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SPATIAL KINETICS IN NUCLEAR REACTOR SYSTEMS

by

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Synopsis

Consideration is given to the construction of analytical and numerical models of xenon-induced nuclear reactor spatial kinetics suitable for stability and control studies based on eigenvalue and frequency domain methods.

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1. Introduction

The specific problem studied in this chapter is the modelling of oscillations of the power density in a nuclear reactor, such that a 'hot spot' moves from one region of the reactor into another and back again as illustrated schematically in Fig.5.1. It is assumed that the purpose of the modelling exercize is to investigate the stability of such oscillations both in the open-loop situation and in the presence of feedback or other control action(1-5) regulating the power distribution about a known steady state.

It is not possible in a chapter of this length to give a detailed account of the whole modelling exercize from the basic physical foundations. Rather we take the stance that, as diffusion models are almost universally used by physicists for static and fuel cycling calculations, and that data is hence often available for such models, we should take a diffusion model as a basis for control studies. Such a diffusion model is generally available in the form of a set of coupled, nonlinear partial differential equations in space and time representing the coupled hydraulic, thermodynamic and neutron flux dynamics within the reactor volume. The two obvious difficulties of such models are firstly the nonlinearities and secondly the distributed nature of the system which, if approximated by the use of finite-difference schemes, yields very large numbers of ordinary differential equations if an adequate description of system dynamics is required.

The modelling problem considered in this chapter is the problem of constructing a low-order linear lumped-parameter model of xenon-induced spatial power oscillations in a large, cylindrical nuclear power reactor to replace an (assumed known) nonlinear distributed parameter model. In this context, the idea of 'low-order model' is relative. Typically(6), a 'low-order model' will consist of several independent sets of 40-90 linear algebraic and ordinary differential equations!

2. Linearization of the Diffusion Equation

For our purposes a thermal nuclear power reactor will be regarded as a cylindrical volume of space (Fig. 5.2) containing uranium fuel, moderator, coolant, control elements and structural members required to hold the system together. The primary variable of interest is the space-time dependent system power density $p(r,\theta,z,t)$ modelled by relations of the form

$$p(r,\theta,z,t) \triangleq E_o \int_{E_{min}}^{E_{max}} \Sigma_f(r,\theta,z,E)\phi(r,\theta,z,E,t)\,dE \quad (MW.cm^{-3}) \qquad (5.1)$$

where E_o = average energy produced per fission, $\Sigma_f(r,\theta,z,E)$ is the macroscopic fission cross-section at the position (r,θ,z) for neutrons of energy E and the neutron flux at the point (r,θ,z) due to neutrons in any energy range $E_1 \leq E \leq E_2$ is represented by

$$\phi(r,\theta,z,E_1,E_2,t) \triangleq \int_{E_1}^{E_2} \phi(r,\theta,z,E,t)\,dE \quad (n.cm^{-2}.sec^{-1}) \qquad (5.2)$$

In practice, it is not possible to obtain data on the fission cross-sections over the whole energy range. The power distribution is modelled by supposing that the energy continuum can be approximated by a number of discrete energy bands $E_{min} = E_1 \le E \le E_2$, $E_2 \le E \le E_3, \ldots, E_\ell \le E \le E_{\ell+1} = E_{max}$ and averaging over these energy bands. For example, using the notation

$$\phi_j(r,\theta,z,t) \triangleq \phi(r,\theta,z,E_j,E_{j+1},t) \quad , \quad 1 \le j \le \ell \tag{5.3}$$

then equation (5.1) is approximated by the linear form

$$p(r,z,\theta,t) = E_o \sum_{j=1}^{\ell} \Sigma_j(r,\theta,z)\phi_j(r,\theta,z,t) \tag{5.4}$$

where the $\Sigma_j(r,\theta,z)$ are average fission cross-sections in the range $E_j \le E \le E_{j+1}$ and are estimated experimentally. The space-time behaviour of the ϕ_j are then modelled by the coupled set of 'multi-group' diffusion equations[1-5].

For illustrative purposes consider the one-group case ($\ell = 1$) and the simplified model of the space-time behaviour of $\phi_1(r,\theta,z,t)$

$$\nabla^2\phi(r,\theta,z,t) + B^2(r,\theta,z,t)\phi(r,\theta,z,t) = \frac{\ell^*}{M^2}\frac{\partial\phi}{\partial t}(r,\theta,z,t) \tag{5.5}$$

where we have dropped the subscript on ϕ_1 for notational simplicity. Term by term

(a) the Laplacian term $\nabla^2\phi$ represents diffusion of neutrons throughout the reactor volume

(b) the term $B^2\phi$ is the net neutron production rate per unit volume, and

(c) the term $\frac{\ell^*}{M^2}\frac{\partial\phi}{\partial t}$ represents the rate of change of neutron population.

The coefficient ℓ^* is the mean neutron-lifetime in the reactor core. The parameter M^2 is the position-dependent migration area of neutrons in the reactor core. The coefficient $B^2(r,\theta,z,t)$ is the space-time (and, as we shall see later, power) dependent geometric buckling of the flux at (r,θ,z) and time t. In principle therefore, equation (5.5) is a nonlinear partial differential equation!

The system is also subjected to boundary conditions of the form

$$\phi(R,\theta,z,t) + \ell_R \frac{\partial\phi}{\partial r}(r,\theta,z,t)\Big|_{r=R} \equiv 0$$

$$\phi(r,\theta,0,t) - \ell_z \frac{\partial\phi}{\partial z}(r,\theta,z,t)\Big|_{z=0} \equiv 0$$

$$\phi(r,\theta,H,t) + \ell_z \frac{\partial\phi}{\partial z}(r,\theta,z,t)\Big|_{z=H} \equiv 0 \qquad (5.6)$$

where ℓ_R and ℓ_z are the radial and axial extrapolation distances respectively, and the rotational invariance condition

$$\phi(r,\theta,z,t) \equiv \phi(r,\theta+2\pi,z,t) \qquad (5.7)$$

The boundary conditions (5.6) are only an approximation to the real physical situation and are based on the construction illustrated in Fig. 5.3. for the radial case ie it is assumed that the neutron flux vanishes at a fixed point $R+\ell_R$ and that this point coincides with that obtained by linear extrapolation of the neutron flux from the reactor boundary.

As the objective is to obtain a linear model of the dynamics of small spatial power perturbations about a fixed operating point, the first job is to characterize the reactor steady state. More precisely, if $B_o^2(r,\theta,z)$ and $\phi_o(r,\theta,z)$ are the steady state buckling and flux distributions respectively, equation (5.5) reduces to the search for <u>positive</u> solutions of the partial differential equation

$$\nabla^2\phi_o(r,\theta,z) + B_o^2(r,\theta,z)\phi_o(r,\theta,z) \equiv 0 \qquad (5.8)$$

with boundary conditions easily derived from equation (5.6). Note, in particular, that the steady state is characterized and governed by the geometric buckling and the boundary conditions.

Following normal procedures, write all variables as the sum of steady state values plus a perturbation from the steady state

$$\phi(r,\theta,z,t) = \phi_o(r,\theta,z) + \delta\phi(r,\theta,z,t)$$

$$B^2(r,\theta,z,t) = B_o^2(r,\theta,z) + \delta B^2(r,\theta,z,t) \qquad (5.9)$$

Substituting into equation (5.5), using the steady state condition (5.8) and neglecting the product term $\delta B^2\delta\phi$ yields the following approximate linear model for small transient perturbations about the specified reactor steady state

$$\nabla^2\delta\phi(r,\theta,z,t) + B_o^2(r,\theta,z)\delta\phi(r,\theta,z,t)+\phi_o(r,\theta,z)\delta B^2(r,\theta,z,t)$$

$$= \frac{\ell^*}{M^2}\frac{\partial\delta\phi}{\partial t}(r,\theta,z,t) \qquad (5.10)$$

The reader should easily verify that $\delta\phi$ is subject to the boundary conditions of the form of equation (5.6) with ϕ replaced by the perturbation $\delta\phi$.

