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A NOTE ON FIXED-TIME LINEAR QUADRATIC CONSTRAINED OPTIMIZATION WITH TERMINAL CONSTRAINTS

by

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Abstract

A simple approach to the Minimum Principle for linear differential systems with fixed-time, quadratic state weighting is presented which requires only the application of elementary matrix manipulations and integration by parts. The results takes the form of sufficient conditions for optimality and could play a useful role in the teaching of the essential concepts of optimal control theory to non-mathematicians.

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1. Statement of the Problem

Consider the linear, time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad , \quad x(0) = x_o \qquad \ldots(1)$$

with $x(t) \in R^n$, $u(t) \in R^\ell$ and $A(t)$, $B(t)$ piecewise continuous functions of time on some fixed interval $[0,T]$. The input signal $u(t)$ is subject to the constraints

$$u(t) \in \Omega(t) \subset R^\ell \qquad \forall \quad t \in [0,T] \qquad \ldots(2)$$

and it is required to transfer the initial state $x(0)$ to a target set $T(x_o) \subset R^n$ using a piecewise continuous controller satisfying relation (2), and minimizing the performance criterion

$$J(u) \triangleq \int_o^T \{\tfrac{1}{2} \ <x(t),\ Q(t)x(t)> \ + g(u(t),t)\}dt \qquad \ldots(3)$$

with $Q(t) = Q^T(t)$, $Q(t) \geq 0 \ \forall \, t \in [0,T]$ and $g(u,t)$ suitably continuous on $R^\ell \times [0,T]$.

2. Sufficient Conditions for Optimality

Defining the Hamiltonian function

$$H[x,p,u,t] \triangleq \tfrac{1}{2}<x,Q(t)x>+g(u,t)+<p,A(t)x+B(t)u> \qquad \ldots(4)$$

the following result is obtained below:

Theorem

If $u(t)$, $t \in [0,T]$, is a solution of the TPBVP

$$\dot{x}^*(t) = A(t)x^*(t) + B(t)u^*(t)$$

$$x^*(0) = x_o , \quad x^*(T) \in T(x_o) \qquad \ldots(5)$$

$$\dot{p}(t) = -A^T(t)p(t) - Q(t)x^*(t) \qquad \ldots(6)$$

$$<p(T), z-x^*(T)> = 0 \quad \forall \quad z \in T(x_o) \qquad \ldots(7)$$

$$H[x^*(t),p(t),u^*(t),t] = \min_{u\ \Omega(t)} H[x^*(t),p(t),u,t], \forall t \in [0,T] \qquad \ldots(8)$$

then $u^*(t)$ is an optimal controller.

Proof

Let $u(t)$ be an admissible controller generating a state trajectory $x(t)$ satisfying $x(0) = x_o$, $x(T) \in T(x_o)$. Using integration by parts, it is easily verified using equation (7) that

$$\int_o^T \{<\dot{p}(t),x(t)-x^*(t)> + <p(t),\dot{x}(t)-\dot{x}^*(t)>\}dt$$

$$= [<p(t),x(t)-x^*(t)>]_o^T = 0 \qquad \ldots(9)$$

Substituting for the state and costate derivatives,

$$\int_o^T \{<-A^Tp-Qx^* , x-x^*> + <p, A(x-x^*)+B(u-u^*)>\}dt$$

$$= \int_o^T \{<B^Tp, u-u^*> - <Qx^*, x-x^*> \}dt$$

$$= 0 \qquad \ldots(10)$$

Using the identity

$$<Qx^*,x-x^*> = \tfrac{1}{2}<x,Qx> - \tfrac{1}{2}<x^*,Qx^*> - \tfrac{1}{2}<x-x^*,Q(x-x^*)> \qquad \ldots(11)$$

it follows that

$$\int_0^T \{\tfrac{1}{2}\langle x,Qx\rangle - \tfrac{1}{2}\langle x^*,Qx^*\rangle\}dt$$

$$= \int_0^T \langle B^T p, u-u^*\rangle dt + \tfrac{1}{2}\int_0^T \langle x-x^*, Q(x-x^*)\rangle dt \qquad \ldots(12)$$

or, by adding $\int_0^T \{g(u(t),t)-g(u^*(t),t)\}dt$ to both sides of the equation and using equation (4),

$$J(u)-J(u^*) = \int_0^T \{H[x^*(t),p(t),u(t),t]-H[x^*(t),p(t),u^*(t),t]\}dt$$

$$+ \int_0^T \tfrac{1}{2}\langle x(t)-x^*(t),\ Q(t)(x(t)-x^*(t))\rangle dt \qquad \ldots(13)$$

The optimality of $u^*(t)$ follows directly from equations (8) and (13), noting that $Q(t) \geqslant 0$.

QED.

3. Comments

The above proof should only be regarded as an outline suitable for teaching. Its main purpose is to indicate to the student the natural source of concepts such as costate (equation (6)), transversality conditions (equation (7)) and the natural interpretation of the Hamiltonian (equation (13)) as a first order measure of the effect of control action on the performance criterion. The power of the approach as a teaching tool lies in the simplicity of the proof, the possibility of inclusion of terminal constraints and controller constraints and the observation that the theorem paves the way for a discussion of the LQP problem, minimum energy and minimum fuel problems normally included in UG and PG control courses.