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ASYMPTOTIC STABILITY OF DIFFERENTIAL MULTIPASS PROCESSES

by

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Abstract

The recently developed concept of the asymptotic stability of linear multipass processes is applied to a general form of linear time-invariant model. The analysis indicates that pass initial or boundary conditons have a significant and measurable effect on system stability. The models include differential-delay systems as a special case in which the derived stability criterion reduces to known frequency domain results.

Following the pioneering work of Edwards[1-4] on the modelling and stability analysis of a number of engineering multipass processes and the more abstract concepts introduced by Owens[5], attempts are now being made to identify system structural properties having a significant effect on stability by the detailed analysis of special classes of systems that are amenable to analysis and of a sufficiently general form to have some relevance to applications. In particular, the initial conditions or end boundary conditions are known in certain special cases[5] to have an important effect on stability. The purpose of this note is to strengthen this observation and hence the importance of modelling end effects adequately by consideration of a general class of differential multipass processes of the form

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\, x_{k+1}(t) + B\, x_k(t) \\ x_k(t) &\in C^n \;,\quad k \geq 0 \;,\quad t \in [0,\alpha] \end{aligned} \qquad \ldots(1)$$

with pass-dependent initial conditions

$$x_{k+1}(o) = \tilde{x}(o) + K_o x_k(o) + \sum_{j=1}^{m} K_j x_k(t_j) + \int_o^{\alpha} K(t) x_k(t)\,dt \qquad \ldots(2)$$

where $A,B,K_o\ldots,K_m$ are constant nxn real matrices, $\tilde{x}(o)$ is a real nx1 vector, $K(t)$ is a piecewise-continuous nxn matrix function of t on the pass interval $0\leq t\leq\alpha$ and $0<t_1<\ldots<t_m\leq\alpha$. That is, in contrast to previous studies[5] where a constant known initial pass condition was assumed, we now assume the general case of pass initial conditions generated as a linear combination of a known constant vector plus samples and a weighted average contribution from the previous pass profile.

Regarding the process as an abstract multipass process[5] in the Banach space E_α of continuous functions from the interval $[0,\alpha]$ into C^n with the usual uniform norm, we have the following result:

Theorem

The multipass process defined above is uniformly asymptotically stable[5] if, and only if, all finite solutions of the relation

$$|\lambda I_n - K_o - \sum_{j=1}^{m} K_j e^{(A+\lambda^{-1}B)t_j} - \int_o^\alpha K(t)e^{(A+\lambda^{-1}B)t}dt| = 0 \qquad \ldots(3)$$

lie in the unit circle in the complex plane.

The proof of the result is outlined at the end of the paper.

Note that the asymptotic stability is independent of the constant component $\tilde{x}(o)$ but is critically dependent on any interaction between pass profiles and initial conditions as represented by the constant matrices K_j, $0\leq j\leq m$, and the weighting matrix $K(t)$. This observation is strengthened by consideration of the case of constant end effects[5] when $K_j = 0$, $0\leq j\leq m$, $K(t) \equiv 0$ and the process is stable independent of the matrices A and B. Hence, in general, we conclude that the presence of an instability can be attributed to interaction between pass profiles and end effects. The need to model end effects adequately is now self-evident.

Before continuing to the proof of the theorem, the results can be illustrated by consideration of the differential-delay system

$$\dot{x}(t) = Ax(t) + Bx(t-\alpha),\ t\geq 0\ ,\ x(t-\alpha) \triangleq x_o(t)\ ,\ 0\leq t\leq\alpha \qquad \ldots(4)$$

regarded as the multipass process of equation (1) with $x_k(t) \triangleq x(t+(k-1)\alpha)$,

$k \geq 0$, $0 \leq t \leq \alpha$ and the initial condition rule $x_{k+1}(o) = x_k(\alpha)$ ie $K_o = 0$, $m = 1$, $t_1 = \alpha$, $K_1 = I_n$, $K(t) \equiv 0$. Equation (3) reduces to $|\lambda I_n - \exp\{(A+\lambda^{-1}B)\alpha\}| = 0$ or, taking $\lambda \neq 0$, writing $\lambda = e^s$ and noting that $|\lambda|<1$ if, and only if, $\mathrm{Re}\{s\}<0$, it follows that the differential-delay system is uniformly asymptotically stable if, and only if, all solutions of the 'characteristic equation'

$$|sI_n - A - e^{-s\alpha}B| = 0 \qquad \ldots(5)$$

lie in the open-left-half complex plane. It is interesting and reassuring to note that, in this case, the concepts of asymptotic stability in the normal and multipass senses coincide.

Outline Proof of Theorem: Using the notation of ref (5) and the transition matrix solution of equation (1), the process is characterized by the operator L_α where $y_2 = L_\alpha y_1$ is defined by the relation,

$$y_2(t) = e^{At}y_2(o) + \int_o^t e^{A(t-s)} By_1(s)ds$$

$$y_2(o) = K_o y_1(o) + \sum_{j=1}^{m} K_j y_1(t_j) + \int_o^\alpha K(t)y_1(t)dt \qquad \ldots(6)$$

Consider now the eigenvalue equation $L_\alpha x = \lambda x$ with $\lambda \neq 0$. That is

$$\lambda x(t) = e^{At}\beta + \int_o^t e^{A(t-s)} B\, x(s)ds$$

$$\beta = K_o x(o) + \sum_{j=1}^{m} K_j x(t_j) + \int_o^\alpha K(t)x(t)dt \qquad \ldots(7)$$

written in the equivalent form,

$$\dot{x}(t) = (A+\lambda^{-1}B)x(t) \quad , \qquad \lambda x(o) = \beta \qquad \ldots(8)$$

from which $x(t) = e^{(A+\lambda^{-1}B)t}x(o)$ and the relation $\lambda x(o) = \beta$ yields

$$\{\lambda I_n - K_o - \sum_{j=1}^{m} K_j e^{(A+\lambda^{-1}B)t_j} - \int_o^{\alpha} K(t) e^{(A+\lambda^{-1}B)t} dt\} x(o) = 0 \qquad \ldots(9)$$

Noting that $x \neq 0$ if, and only if, $x(o) \neq 0$ it follows directly that the solutions of equation (3) are the eigenvalues and hence the spectral values of L_α (which is compact[6]). The multipass process is stable if, and only if, the spectral radius of L_α is strictly less than one[5]. Noting that every non-zero point of the spectrum of a compact operator is isolated[6] and that the spectrum is compact, this is equivalent to every eigenvalue lying in the open unit circle of the complex plane. This completes the proof of the result.

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