Finally, if we restrict our attention to slow, long term transients the term $\frac{\ell^*}{M^2}\frac{\partial\delta\phi}{\partial t}$ can be neglected leaving the model

$$\nabla^2\delta\phi(r,\theta,z,t)+B_o^2(r,\theta,z)\delta\phi(r,\theta,z,t)+\phi_o(r,\theta,z)\delta B^2(r,\theta,z,t) = 0 \qquad (5.11)$$

3. Linearization of the Iodine/Xenon Equations

A major destabilizing influence[4-6] on large thermal reactor systems is the slow dynamic effect of the fission xenon-135 produced by the decay chain shown in Fig. 5.4. The significance of xenon lies in the dual property of being produced by neutron bombardment of uranium-235 at a rate proportional to the neutron flux and its simultaneous decay by β-decay and destruction by absorbtion of neutrons at a rate proportional to the product of the neutron flux and the xenon concentration (cm^{-3}). In effect the iodine/xenon dynamics are a nonlinear inherent feedback loop within the reactor system.

Restricting attention to slow, long term transients (with periods measured in hours) the Te-135 transition can be neglected in the xenon decay chain. The decay chain can then be represented by the two partial differential equations

$$\gamma_I \Sigma_f(r,\theta,z)\phi(r,\theta,z,t) - \lambda_I I(r,\theta,z,t) = \frac{\partial I}{\partial t}(r,\theta,z,t) \tag{5.12}$$

and

$$\gamma_X \Sigma_f(r,\theta,z)\phi(r,\theta,z,t) + \lambda_I I(r,\theta,z,t) - (\lambda_X + \sigma_X\phi(r,\theta,z,t))X(r,\theta,z,t) = \frac{\partial X}{\partial t}(r,\theta,z,t) \tag{5.13}$$

where γ_I and γ_X are the fractional yields of iodine and xenon in fission respectively, λ_I and λ_X are the iodine and xenon nuclear decay constants respectively, σ_X is the microscopic absorbtion cross-section of Xenon-135 for thermal neutrons and $I(r,\theta,z,t)$ and $X(r,\theta,z,t)$ are the space time concentrations (cm^{-3}) of iodine and xenon respectively in the reactor core.

It is easily verified that the steady state iodine and xenon concentrations corresponding to the steady state neutron flux distribution ϕ_o are given by the formulae

$$I_o(r,\theta,z) = \gamma_I \Sigma_f(r,\theta,z)\phi_o(r,\theta,z)/\lambda_I$$

$$X_o(r,\theta,z) = \frac{(\gamma_I + \gamma_X)\Sigma_f(r,\theta,z)\phi_o(r,\theta,z)}{\lambda_X + \sigma_X\phi_o(r,\theta,z)} \tag{5.14}$$

respectively. Writing

$$\begin{aligned} I(r,\theta,z,t) &= I_o(r,\theta,z) + \delta I(r,\theta,z,t) \\ X(r,\theta,z,t) &= X_o(r,\theta,z) + \delta X(r,\theta,z,t) \end{aligned} \tag{5.15}$$

then the linearized versions of (5.12) and (5.13) take the form

$$\gamma_I \Sigma_f(r,\theta,z)\delta\phi(r,\theta,z,t) - \lambda_I \delta I(r,\theta,z,t) = \frac{\partial \delta I(r,\theta,z,t)}{\partial t} \tag{5.16}$$

and

$$\begin{aligned} &\{\gamma_X \Sigma_f(r,\theta,z) - \sigma_X X_o(r,\theta,z)\}\delta\phi(r,\theta,z,t) + \lambda_I \delta I(r,\theta,z,t) \\ &\quad -\{\lambda_X + \sigma_X \phi_o(r,\theta,z)\}\delta X(r,\theta,z,t) = \frac{\partial \delta X}{\partial t}(r,\theta,z,t) \end{aligned} \tag{5.17}$$

As these equations contain no spatial derivatives, there are no spatial boundary conditions on δI and δX.

4. Inherent Feedback and the Homogeneous Model

The final step in the construction of the linear, distributed parameter model of slow, small transient spatial power perturbations about the specified steady state is the characterization of the buckling perturbation δB^2 occurring in equation (5.11). This is a highly complex task as this change is a complex nonlinear function of reactor temperature distributions, coolant dynamics, xenon concentrations and control action. A common approach is to argue that all dynamic effects other than xenon will have reached steady state if we consider slow transients only, suggesting that their effects can be represented by a simple gain ie our model is

$$\delta B^2(r,\theta,z,t) \triangleq K_p \delta\phi(r,\theta,z,t) + K_X \delta X(r,\theta,z,t) + u(r,\theta,z,t) \qquad (5.18)$$

where $u(r,\theta,z,t)$ is a space-time independent control input function representing the effect of control rods or other control devices, K_X is the xenon 'reactivity coefficient' and K_p is the reactor 'power coefficient'. Both K_X and K_p could be position dependent but, for our purposes, it is assumed that they are constant.

The final homogeneous model (obtained by setting the control action $u \equiv 0$) takes the form of equations (5.11), (5.16), (5.17) and (5.18) or the operator-theoretic form

$$L\,\underline{\phi} = \mu \frac{\partial \underline{\phi}}{\partial t} \qquad (5.19)$$

where the space-dependent operator

$$L \triangleq \begin{pmatrix} \nabla^2 + B_o^2 + \phi_o K_p & 0 & \phi_o K_X \\ \gamma_I \Sigma_f & -\lambda_I & 0 \\ \gamma_X \Sigma_f - \sigma_X X_o & \lambda_I & -(\lambda_X + \sigma_X \phi_o) \end{pmatrix} \qquad (5.20)$$

and

$$\mu \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\phi}^T \triangleq (\delta\phi(r,\theta,z,t),\ \delta I(r,\theta,z,t),\ \delta X(r,\theta,z,t))^T \qquad (5.21)$$

together with spatial boundary conditions on $\delta\phi$. The feedback structure of the system is illustrated in Fig. 5.5 and indicates the potentially destabilizing effects of both the xenon and power feedback loops.

5. Xenon-induced Instability: Analytical Methods

Despite its formal simplicity, the linear model is not in a form suitable for stability calculations. It can however be used to suggest useful approximation schemes and general procedures. The investigation of stability is initiated by looking for non-trivial exponential solutions of (5.19) of the form

$$\underline{\phi}(r,\theta,z,t) = \underline{\phi}_\lambda(r,\theta,z)e^{\lambda t} \tag{5.22}$$

and assuming that the linear model is asymptotically stable if and only if all solutions satisfy $\mathrm{Re}\lambda < 0$. Substituting equation (5.22) into (5.19) yields

$$L\underline{\phi}_\lambda = \lambda\mu\underline{\phi}_\lambda \tag{5.23}$$

or, equivalently, the stability problem can be regarded as the evaluation of the dominant generalized eigenvalues of L. This must be undertaken numerically.

The numerical problems arising in the analysis of (5.23) can be very severe unless care is given in the choice of numerical approximations. The use of finite difference approximations to the spatial derivatives occurring in L can be immediately eliminated due to excessive dimensionality. For example, reasonable accuracy would require of the order of 150 mesh points in the (r,θ) plane and 10 in the axial direction which, bearing in mind the fact that there are three variables at each mesh point, leads to a generalized eigenvalue problem of dimension 150x10x3 = 4500! In this section we consider how modal analysis methods can help reduce this severe problem.

