n\to\infty} s(p_n)\{p_n(s(p_n))^{q-k} - \lambda_{k-q}\} = -\lambda_{k-q}(k-q) \lim_{n\to\infty} \mu(p_n) \qquad \ldots (29)$$

That is, the pivot $\alpha_{k-q} \triangleq \lim_{n\to\infty} \mu(p_n)$ exists and is finite,

$$\beta_q = -\lambda_{k-q}(k-q)\alpha_{k-q}. \qquad \ldots (30)$$

and

$$s(p_n) = (\lambda_{k-q})^{-1/k-q}\, p_n^{1/k-q} + \alpha_{k-q} + \varepsilon(p_n)$$

$$\lim_{n\to\infty} \varepsilon(p_n) = 0 \qquad \ldots (31)$$

The two cases of greatest practical interest are the cases of q=o, 1. For q=o theorem 1 indicate that β_o is a solution of the relations,

$$\{A_o + \lambda_k K_\infty^{(0)}\}x_o = 0 \qquad , x_o \neq 0$$

$$\{\beta_o K_\infty^{(0)} + A_1 + \lambda_k K_{1\infty}^{(1)}\}x_o + \{\lambda_k K_\infty^{(0)} + A_o\}x_1 = 0$$

$$x_1 \in V_1(\lambda_k) = N(A_o + \lambda_k K_\infty^{(0)})^{\perp}$$

$$\ldots (32)$$

In the case of q=1, assuming that (12) is valid , β_1 is a solution of the relations

$$\{\beta_1 K_\infty^{(0)} + A_2 + \lambda_{k-1} K_{1\infty}^{(1)}\}x_0 + \{A_1 + \lambda_{k-1} K_\infty^{(0)}\}\{\check{x}_1 + y_1\}$$
$$+ A_0 x_2 = 0$$
$$\{A_1 + \lambda_{k-1} K_\infty^{(0)}\}x_0 + A_0 \check{x}_1 = 0$$
$$A_0 x_0 = 0 \quad , \quad x_0 \neq 0$$
$$y_1 \in V_1(0)^\perp = N(A_0)$$
$$x_2 \in V_1(0)$$
$$\check{x}_1 \in V_1(0)$$

... (33)

If (15) is a direct sum, then these relations are replaced by (equation (16))

$$\{\beta_1 K_\infty^{(0)} + A_2 + \lambda_{k-1} K_{1\infty}^{(1)}\}x_0 + \{A_1 + \lambda_{k-1} K_\infty^{(0)}\}x_1 + A_0 x_2 = 0$$
$$\{A_1 + \lambda_{k-1} K_\infty^{(0)}\}x_0 + A_0 x_1 = 0$$
$$A_0 x_0 = 0 \quad , \quad x_0 \neq 0$$
$$x_1 \in V_1(0) \oplus V_2(\lambda_{k-1})$$
$$x_2 \in V_1(0)$$

... (34)

2.4 Illustrative Example

Consider a system described by the inverse transfer function matrix, k=2,

$$G^{-1}(s) \triangleq s^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + s \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

... (35)

with controller

$$K(s) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = K_\infty^{(0)} \quad , \quad K_1(s) \equiv 0$$

... (36)

To compute the second order asymptote, case (25) i.e.

$$\left\{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \lambda_2\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right\}x_0 = 0 \quad , \quad x_0 \neq 0$$

... (37)

which has only one non-zero solution, namely

$$\lambda_2 = -\frac{3}{2} \quad , \quad x_0 = \begin{bmatrix}1\\ 2\end{bmatrix}$$

... (38)

Noting that $W_1(\lambda_2) = N(A_o + \lambda_2 K_\infty^{(o)})$ and $R(A_o + \lambda_2 K^{(o)}) \cap W_1(\lambda_2) = \{o\}$ then (Theorem 1) the corresponding pivot α_2 is obtained from (30),(32) as $\beta_o = 3\alpha_2$ where

$$\left\{\beta_o\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}\right\}\begin{bmatrix}1\\ 2\end{bmatrix} + \begin{bmatrix}-2 & 1\\ 1 & -1/2\end{bmatrix}x_1 = 0$$

$$x_1 \in V_1(\lambda_2) = \operatorname{span}\left\{\begin{bmatrix}2\\ -1\end{bmatrix}\right\}$$

... (39)

Calculation of β_o is simplified by premultiplication of (39) by $\{1,2\}$ which eliminates the last term in (39) i.e. $\beta_o = -13/6$ or $\alpha_2 = -13/18$ and the second order asymptotes take the form

$$s(p) = \pm i\sqrt{\frac{2}{3}}\,p^{1/2} - \frac{13}{18} + \varepsilon_2(p)$$

... (40)

where $\varepsilon_2 \to 0$ as $p \to \infty$.