It is clear that knowledge of $\underline{\phi}_\lambda$ is equivalent to knowledge of λ. It is also clear from the complex space-dependence of the elements of L that the analytical determination of $\underline{\phi}_\lambda$ is an impossible task. We therefore attempt to find a solution by expanding $\underline{\phi}_\lambda$ as the sum of known functions[5,7,8]

$$\underline{\phi}_\lambda = \sum_{j=1}^{\infty} v_j \psi_j(r,\theta,z) \qquad (v_j \in R^3 \ , \ j \geq 1) \tag{5.24}$$

where, for example, the scalar functions ψ_j are the eigenfunctions of the operator $\nabla^2 + B_o^{\ 2}$ ie

$$\nabla^2 \psi_j + B_o^{\ 2} \psi_j = \gamma_j \psi_j \qquad j \geq 1 \tag{5.25}$$

In particular we can guarantee that $\underline{\phi}_\lambda$ satisfies the spatial boundary conditions if we ensure that each ψ_j satisfies the boundary conditions.

Taking for simplicity the case of $\ell_R = \ell_z = 0$, the identity

$$\begin{aligned}
(\gamma_j - \gamma_k) \int_V \psi_j \psi_k dV &= \int_V \{\psi_j \nabla^2 \psi_k - \psi_k \nabla^2 \psi_j\} dV \\
&= \int_V \text{div}\{\psi_j \nabla \psi_k - \psi_k \nabla \psi_j\} dV \\
&= \int_{\substack{\text{reactor} \\ \text{boundary}}} \{\psi_j \nabla \psi_k - \psi_k \nabla \psi_j\} . \underline{ds} \\
&= 0
\end{aligned} \tag{5.26}$$

indicates that we can assume that the set $\{\psi_j\}_{j\geq 1}$ is orthonormal ie

$$\int_V \psi_j \psi_k \, dV = \delta_{jk} \tag{5.27}$$

where δ_{jk} is the Kronecker delta and V is the volume of the reactor core. The following development will, of course, be valid for any orthonormal choice of the $\{\psi_j\}_{j\geq 1}$. In particular we note that any choice of linearly independent choice of ψ_j can be replaced by an orthonormal set using Gram-Schmidt orthonormalization[11].

Substituting (5.24) into (5.23) yields the relation

$$\sum_{j=1}^{\infty} L\,\psi_j v_j = \lambda\,\mu \sum_{j=1}^{\infty} \psi_j v_j \tag{5.28}$$

Multiplying by ψ_k, integrating over the reactor volume and using (5.27) yields the relations

$$\sum_{j=1}^{\infty} L_{kj} v_j = \lambda\mu\, v_k \quad , \qquad k\geq 1 \tag{5.29}$$

where the 3x3 constant matrices

$$L_{kj} \triangleq \int_V \psi_k L \psi_j \, dV$$

$$= \int_V \psi_k \begin{pmatrix} \gamma_j+\phi_o K_p & 0 & \phi_o K_X \\ \gamma_I \Sigma_f & -\lambda_I & 0 \\ \gamma_\chi \Sigma_f - \sigma_\chi X_o & \lambda_I & -(\lambda_\chi+\sigma_\chi \phi_o) \end{pmatrix} \psi_j \, dV \tag{5.30}$$

are easily evaluated numerically.

The infinite set of equations are still numerically intractable but approximate solutions can be obtained by a variety of means.

Method 1: If the elements of the 3x3 matrix appearing in the integral of equation (5.30) are constant over a large volume of the reactor it is intuitively plausible that

$$L_{jk} \simeq 0 \qquad j \neq k \tag{5.31}$$

when the eigenvalue equation (5.29) has the approximate, but highly simple, form

$$L_{kk}\, v_k = \lambda\mu v_k \quad , \quad k \geq 1 \tag{5.32}$$

Equivalently the eigenvalues are the solutions of relations of the form

$$|L_{kk} - \lambda\mu| = 0 \tag{5.33}$$

which are quadratics in λ that are easily analysed by pencil and paper methods. The approximation of equation (5.31) is fairly severe, however, and the results can only be regarded as giving rough estimates and indicating parametric trends.

Method 2: If the series in the above converge rapidly enough, equation (5.29) can be approximated by a truncated form

$$\sum_{j=1}^{M} L_{kj} v_j = \lambda\mu\, v_k \quad , \quad 1 \leq k \leq M \tag{5.34}$$

or, equivalently, the 3Mx3M generalized eigenvalue problem

$$\begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1M} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ L_{M1} & \cdot\;\cdot\;\cdot & & L_{MM} \end{pmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_M \end{pmatrix} = \lambda \begin{pmatrix} \mu & 0 & \cdots & 0 \\ 0 & \mu & & \\ \cdot & & & \\ \cdot & & & 0 \\ \cdot & & & \\ 0 & \cdots & 0 & \mu \end{pmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_M \end{pmatrix} \tag{5.35}$$

for which known solutions methods exist[9]. For example, it is observed that equation (5.35) is equivalent to the determinental relation

$$\det \left\{ \begin{pmatrix} L_{11} & \cdots & L_{1M} \\ \vdots & & \vdots \\ L_{M1} & \cdots & L_{MM} \end{pmatrix} - \lambda \begin{pmatrix} \mu & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mu \end{pmatrix} \right\} = 0 \qquad (5.36)$$

which, by suitable row and column operations, reduces to the form

$$\det \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}-\lambda I_{2M} \end{pmatrix} \right\} = 0 \qquad (5.37)$$

or, if $|A_{11}| \neq 0$,

$$|\lambda I_{2M} - A_{22} + A_{21}A_{11}^{-1}A_{12}| = 0 \qquad (5.38)$$

which is a 2Mx2M standard eigenvalue problem. In practice good estimates of the stability of the configuration can be achieved using fairly small numbers of ψ_j, typically 20-40. The corresponding eigenvalue calculations are hence of dimension 40-80 which is fairly large, but managable as, in practice[6,10], the matrices are fairly well-conditioned.

6. Xenon-induced Transients: State Space Models

The techniques described in the previous section for obtaining models for use in stability calculations based on eigenfunction (or 'modal') methods is easily extended[5,8] to produce state space models[12] describing the reactor response to control inputs. In general terms, suppose that the control input distribution $u(r,\theta,z,t)$ can be represented as the sum of m independent contributions with a given spatial distribution but manipulable amplitudes ie

$$u(r,\theta,z,t) = \sum_{j=1}^{m} f_j(r,\theta,z)\, u_j(t) \tag{5.39}$$

and let $u(t) = (u_1(t),\ldots,u_m(t))^T$ be the system manipulable input vector. The input-driven model can now be obtained from equations (5.11) and (5.16-18) to be of the form

$$\mu \frac{\partial \underline{\phi}}{\partial t} = L\,\underline{\phi} + Fu \tag{5.40}$$

where

$$F = \begin{bmatrix} \phi_o f_1 & \phi_o f_2 & \ldots & \phi_o f_m \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \tag{5.41}$$

In a similar manner to equation (5.24) suppose that $\underline{\phi}$ can be expanded as a linear combination of known orthonormal functions $\psi_j(r,\theta,z)$, $j\geq 1$, with time dependent amplitudes $v_j(t) \in R^3$ ie

$$\underline{\phi} = \sum_{j=1}^{\infty} \psi_j(r,\theta,z) v_j(t) \tag{5.42}$$

Substituting into equation (5.40), similar techniques to those used in section 5 can be used to replace this model by an infinite set of first order vector ordinary differential equations

$$\mu \frac{dv_k(t)}{dt} = \sum_{j=1}^{\infty} L_{kj} v_j(t) + F_k u(t) \quad , \quad k\geq 1 \tag{5.43}$$

where F_k is the 3xm constant matrix

$$F_k \triangleq \int_V \psi_k\, F\, dV \quad , \quad k\geq 1 \tag{5.44}$$

A finite-dimensional state space model approximating (5.43) is obtained by the truncation technique of method II of section 5 ie replace (5.43) by the truncated set

$$\mu \frac{dv_k(t)}{dt} = \sum_{j=1}^{M} L_{kj} v_j(t) + F_k u(t) \quad , \quad 1 \le k \le M \tag{5.45}$$

which can be written in the (nonstandard) form

$$\mu_e \dot{x}(t) = Ax(t) + Bu(t) \tag{5.46}$$

where

$$x(t) = (v_1^T(t), v_2^T(t), \ldots, v_M^T(t))^T$$

$$A = \begin{pmatrix} L_{11} & \cdot\cdot\cdot & L_{1M} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ L_{M1} & \cdot\cdot\cdot & L_{MM} \end{pmatrix} , \quad B = \begin{pmatrix} F_1 \\ \cdot \\ \cdot \\ \cdot \\ F_M \end{pmatrix} , \quad \mu_e = \begin{pmatrix} \mu & 0 & .. & 0 \\ 0 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ 0 & & 0 & \mu \end{pmatrix} \tag{5.47}$$

If we also suppose that m output measurements of the form, $1 \le k \le m$,

$$y_k(t) = \int_{V_k} p(r,\theta,z,t)\,dV \tag{5.48}$$

equal to the total power generated in a volume V_k of the reactor core, then substituting from equations (5.4) and (5.42) we obtain

$$y_k(t) = \sum_{j=1}^{\infty} C_{kj} v_j(t) \quad , \quad C_{kj} = E_o \int_{V_k} \Sigma_f(r,\theta,z)\psi_j(r,\theta,z)\,dV \; (1\ 0\ 0) \tag{5.49}$$

Truncating after M terms, and defining the output vector $y(t) = (y_1(t), \ldots y_m(t))^T$ we obtain the standard form of output equation

$$y(t) = C\,x(t) \tag{5.50}$$

where

$$C = \begin{pmatrix} C_{11} & \cdot & \cdot & \cdot & C_{1M} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ C_{m1} & \cdot & \cdot & \cdot & C_{mM} \end{pmatrix} \qquad (5.51)$$

The combined state space model of equations (5.46) and (5.50) can be used for simulation purposes, the accuracy increasing as M increases. The model can also be used as the basis of control studies[4-6,8,12] based on time-domain control synthesis procedures. Controller design could also proceed[6] based on multivariable frequency domain design techniques[12] using the reactor transfer function matrix

$$G(s) = C(s\mu_e - A)^{-1}B \qquad (5.52)$$

(obtained by taking Laplace transforms of equations (5.46) and (5.50) with zero initial conditions).

7. The Important Case of Azimuthal Symmetry

The use of eigenfunction/modal methods makes possible a significant reduction in model dimension when compared with finite-difference models but, for good results, requires careful choice of functions ψ_j, $1 \leq j \leq M$. The use of the eigenfunction defined by (5.25) is a good choice but they must, in general, be computed numerically. This is a difficult task in its own right! In practice, therefore, a compromise can be reached by representing certain spatial distributions by modal expansions (when such modal expansions can be easily defined) and other spatial distributions by finite difference approximation schemes[10]. A case of particular interest is described below.

In many practical situations, the reactor steady state is azimuthally symmetric in the sense that the steady state flux and fuel distributions $\phi_o(r,\theta,z)$ and $\Sigma_f(r,\theta,z)$ are independent of θ. This immediately has some implications for the structure of the solutions of equation (5.40). More precisely, remembering that the solutions are periodic in θ (equation (5.7)), expand $\underline{\phi}$ as a Fourier series

$$\underline{\phi}(r,\theta,z,t) = \psi_o(r,z,t) + \sum_{k=1}^{\infty} \{\psi_{jc}(r,z,t)\cos n\theta + \psi_{js}(r,z,t)\sin n\theta\} \tag{5.53}$$

where $\psi_o(r,z,t)$, $\psi_{jc}(r,z,t)$ and $\psi_{js}(r,z,t)$ are unknown functions to be determined. Substituting into equation (5.40) and noting that

$$\frac{\partial^2}{\partial\theta^2}\begin{pmatrix}\cos n\theta \\ \sin n\theta\end{pmatrix} = -n^2\begin{pmatrix}\cos n\theta \\ \sin n\theta\end{pmatrix} \quad , \quad n\geq 0 \tag{5.54}$$

yields the relations (after a little manipulation)

$$\mu \frac{\partial\psi_o}{\partial t} + \sum_{k=1}^{\infty} \{\mu\frac{\partial\psi_{jc}}{\partial t}\cos n\theta + \mu\frac{\partial\psi_{js}}{\partial t}\sin n\theta\}$$

$$= L_o\psi_o + \sum_{k=1}^{\infty} \{\cos n\theta \; L_n\psi_{jc} + \sin n\theta \; L_n\psi_{js}\} + Fu \tag{5.55}$$

where, for $n\geq 0$, the operator L_n is obtained from L by replacing ∇^2 by

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{n^2}{r^2} \tag{5.56}$$

Defining the Fourier coefficients of F

$$F_o = \frac{1}{2\pi}\int_o^{2\pi} F(r,\theta,z)\,d\theta$$

$$F_{nc} = \frac{1}{\pi} \int_{o}^{2\pi} F(r,\theta,z) \cos n\theta d\theta \quad , \quad n \geq 1$$

$$F_{ns} = \frac{1}{\pi} \int_{o}^{2\pi} F(r,\theta,z) \sin n\theta d\theta \quad , \quad n \geq 1 \qquad (5.57)$$

then the linear independence of the trigonometric functions indicates that the model of equation (5.40) can be reduced to the model,

$$\mu \frac{\partial \psi_o}{\partial t} = L_o \psi_o + F_o u$$

$$\mu \frac{\partial \psi_{nc}}{\partial t} = L_n \psi_{nc} + F_{nc} u \quad , \quad n \geq 1$$

$$\mu \frac{\partial \psi_{ns}}{\partial t} = L_n \psi_{ns} + F_{ns} u \quad , \quad n \geq 1 \qquad (5.58)$$

Assuming, for simplicity, zero extrapolation lengths, the spatial boundary conditions on these equations take the form,

$$\psi_o(R,z,t) \equiv \psi_{nc}(R,z,t) \equiv \psi_{ns}(R,z,t) \equiv 0 \quad , \quad n \geq 1$$

$$\psi_o(r,0,t) \equiv \psi_{nc}(r,0,t) \equiv \psi_{ns}(r,0,t) \equiv 0 \quad , \quad n \geq 1$$

$$\psi_o(r,H,t) \equiv \psi_{nc}(r,H,t) \equiv \psi_{ns}(r,H,t) \equiv 0 \quad , \quad n \geq 1 \qquad (5.59)$$

together with continuity and differentiability requirements on the axis (r = 0),

$$\psi_{nc}(0,z,t) \equiv \psi_{ns}(0,z,t) \equiv 0 \quad , \quad n \geq 1$$

$$\left. \frac{\partial \psi_{nc}}{\partial r}(r,z,t) \right|_{r=0} \equiv \left. \frac{\partial \psi_{ns}}{\partial r}(r,z,t) \right|_{r=0} \equiv 0 \quad , \quad n = 2k,\ k \geq 0 \qquad (5.60)$$

The output equation (5.48) takes the vector form,

$$y_k(t) = y_o(t) + \sum_{n=1}^{\infty} \{ y_{nc}(t) + y_{ns}(t) \} \tag{5.61}$$

where

$$(y_o(t))_k \triangleq E_o \int_{V_k} \Sigma_f(r,z)\psi_o(r,z,t)\,dV$$

$$(y_{nc}(t))_k \triangleq E_o \int_{V_k} \Sigma_f(r,z)\cos n\theta\ \psi_{nc}(r,z,t)\,dV \quad , \quad n\geq 1$$

$$(y_{ns}(t))_k \triangleq E_o \int_{V_k} \Sigma_f(r,z)\sin n\theta\ \psi_{ns}(r,z,t)\,dV \quad , \quad n\geq 1 \tag{5.62}$$

represent the contributions to the kth output from the various trigonometric modes.