To compute the first order asymptote, note that $W_1(o) = N(A_o) = \operatorname{span}\left\{\begin{bmatrix}-1\\ 1\end{bmatrix}\right\}$ and $R(A_o) \cap K_\infty^{(o)} W_1(o) = \{o\}$ so that, applying theorem 1, λ, is a solution of the relations (26),

$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}x_0 = 0 \quad , \quad x_0 \neq 0$$

$$\{\lambda_1 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\} x_0 + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_1 = 0$$

$$x_1 \in V_1(0) = \text{span } \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$$

... (41)

i.e. $x_0 = \{-1, 1\}^T$, $\lambda_1 = -1/3$, $x_1 = \{2/3, 2/3\}^T$

Noting that

$$W_2(\lambda_1) = \text{span } \{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \} \text{ and } R(A_0) + \{A_1 + \lambda_1 K_\infty^{(0)}\} V_1(0)^{\perp} = \text{span } \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$$

then condition (12) is satisfied i.e. using (33), β_1 is a solution of the relation,

$$\{\beta_1 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7/3 & 1 \\ 2 & 2/3 \end{bmatrix} \{ \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix} + y_1 \}$$

$$+ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0 \quad , \quad y_1 \in N(A_0) = \text{span } \{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \} \qquad \text{... (42)}$$

Multiplying for the left by {-1,1} to eliminate the terms in y_1 and x_2 yields the value $\beta_1 = -14/27$ i.e. $\alpha_1 = -42/27$, so that the first order asymptote takes the form

$$s(p) = -3p - \frac{42}{27} + \varepsilon_1(p) \qquad \text{... (43)}$$

where $\varepsilon_1 \to 0$ as $p \to \infty$.

2.4 Summary and General Remarks

The analysis of sections 2.1 - 2.3 illustrates how the geometric analysis of the eigenvalue equation (5) in the input-output space leads to an algebraic characterization of both the asymptotic directions and pivots of the system root-locus. Examination of the defining relationships, as illustrated in sections 2.2 - 2.3., indicate that the controller appears in a quasi-linear form and is separated from the direct effect of plant parameters. This property (Owens 1976a) can enable the direct investigation of the effect of controller structure on the root-locus and give insight into compensation techniques without the use of spectral expansion methods. Some preliminary results on compensation methods are discussed in section 3.

Of particular interest are the following observations:

(a) Subject to the conditions of Theorem 1, it should be noted that the strictly proper component H(s) of the inverse system has no effect on the asymptotic directions and pivots of the root-locus nor any effect on the asymptotic behaviour of the eigenvector x(p). In numerical calculations or systems analysis this term can hence be neglected. It will however, have an important effect on the approach to the finite zero of the root-locus and the approach to the asymptotes.

(b) The existence of finite asymptotic directions and pivots of order k-q-1 requires that the constraint (12) be satisfied for $\lambda = o$. Intuitively the constraint indicates that there must be some 'compatability' between the structures of the plant and controller, analogous to the requirement that the G_i (Kouvaritakis and Shaked, 1976) have simple structure. It is apparent that a suitable choice of $K_\infty^{(o)}$ will guarantee compatability of plant and controller. Also, for the inductive algorithm defined by Theorem 1 to continue to the analysis of the $(k-q-2)^{th}$ order infinite zeros, the subspace sum defined by (15) must be direct for $\lambda = o$. This requirement is a function only of plant data and hence represents a constraint on the applicability of the results. Fortunately, it can easily be shown that arbitrarily small perturbations of the plant coefficient matrices $A_o, A_1, \ldots$ ensure that (15) is direct so the results are valid for 'almost all' systems. It should be noted that previous results (Kouvaritakis and Shaked, 1976) suffer from a similar constraint (Owens 1976 b).