It is important to note that the decomposition of the model expressed by (5.58) and (5.61) represents a significant potential numerical advantage. For example, in stability studies, it is easily seen that solutions λ of the generalized eigenvalue problem (5.23) are also solutions of generalized eigenvalue problems of the form of

$$L_n \eta_n(r,z) = \lambda \mu \eta_n(r,z) \tag{5.63}$$

for some $n\geq 0$. Conversely, if λ satisfies (5.63), it also satisfies (5.23) with $\underline{\phi}_\lambda = \eta_n \cos n\theta$ or $\underline{\phi}_\lambda = \eta_n \sin n\theta$. Moreover, it can be shown[(6)] that, in rough terms,

$$\lim_{n\to+\infty} \lambda < 0 \tag{5.64}$$

and hence, for stability studies, the eigenvalue equation (5.63) need only be considered in some range $0\leq n\leq M$ (typically 2 or 3).

In a similar manner, it can be shown[6] that the series in (5.53) converges at least as fast as $0\ (1/n^2)$ and hence, without too much loss of accuracy, can be truncated to yield the approximate model

$$\mu \frac{\partial \psi_o}{\partial t} = L_o \psi_o + F_o u$$

$$\mu \frac{\partial \psi_{nc}}{\partial t} = L_n \psi_{nc} + F_{nc} u \quad , \quad 1 \leq n \leq M$$

$$\mu \frac{\partial \psi_{ns}}{\partial t} = L_n \psi_{ns} + F_{ns} u \quad , \quad 1 \leq n \leq M \tag{5.65}$$

with the approximate output model

$$y(t) = y_o(t) + \sum_{n=1}^{M} \{ y_{nc}(t) + y_{ns}(t) \} \tag{5.66}$$

and the relevant boundary conditions (equations (5.59) and (5.60)).

The truncated model also gives some insight into the structure of the system transfer function matrix[12] $G(s)$ relating the output $y(s)$ to the input $u(s)$. Namely, if $G_o(s)$, $G_{nc}(s)$ and $G_{ns}(s)$ are the transfer function matrices relating y_o, y_{nc} and y_{ns} respectively to the input $u(t)$, then a simple calculation yields the identity

$$G(s) = G_o(s) + \sum_{n=1}^{M} \{ G_{nc}(s) + G_{ns}(s) \} \tag{5.67}$$

Lumped-parameter state-space models of (5.65) and (5.66) could be derived by the use of finite-difference approximations to spatial derivatives or by modal techniques analogous to those described earlier in the chapter. The models have the natural 'block diagonal' form,

$$\mu_e \dot{x}_o(t) = A_o x_o(t) + B_o u(t) \quad , \quad y_o(t) = C_o x_o(t)$$

$$\mu_e \dot{x}_{nc}(t) = A_{nc} x_{nc}(t) + B_{nc} u(t) \quad , \quad y_{nc}(t) = C_{nc} x_{nc}(t) \quad , \quad n \geq 1$$

$$\mu_e \dot{x}_{ns}(t) = A_{ns} x_{ns}(t) + B_{ns} u(t) \quad , \quad y_{ns}(t) = C_{ns} x_{ns}(t) \quad , \quad n \geq 1 \tag{5.68}$$

yielding the transfer function matrix model

$$G(s) = C_o(s\mu_e - A_o)^{-1} B_o + \sum_{n=1}^{M} \{ C_{nc}(s\mu_e - A_{nc})^{-1} B_{nc} + C_{ns}(s\mu_e - A_{ns})^{-1} B_{ns} \} \tag{5.69}$$

Remembering that the partial differential equations of (5.65) are defined in the two-dimensional (r,z) rather than the original three-dimensional (r,θ,z) reactor volume, the numerical advantages are apparent. For example, if the (r,z) plane is divided into 150 mesh areas, each block model of (5.68) will have dimension 450 (still large but much better than the dimensions of three-dimensional finite-difference models). The use of modal series expansion approximations will reduce these dimensions considerably more although, in practice, a combination of the techniques is probably most efficient(10).

8. Combined Modal and Finite Difference Models(6,10)

A combined modal and finite-difference scheme for approximating equations (5.65) can be derived based on radial zoning of the (r,z) plane shown in Fig. 5.6. It is motivated by the need to produce accurate lumped-parameter approximations of managable dimension for numerical calculations. It is based on the observation[6,10] that radial dynamics are more important than axial dynamics in reactor stability and control studies and hence that it is possible to use fairly crude approximations to axial spatial dynamics. More precisely, it is assumed that, in radial zone j ($1 \leq j \leq NR$), the vectors of interest can be expressed as a finite linear combination of spatially separable forms. Taking, for illustrative purposes, the case of ψ_o, we write

$$\psi_o(r,z,t) = \sum_{k=1}^{NA} D_{jk}(z)\Phi_{jk}(r,t) \qquad (5.70)$$

in radial zone j. The 3x3 matrices $D_{jk}(z)$ are all diagonal matrices of known axial 'synthesis modes'. Without loss of generality, we can assume that they are orthonormal in the sense that

$$\int_0^H D_{jk}(z)D_{ji}(z)\,dz = \delta_{ki}I_3 \quad , \quad 1 \leq i,\ k \leq NA \qquad (5.71)$$

The terms $\Phi_{jk}(r,t)$, $1 \leq k \leq NA$, are unknown vector functions of radial position and time.

Substituting into (5.65), multiplying by $D_{ji}(z)$ and integrating over the interval $0 \leq z \leq H$ will eliminate the axial dependence from the equations. Repeating this for $1 \leq i \leq NA$ and for each radial zone yields the model

$$\mu \frac{\partial \Phi_{ji}(r,t)}{\partial t} = \sum_{k=1}^{NA} L_o^{jik}\Phi_{jk}(r,t) + F_o^{ji}(r)u(t)$$

$$1 \leq i \leq NA \quad , \quad 1 \leq j \leq NR \qquad (5.72)$$

where

$$L_o^{jik} \triangleq \int_o^H D_{ji}(z) L_o D_{jk}(z)\,dz$$

is a 3x3 spatial operator involving radial derivatives only and radial dependent coefficients, and

$$F_o^{ji}(r) \triangleq \int_o^H D_{ji}(z) F_o \, dz \tag{5.73}$$

Typically, good accuracy in stability and frequency response calculations can be obtained with NA = 1 or 2 so that a significant reduction in the dimension of the representation of axial dynamics is possible compared with direct finite-difference methods.

Defining the spatial average over zone j as

$$\Phi_{ji}(t) \triangleq \frac{\int_{\text{zone } j} \Phi_{ji}(r,t)\, r\,dr}{\int_{\text{zone } j} r\,dr} \tag{5.74}$$

averaging both sides of equation (5.72) in the same way and approximating the radial derivatives by finite difference methods[6,10] in terms of the $\Phi_{ji}(t)$, leads, after much manipulation, to a state space model of dimension N = (3NA)NR. Typically NA = 2 and NR = 15 leading to a model of dimension 90. This is large but managable as the matrices involved tend not to be ill-conditioned. If N = 1, the dimension reduces to 45 which is easily coped with.

Finally, the choice of axial synthesis modes requires careful thought[6,10,13] and should be based on the purposes for which the model was constructed. On the assumption that the model is required for stability and control studies, techniques can be derived based on the use of steady state data, a little guesswork and some involved calculations aimed at minimizing the errors in estimating the xenon feedback term in equation (5.18). The interested reader is referred to the references for more detail.

9. Summary

In many ways the modelling exercize in the analysis of spatial kinetics in nuclear reactor systems is an exercize in physical approximation and numerical reduction of the complex nonlinear partial differential equations describing the space-time behaviour of the reactor power distribution within the core. This is particularly important in the area of stability and control studies using eigenvalue[5,8,9,10], frequency domain[6,10,12,14,15] and optimization methods[4,5,6,8] which, for numerical feasibility, require linear state space models of relatively low dimension. There are many ways[5] of achieving this objective depending upon the accuracy required and the available data. This chapter has outlined, in the context of the modelling of xenon-induced oscillations in thermal reactor systems, how the ideas of modal expansion and finite difference methods can be used together to provide a successful solution to the problem.