(c) An essential difference between the approach adopted in this paper and that adopted by Kouvaritakis and Shaked is that Kouvaritakis and Shaked construct the j^{th} order infinite zeros in the order $j=1,2,3,\ldots$, whereas the results of this paper use the order $j=k, k-1, \ldots$. Thus the results of this paper can be used to evaluate high order infinite zeros when the algorithm of Kouvaritakis and Shaked breaks down (Owens 1976 b), and the results of Kouvaritakis and Shaked can be used to calculate the low order infinite zeros when the analysis of this paper fails. In this sense the two approaches are complementary.

3. Root-loci Structure and Dynamic Compensation

The analysis of sections 2.1 - 2.2. indicate that the two basic control parameters governing the asymptotic directions and pivots are $K_\infty^{(o)}$ and $K_1^{(1)}$. It is the purpose of this section to illustrate how the asymptotes can be systematically manipulated by suitable choice of these parameters and to discuss the physical interpretation of the eigenvector $x(p)$.

3.1 Interpretation of the Eigenvector $x(p)$

Let $Y(s)$ be the mxm matrix whose j^{th} column is the closed-loop response to a unit step demand in channel j i.e.

$$\{ G^{-1}(s) + p K(s) \} Y(s) = \frac{1}{s} p K(s) \qquad \ldots. (44)$$

Assuming, for simplicity that the asymptotic directions are distinct, partial fraction analysis of $Y(s)$ indicates that

$$\{ G^{-1}(s(p_n)) + p_n K(s(p_n)) \} R(s(p_n)) = 0 \qquad \ldots (45)$$

for any unbounded pole sequence $\{s(p_n)\}_{n>o}$ where the mxm matrix $R(s(p_n))$ is the residue in $Y(s)$ of the pole $s(p_n)$. It follows directly that the range of $R(s(p_n))$ lies in the subspace generated by taking linear combinations of the eigenvectors $x(p_n)$ corresponding to $s(p_n)$. Equivalently, the eigenvectors $x(p_n)$ generated by $s(p_n)$ describe the outputs affected by that pole. In particular, the limit eigenvector x_o describes the outputs affected by the asymptote $s(p_n)$ in the limit as $n \to \infty$. In this sense $x(p_n)$, and, in particular, x_o can be regarded as design parameters.

The following theorem describes some properties of the limit eigenvectors which will prove to be useful in illustrating compensation methods.

Theorem 2

Suppose that the closed loop system has infinite zeros of order $k, \ldots, k-\ell+1$ then, if (12) holds and (15) is a direct sum in the relevant cases,

(a) If $V(\lambda_j, \alpha_{ij})$ is the subspace generated by limit eigenvectors corresponding to the j^{th} order asymptote with asymptotic direction λ_j^{-1} and pivot α_{ij} then the subspace sum

$$\sum_{i,j} V(\lambda_j, \alpha_{ij})$$

is a direct sum.

(b) The number of distinct solution λ of (25) is finite number, and for a given solution x_o, both λ and $x_1, \ldots, x_q$ are uniquely defined.

Proof

Let $x_o^{(j)}$, $o \leq j \leq q$, be a non-zero limit eigenvector corresponding to a $(k-j)^{th}$ order asymptote and suppose that

$$x_o^{(q)} = \alpha_1 x_o^{(o)} + \cdots + \alpha_q x_o^{(q-1)} \qquad \ldots. (46)$$

i.e. $A_o x_o^{(q)} = o = \alpha_1 A_o x_o^{(o)} = -\alpha_1 \lambda_k K_\infty^{(o)} x_o^{(o)}$ (by (25)) so that $\alpha_1 = o$. Using induction, suppose that $\alpha_1 = \alpha_2 = \ldots = \alpha_j = o$ so that

$$x_o^{(q)} = \alpha_{j+1} x_o^{(j)} + \cdots + \alpha_q x_o^{(q-1)} \qquad \ldots. (47)$$

It follows from (25) that

$$A_j x_o^{(q)} + A_{j-1} x_1 + \cdots + A_o x_j = 0$$

$$A_{j-1} x_o^{(q)} + \cdots \qquad + A_o x_{j-1} = 0$$

$$\vdots \quad \vdots$$

$$A_o x_o^{(q)} = 0$$

$$x_i \in V_1(0) \oplus \cdots \oplus V_{j+1-i}(0) \quad , 1 \leq i \leq j \qquad \ldots (48)$$

Substituting from (47) and using (25), lead to

$$\alpha_{j+1} \lambda_{k-j} K_\infty^{(0)} x_o^{(j)} + A_{j-1} z_1 + \cdots + A_o z_j = 0$$

$$A_{j-2} z_1 + \cdots + A_o z_{j-1} = 0$$

$$\vdots \quad \vdots$$

$$A_o z_1 = 0$$

$$z_i \in V_1(0) \oplus \cdots \oplus V_{j+1-i}(0) \quad , 1 \leq i \leq j \qquad \ldots (49)$$

Note that $A_o z_1 = o$ implies that $z_1 \in V_2(o) \oplus \ldots \oplus V_j(o)$. Using induction suppose now that there exists an integer $r \geqslant 1$ such that $z_i \in V_{r+2-i}(o)$ $\oplus \ldots \oplus V_{j+1-i}(o)$, $1 \leq i \leq r$ Examine the relation $A_r z_1 + \cdots + A_o z_{r+1} = 0$ and note that, $1 \leq i \leq r$,

$$\begin{aligned} A_{r+1-i} z_i &\in A_{r+1-i} \{ V_{r+2-i}(0) \oplus \cdots \oplus V_{j+1-i}(0) \} \\ &= A_{r+1-i} V_{r+2-i}(0) \\ &\subset A_{r+1-i} \{ \{ V_1(0) \oplus \cdots \oplus V_{r+1-i}(0) \}^{\perp} \} \end{aligned} \qquad \ldots (50)$$

and $A_o z_{r+1} \in R(A_o)$. Applying (15), it follows that $A_{r+1-i} z_i = o, 1 \leqslant i \leqslant r+1$, or, from the properties of the subspace $V_i(o)$, $i \geqslant 1$, we have $z_i \in$ $V_{(r+1)+2-i}(0) \oplus \cdots \oplus V_{j+1-i}(0)$, $1 \leq i \leq r+1$. Finally, taking r=j-1, then

$z_i \in V_{j+1-i}(o) \subset \{V_1(o) \oplus \ldots \oplus V_{j-i}(o)\}^{\perp}$, $1 \leqslant i \leqslant j-1$ and applying (12) to the relation

$$\alpha_{j+1}\lambda_{k-j}K_{\infty}^{(o)}x_o^{(j)} + A_{j-1}z_1 + \ldots + A_o z_j = 0 \qquad \ldots\ldots (51)$$

indicates that $\alpha_{j+1}\lambda_{k-j}K_{\infty}^{(o)}x_o^{(j)} = o$ i.e. $\alpha_{j+1} = o$. From the inductive hypthesis, $\alpha_1 = \alpha_2 = \ldots = \alpha_q = o$ or $x_o^{(q)} = o$, contrary to assumption. It follows directly that $x_o^{(j)}$, $o \leqslant j \leqslant q$ are linearly independent or, equivalently, limit eigenvectors corresponding to different order infinite zeros are linearly independent.

Similar, but tedious, arguments can be applied to show that limit eigenvectors corresponding to distinct asymptotic directions of asymptotes of the same order and also to limit eigenvectors with the same asymptotic directions but different pivots are linearly independent, hence proving (a).

To prove (b) note that (a) implies that the number of distinct solutions λ of (25) is finite due to the finite-dimensionality of the input-output space. The fact that, for a given x_o, both λ and $x_1,\ldots,x_q$ are uniquely defined is proved using an argument similar to that used in (a).

In general (Kouvaritakis and Shaked, 1976) the system will possess m distinct asymptotes and hence the results described in section 2 will generate m linearly independent eigenvectors (Theorem 2). It has been noted that the eigenvectors denote the effect of various asymptotes on the outputs at high gain. Consider the problem of the allocation of non-zero limit eigenvectors $x_o^{(j)}$, $1 \leqslant j \leqslant m$, by choice of a suitable trial controller $K_o(s)$. To do this, choose m distinct non-zero numbers $\lambda^{(j)}$ and choose $x_o^{(j)}$, $1 \leqslant j \leqslant m$, by the following procedure. Examine the eigenvalue problem (25) with q = o i.e.

$$\{A_o + \lambda^{(1)}K_{\infty}^{(o)}\}x_o^{(1)} = 0 \qquad \ldots (52)$$

Choosing $x_o^{(1)} \notin N(A_o)$ then, if $-b_1 \triangleq (\lambda^{(1)})^{-1}A_o x_o^{(1)}$,

$$K_{0\infty}^{(0)} x_0^{(1)} = b_1 \qquad \ldots (53)$$

This procedure can be repeated for $x_0^{(1)}, \ldots, x_0^{(j)}$, where $j = m - \dim N(A_0)$, to generate relations of the form

$$K_{0\infty}^{(0)} x_0^{(i)} = b_i \quad , 1 \leq i \leq j \qquad \ldots (54)$$

and the system will have j k^{th} order infinite zeros with asymptotic directions $(\lambda^{(i)})^{-1}$, $1 \leq i \leq j$. Moving onto the $(k-1)^{th}$ order infinite zeros, choose $x_0^{(j+1)}$ as a solution of the relations (c.f. (25)).