10. Acknowledgement

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11. References

(1) Syrett, J.J.: 'Nuclear reactor theory', Nuclear Engineering Monographs, Temple Press, London, 1959.

(2) Hitchcock, A.: 'Nuclear reactor stability', ibid, 1960.

(3) Schultz, M.A.: 'Control of nuclear reactors and power plants', McGraw-Hill, 1955.

(4) Weaver, L.E.: 'Reactor dynamics and control', American Elsevier Publishing Company, New York, 1968.

(5) Stacey, W.M. (Jnr): 'Space-time nuclear reactor kinetics', Nuclear Science and Engineering Monograph 5, Academic Press, 1969.

(6) Owens, D.H.: 'Multivariable control analysis of distributed parameter nuclear reactors', Ph.D. Thesis, Imperial College, 1973.

(7) Kaplan, S.: 'The property of finality and the analysis of problems in reactor space-time kinetics by various modal expansions', Nucl. Sci. Eng., 9, 1961, pp.357-361.

(8) Wiberg, D.M.: 'Optimal control of nuclear reactor systems', Advances in control systems 5, Academic Press, 1967.

(9) Sumner, H.M.: 'ZIP Mk 2: A FORTRAN code for calculating the eigenvalues of large sets of linear equations', AEEW-R 543, 1969, HMSO.

(10) Owens, D.H.: 'XENFER: A FORTRAN code for the calculation of spatial transfer functions of thermal reactors subject to xenon poison effects', AEEW-R 816, 1973, HMSO.

(11) Halmos, P.R.: 'Finite-dimensional vector spaces', Van Nostrand, 1958.

(12) Owens, D.H.: 'Feedback and multivariable systems', Control Engineering Series 7, Peter Peregrinus, 1978.

(13) Owens, D.H.: 'Calculation of nuclear reactor spatial transfer functions using the computer programme XENFER', AEEW-R 817, 1973, HMSO.

(14) Owens, D.H.: 'Multivariable-control-systems design concepts in the failure analysis of a class of nuclear reactor spatial control systems', Proc.IEE, 120(1), 1973, pp.119-125.

(15) Owens, D.H.: 'Dyadic approximation method for multivariable control-systems design with a nuclear reactor application', ibid, 120(7), 1973, pp.801-809.

(16) Bennet, D.J.: 'The elements of nuclear power', Longmans, 1972.

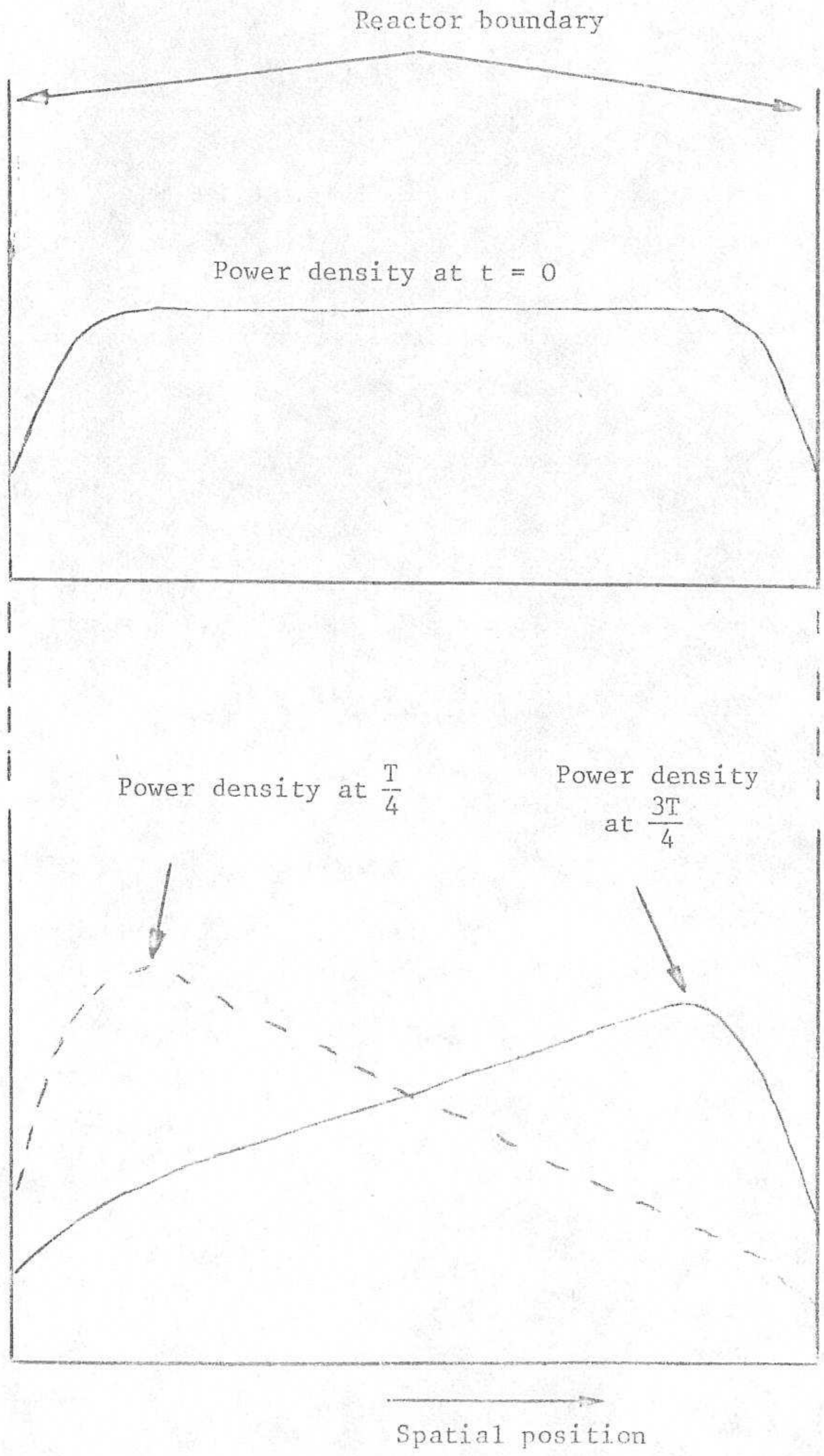

Fig. 5.1. Power density oscillation of period T in a one-dimensional nuclear reactor