$$\{A_1 + \lambda^{(j+1)} K_{0\infty}^{(0)}\} x_0^{(j+1)} + A_0 x_1 = 0$$

$$A_0 x_0^{(j+1)} = 0$$

$$x_1 \in V_1(0) \qquad \ldots (55)$$

i.e. $x_0^{(j+1)} \in N(A_0)$. Choosing x_1 'at random' (55) reduces to (c.f. (54))

$$K_{0\infty}^{(0)} x_0^{(j+1)} = b_{j+1} \qquad \ldots (56)$$

In general, if $x_0^{(i)}$ is to correspond to a $(k-q_i)^{th}$ order infinite zero, choose $x_0^{(i)}$ as a solution of the relations (c.f.(25))

$$A_{q_i-1} x_0^{(i)} + A_{q_i-2} x_1 + \cdots + A_0 x_{q_i-1}$$

$$\vdots \quad \vdots$$

$$A_0 x_0^{(i)} = 0$$

$$x_r \in V_1(0) \oplus \cdots \oplus V_{q_i+1-r}(0) \quad , 1 \leq r \leq q_i - 1. \qquad \ldots (57)$$

and specify $K_{0\infty}^{(0)}$ by the relation

$$\{A_{q_i} + \lambda^{(i)} K_{0\infty}^{(0)}\} x_0^{(i)} + A_{q_i-1} x_1 + \cdots + A_0 x_{q_i} = 0 \qquad \ldots (58)$$

Specifications of $x_1, \ldots, x_{q_i}$ leads to a relation $K_{0\infty}^{(0)} x_0^{(i)} = b_i$ as in (54) and (56).

Collecting all such relations in matrix form leads to

$$K_{o\infty}^{(o)}\left[x_o^{(1)},\dots,x_o^{(m)}\right] = \left[b_1,\dots,b_m\right] \quad \dots (59)$$

or, as, by construction, $[x_o^{(1)}, \dots, x_o^{(m)}]$ is invertible, a suitable controller is

$$K_{o\infty}^{(o)} = \left[b_1,\dots,b_m\right]\left[x_o^{(1)},\dots,x_o^{(m)}\right]^{-1} \quad \dots (60)$$

It simply remains to check that $K_{o\infty}^{(o)}$ is nonsingular and satisfies (12) and (15).

Finally, note that the possible choice of $x_o^{(i)}$ are constrained. For example, from (52), for $x_o^{(i)}$ to correspond to a k^{th} order asymptote, it is necessary that $x_o^{(i)} \notin N(A_o)$. Similarly, for $x_o^{(i)}$ to correspond to a $(k-1)^{th}$ order asymptote, it is necessary that $x_o^{(i)} \in N(A_o)$. In general, for $x_o^{(i)}$ to correspond to a $(k-q_i)^{th}$ order asymptote ($q_i \geq 1$) it is necessary that $x_o \in W_{q_i}(o)$ (see (57)). This observation has a direct interpretation in terms of constraints in terms of the achievable degree of interaction in the closed-loop system. For example, consider the example of section 2.4. where the only possible choice of $x_o \in N(A_o)$ is $x_o = \{-1, 1\}^T$. Equivalently, at high gain, the effect of the first order infinite zero is constrained to lie in span $\{-1,1\}^T$. As higher order infinite zeros respond more slowly that the first, it follows that the response to a step demand takes the form

$$y(t) \simeq \gamma\{1 - e^{-3pt}\}\begin{bmatrix}-1\\1\end{bmatrix}, \; \gamma = \text{real constant} \quad \dots (61)$$

in the vicinity of t=o+, indicating a large degree of transient interaction. Noting that the second order asymptote (equation (40)) will also cause oscillation at high gain, it is obviously necessary either to use low gains or phase compensate the second order component to speed up its response.

3.2 Manipulation of the Asymptotic Directions

Let $K_o(s) = K_{o\infty}^{(o)}$ be a trial proportional controller with linearly independent limit eigenvectors $x_o^{(i)}$, $1 \leq i \leq m$ corresponding to a cluster point $\hat{\lambda}_{k-q_i}$, $1 \leq i \leq m$. This could be a simple guess or K_o may be chosen to specify the limit eigenvectors as discussed in section 3.1. Suppose that it is required to redesign $K_o(s)$ to produce a controller $K(s) = K_\infty^{(o)}$ with corresponding cluster points λ_{k-q_i}, $1 \leq i \leq m$. Using a parallel analysis to Owens (1976a), choose

$$K_\infty^{(o)} = K_{o\infty}^{(o)} \sum_{i=1}^{m} \frac{\hat{\lambda}_{k-q_i}}{\lambda_{k-q_i}} x_o^{(i)} v_o^{(i)} \qquad \ldots (62)$$

where $v_i^{(i)} x_o^{(j)} = \delta_{ij}$, $1 \leq i,j \leq m$,

then it is easily seen that $\lambda_{k-q_i} K_\infty^{(o)} x_o^{(i)} = \hat{\lambda}_{k-q_i} K_{o\infty}^{(o)} x_o^{(i)}$

so that (equation (28)) λ_{k-q_i} replaces $\hat{\lambda}_{k-q_i}$, $1 \leq i \leq m$, as required. Note that the limit eigenvectors remain unchanged, so that (62) represents just one of an infinity of possible solutions to the problem.

3.3 Manipulation of Pivots by Dynamic Compensation

Suppose that the proportional controller $K(s) = K_\infty^{(o)}$ has been designed to produce the required limit eigenvectors $x_o^{(i)}$, $1 \leq i \leq m$, and asymptotic directions and, by direct calculation, suppose that the corresponding pivots have been determined as $\hat{\alpha}_1^{(1)}, \ldots, \hat{\alpha}_m^{(m)}$. Consider the problem of choosing $K_1(s)$ (equation (4)) to move the pivots to $\alpha^{(1)}, \ldots, \alpha^{(m)}$. A solution to this problem is obtained by choosing $a_i > o$, $1 \leq i \leq m$, and

$$K_1(s) = K_\infty^{(o)} \sum_{i=1}^{m} (k-q_i) \frac{(\alpha^{(i)} - \hat{\alpha}^{(i)})}{s + a_i} x_o^{(i)} v_o^{(i)} \qquad \ldots (63)$$

from which

$$K_{1\infty}^{(1)} = K_\infty^{(o)} \sum_{i=1}^{m} (k-q_i)(\alpha^{(i)} - \hat{\alpha}^{(i)}) x_o^{(i)} v_o^{(i)} \qquad \ldots (64)$$

and

$$\{-\lambda_{k-q_i}(k-q_i)\alpha^{(i)}K_\infty^{(o)} + A_2 + \lambda_{k-q_i} K_{1\infty}^{(1)}\}x_o^{(i)}$$

$$= \{-\lambda_{k-q_i}(k-q_i)\hat{\alpha}^{(i)}K_\infty^{(o)} + A_2\}x_o^{(i)} \quad \ldots (65)$$

Using section 2.3, equations (30), (33), (34) it follows that the pivots of the compensated system are $\alpha^{(1)}$, ..., $\alpha^{(m)}$ as required and also that the limit eigenvectors $x_o^{(i)}$ remain unchanged.

Noting that $\sum_{i=1}^{m} x_o^{(i)} v_o^{(i)} = I_m$

the overall controller take the form

$$K(s) = K_\infty^{(o)} + K_1(s)$$

$$= K_\infty^{(o)} \sum_{i=1}^{m} \frac{(s+c_i)}{(s+a_i)} x_o^{(i)} v_i^{(i)}$$

$$= K_\infty^{(o)} [x_o^{(1)}, \ldots, x_o^{(m)}] \, diag\left\{\frac{(s+c_i)}{(s+a_i)}\right\}_{1\le i\le m} \begin{bmatrix} v_o^{(1)} \\ \vdots \\ v_o^{(m)} \end{bmatrix} \quad \ldots (66)$$

where

$$c_i = a_i + (k-q_i)(\alpha^{(i)} - \hat{\alpha}^{(i)}) \quad , 1 \le i \le m \quad \ldots (67)$$

so that the resulting system can be realized using simple phase compensation networks.