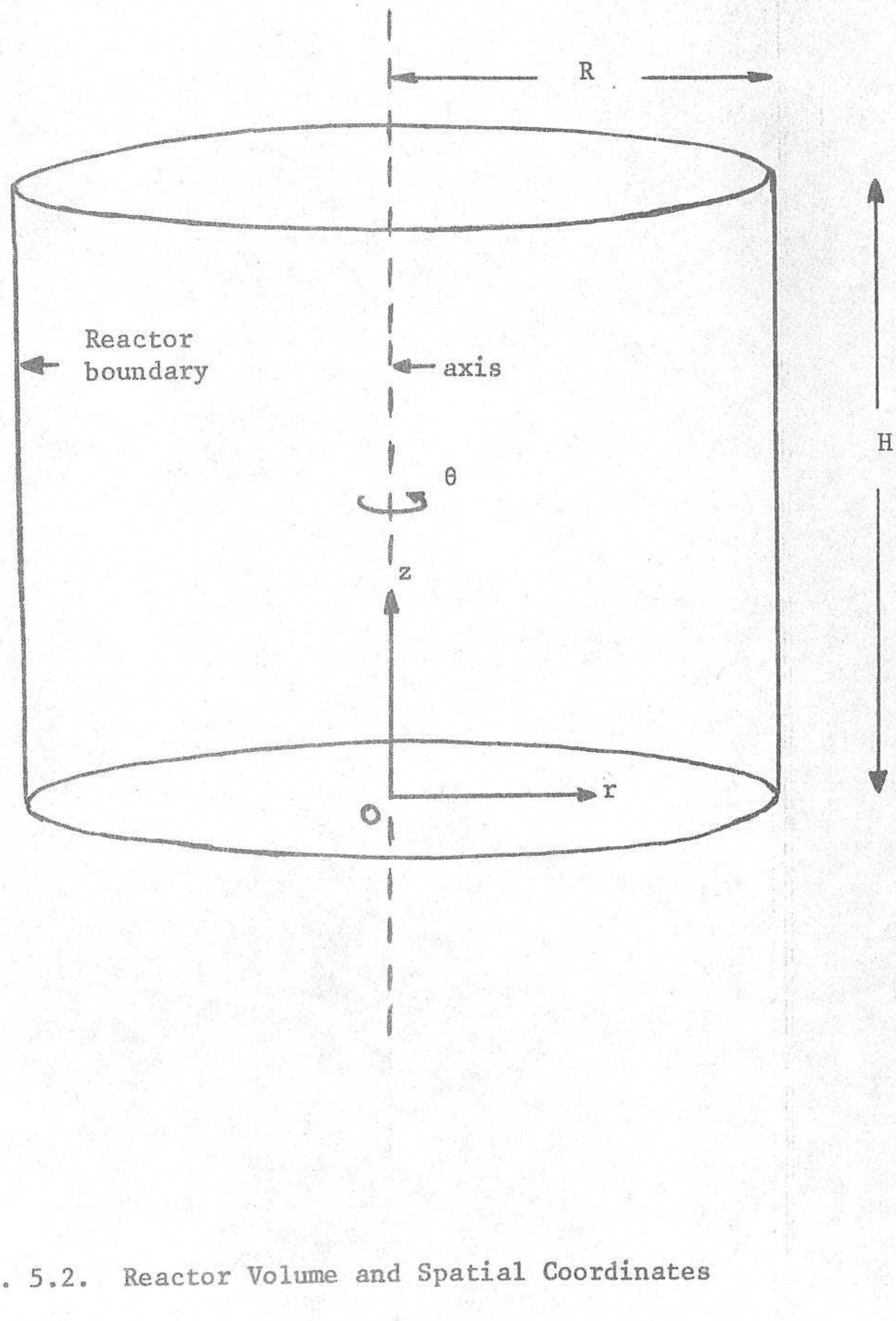

Fig. 5.2. Reactor Volume and Spatial Coordinates

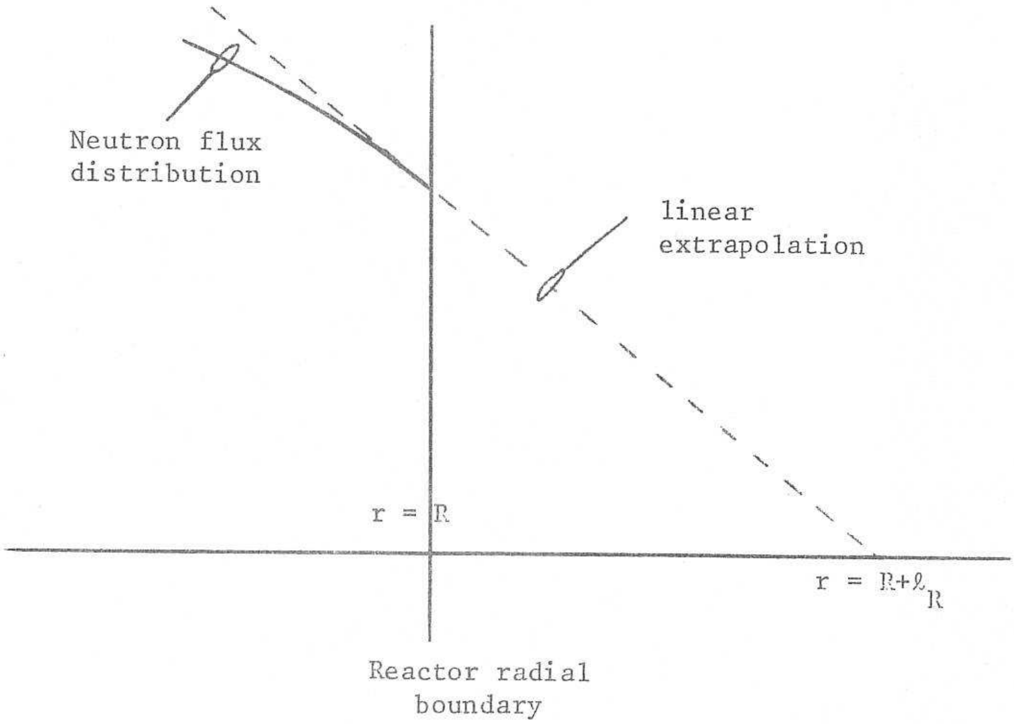

Fig. 5.3. Boundary condition construction

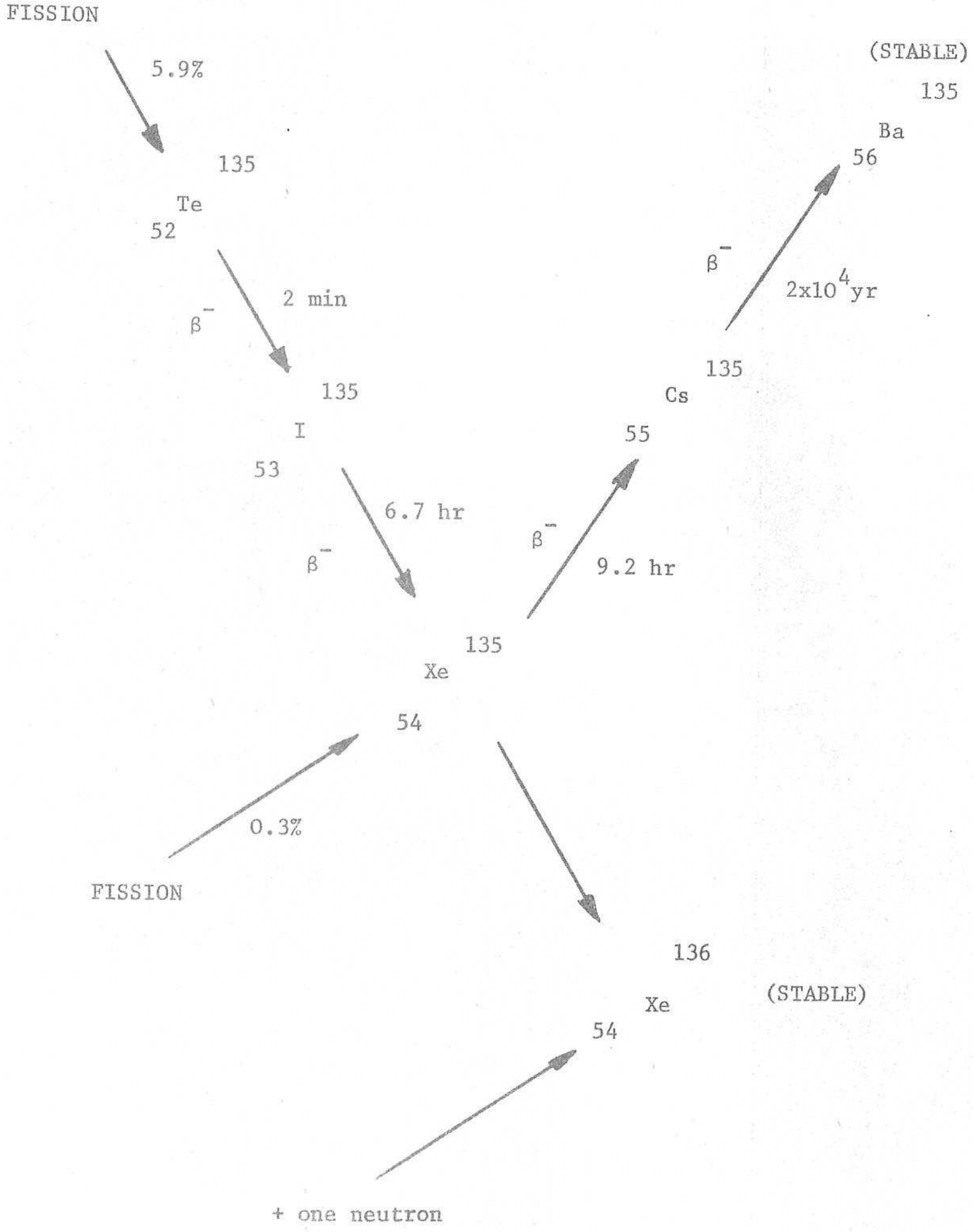