Finally, if integral action is required, set

$$K(s) = \{K_\infty^{(o)} + K_1(s)\}\{I_m + \frac{K_2}{s}\} \quad \ldots (68)$$

when K_2 is a real, constant mxm matrix. Noting that

$$K(s) = K_\infty^{(o)} + \frac{1}{s}\{K_{1\infty}^{(1)} + K_\infty^{(o)} K_2\} + o(s^{-2}) \quad \ldots (69)$$

then $K_{1\infty}^{(1)} + K_\infty^{(o)}K_2$ replaces $K_{1\infty}^{(1)}$ in the calculation of the pivots. It can be shown in a manner similar to (65) that the choice of

$$K_2 = \sum_{i=1}^{m} (k-q_i)\psi_i \, x_o^{(i)} v_o^{(i)} \quad \ldots (70)$$

in conjunction with $K_1(s)$ given by (63) moves the pivots $\hat{\alpha}^{(i)}$ to the positions $\alpha^{(i)} + \psi_i$, $1 \le i \le m$.

3.4 An Intuitive Approach to Feedback Design

The discussion of sections 3.2, 3.3 illustrate the fact that, by suitable choice of proportional control and phase compensation, the asymptotes of the root-loci of the large class of systems considered here can be manipulated at will. Also, it has been demonstrated that the asymptotic eigenvectors of the eigenvalue equation (5) can provide insight into interaction behaviour and interaction problems in the closed-loop system at high gain and that, within certain constraints specified by the plant structure, these eigenvectors can be manipulated as design parameters. These observations suggest the following intuitive approach to design:

STEP ONE: compute $G^{-1}(s)$ and identify the matrices $A_o, A_1, \ldots, A_k$. If $|A_o| \neq 0$ then previous results apply (Owens 1976a). If $|A_o| = 0$, then an examination of $W_j(o)$, $j \geq 1$, will provide some insight into interaction problems in the closed-loop system at high gains, and the technique outlined in section 3.1 can be used to allocate the limit eigenvectors.

STEP TWO: adjust the cluster points λ_{k-q_i} using the technique of section 3.2

STEP THREE: manipulate the pivots using phase compensation networks as described in section 3.3. and choose the overall gain p by examination of the root-locus plot. If the root-locus is unsatisfactory, return to step two.

It is recognized that the analysis of this paper provides little insight into the specific choice of limit eigenvectors, asymptotic directions and pivots unless $|A_o| \neq 0$ (Owens 1976a). This problem can only be resolved by futher analysis or, more probably, by rules of thumb derived after experience with its application.

4. Conclusions

The paper is presented as a generalization of previous work (Owens 1976a) on the analysis of the asymptotic behaviour of the root-loci of linear multivariable systems using the inverse transfer function matrix to systems possessing more than one order of infinite zeros, and represents a parallel and complementary description to that of Kouvaritakis and Shaked (1976) (section 2.4). The formulation and results of this paper can be summarized as follows;

a) The analysis of asymptotic root-loci via the eigen-analysis of $G^{-1} + pK$ leads to geometric relationships in the system input output space in terms of the coefficient matrices $A_o, .., A_k$ of the polynomial part of the inverse system, and controller parameters.

b) The strictly proper component H(s) of the inverse system has no effect on the asymptotes of the root-locus.

c) In order to guarantee the existence of the various orders of infinite zeros, it is necessary to assume a compatability condition (12) between plant and controller structure. Moreover, in order to, continue the inductive algorithm defined by Theorem 1, the plant must satisfy a 'regularity' constraint such that (15) is a direct sum. This illustrates the interesting fact that the existence and structure of asymptotes (as defined by Kouvaritakis and Shaked) are plant induced phenomena.

d) The eigenvector of $G^{-1} + pK$ have a direct interpretation in terms of the residue structure of the closed-loop system and indicate that systems with more than one order of infinite zeros will suffer from severe interactions problems at high gain.

e) The effect of plant and controller on the root-locus is separated and the controller appears on the relations defining the asymptotes in a quasi-linear manner. This formulation simplifies the analysis and choice of controller structure and makes possible the systematic allocation of limit eigenvectors, asymptotic directions and pivots.

Finally, by replacing G by GK and pK by pIm in the analysis of the paper, the asymptotic behaviour of the root-locus can be analysed in terms of the eigenproperties of $(GK)^{-1} + p I_m$. In this form the results may be used to guide and suggest feedback design using the Inverse Nyquist Array (Rosenbrock, 1969).

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