Fig. 5.4. Xenon decay chain

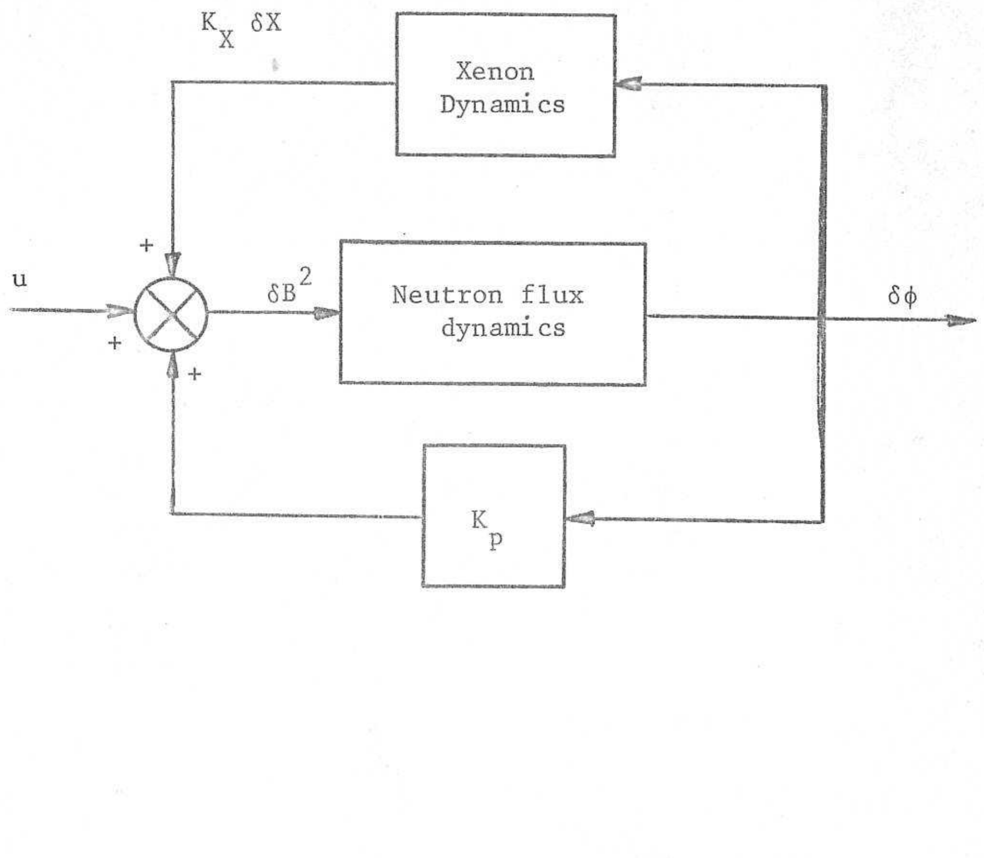

Fig. 5.5. Inherent feedback loops in long-term reactor spatial dynamics

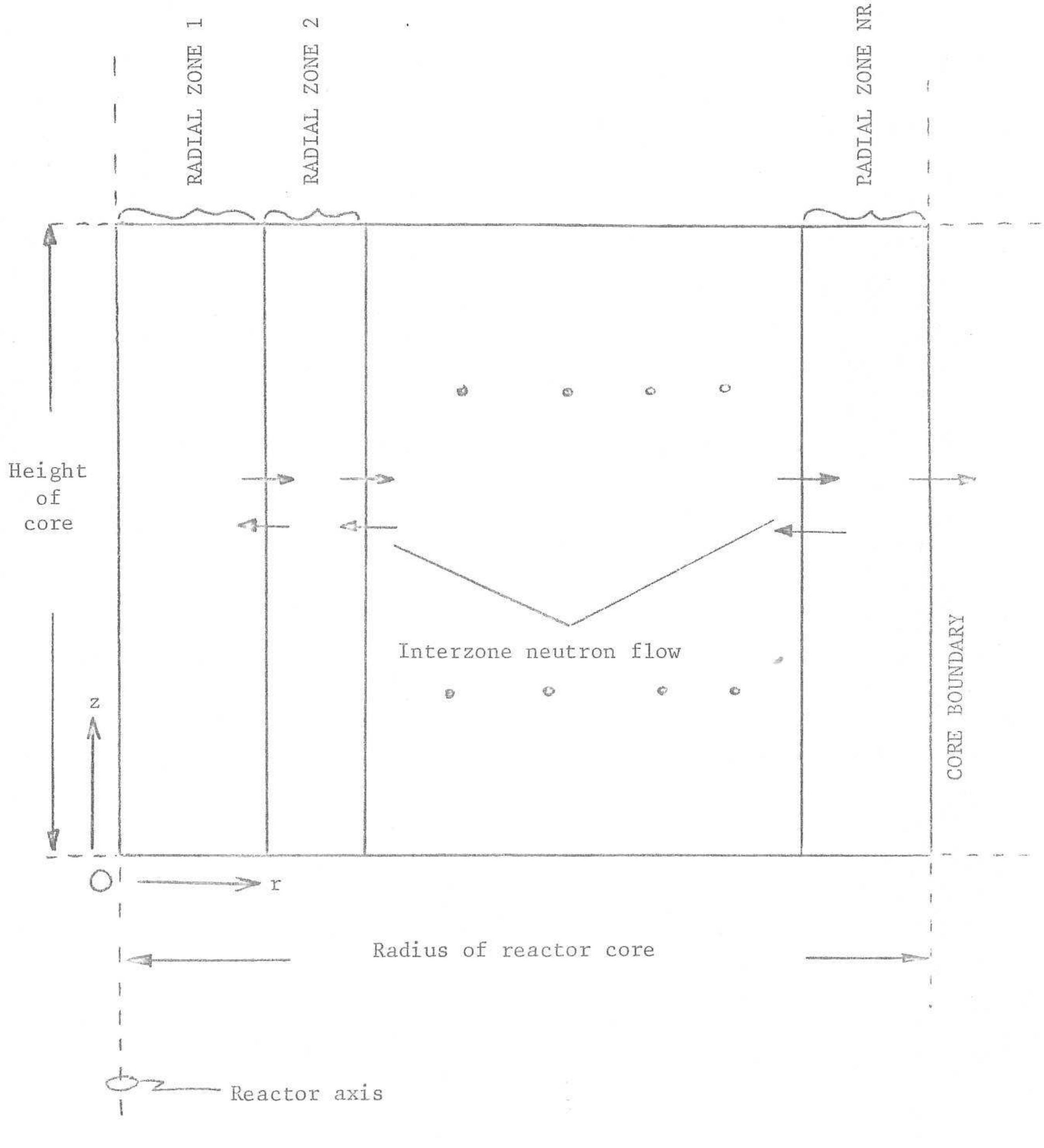

Fig. 5.6. Radial zoning system