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BOUNDS FOR THE ZEROS OF PARTIALLY SYMMETRIC MULTIVARIABLE SYSTEMS

by

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Abstract

The real parts of the invariant zeros of an $m \times \ell$ system S(A,B,C) satisfying a partial symmetry condition can be bounded from above and below by the eigenvalues of the symmetric part of A. Where applicable the results provide a simple computational technique for assessing the minimum phase structure of the system.

The concept of invariant zeros of an $m \times \ell$ $(m \geq \ell)$ left-invertible linear time-invariant system S(A,B,C) of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \qquad \ldots(1)$$

is now well-established[1,2]. More recently the concept of the root-locus of the system subjected to unity negative feedback with constant forward path controller K has been established[1,3-5] and, in particular, it has been noted that the system invariant zeros are a subset of the finite cluster points of the root-locus. Moreover they are independent of the choice of K and, as in classical control, the presence of right-half-plane zeros is a severe limitation on attainable closed-loop transient performance. It follows that the computation of invariant zeros is an important practical problem[6]. It is also clear that, in applications to large-scale or ill-conditioned problems, direct calculation of the invariant zeros may not be feasible. In such cases the availability of easily computed estimates of zero positions could be of great value.

This paper presents the following theorem describing upper and lower bounds on the real part of the invariant zeros in the special case when the system matrices possess a partial symmetry.

Theorem 1

If there exists an ℓxm constant matrix K such that the system S(A,B,KC) is invertible and BKC is positive semi-definite, then the invariant zeros $z_1, z_2, \ldots, z_{n_z}$ of S(A,B,C) satisfy the relation

$$\lambda_1 \leq \text{Re } z_j \leq \lambda_n \quad , \quad 1 \leq j \leq n_z \qquad \ldots(2)$$

where λ_1 and λ_n are the smallest and largest eigenvalues of $(A+A^T)/2$.

Proof

Consider the system S(A,B,C) subjected to unity negative output feedback with constant forward path controller pK, where p is a real scalar gain constant. The closed-loop eigenvalue equation takes the form,

$$\{s(p)I_n - A + pBKC\}x(p) = 0 \qquad \ldots(3)$$

where s(p) and x(p) are eigenvalues and eigenvector of A-pBKC respectively. Without loss of generality we suppose that $x^+(p)x(p) \equiv 1$. Multiplying equation (3) from the left by $x^+(p)$, letting p tend to $+\infty$ and supposing that $s(p) \to z_j$ and $x(p) \to x_\infty$ with $x_\infty^+ x_\infty = 1$ yields the relation

$$z_j - x_\infty^+ A\, x_\infty = -\lim_{p \to +\infty} px^+(p)BKCx(p) \qquad \ldots(4)$$

and hence, by taking real parts, the relation

$$\text{Re}(z_j) - x_\infty^+ \frac{(A+A^T)}{2} x_\infty \leq 0 \qquad \ldots(5)$$

It follows directly that $\text{Re}z_j \leq \lambda_n$. The argument can now be repeated letting p tend to $-\infty$, when the inequality in equation (5) is reversed and it follows directly that $\text{Re}z_j \geq \lambda_1$ which proves the result.

When the conditions of the theorem are valid, equation (2) provides preliminary bounds on the positions of the invariant zeros. For example, if $\lambda_n < 0$, it is clear that all invariant zeros lie in the open left-half complex plane and the system is hence minimum phase. In particular, the bounds are obtained by computation of the largest and smallest eigenvalues of a symmetric matrix. For the case of large scale systems, such calculations can often be performed very accurately even though the direct calculation of the zeros may not be feasible.

Finally, the following result indicates that the conditions required for application of theorem one are equivalent to a form of symmetry in the structures of B and C.

Theorem 2

The conditions required for application of theorem 1 are valid if, and only if, there exists a constant $\ell \times m$ matrix $\tilde{K}$ such that $\tilde{K}C = B^T$.

Proof

If $\tilde{K}C = B^T$ then, choosing $K = \tilde{K}$, it follows that $BKC = BB^T \geq 0$ and $S(A,B,KC)$ is invertible as KCB is nonsingular.

Conversely, if $BKC \geq 0$, it follows that $KC = QB^T$ for some $\ell \times \ell$ matrix Q. To prove this suppose the contrary when there exists vectors $x_1, x_2 \in R^n$ such that $KCx_1 = 0$, $x_1^T B > 0$, $KCx_2 > 0$, $x_2^T B = 0$. It follows that $(x_1+x_2)^T BKC(x_1+x_2) > 0$ and that $(x_1-x_2)^T BKC(x_1-x_2) < 0$ which contradicts the statement that $BKC \geq 0$. The theorem is now proven by noting that the invertibility of $S(A,B,KC)$ requires that rank B = rank $KC = \ell$ and hence that $|Q| \neq 0$ and $B^T = \tilde{K}C$ with $\tilde{K} = Q^{-1}K$.

To illustrate the results, consider the case of the single-input-single-output system

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad C = |1 \quad 1| \qquad \ldots(6)$$

where $B^T = C$. It is easily verified that the system has one zero at the point $z_1 = 0$. Also $(A+A^T)/2 = 0$ so that $\lambda_1 = \lambda_2 = 0$ ie, in this simple case, the upper and lower bounds are identical and equal to the zero in question.

Consider now the multi-input-multi-output system

$$A = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -3 & 1 \\ 0 & 1 & -3 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.4 \\ -0.4 & 0.6 \end{pmatrix} C \qquad \ldots(7)$$

which has one zero at the point $z_1 = -3$. It is easily verified that $(A+A^T)/2$ has eigenvalues $\lambda_1 = \lambda_2 = -4$, $\lambda_3 = -1$ and hence that $\lambda_1 \leq \text{Re}z_1 \leq \lambda_3$ as predicted. Note that the minimum phase structure of the system follows directly from the inequality $\lambda_3 = -1 < 0$ without the need to compute z_1.

References

(1) OWENS, D.H.: 'Feedback and multivariable systems', IEE Control Engineering Monograph 7, Peter Peregrinus Ltd, 1978.

(2) OWENS, D.H.: 'Invariant zeros of linear multivariable systems: A geometric analysis', Int. J. Control, 1977, 26, 4, 537-548.

(3) MACFARLANE, A.G.J., POSTLETHWAITE, I.: 'The generalized Nyquist stability criterion and multivariable root-loci', Int. J. Control, 1977, 25, 837-874.

(4) OWENS, D.H.: 'Dynamic transformations and the calculation of multivariable root-loci', Int. J. Control, 1978, 28, 333-343.

(5) OWENS, D.H.: 'Multivariable root-loci and the inverse transfer function matrix', Int. J. Control, 28, 345-351.

(6) KOUVARITAKIS, B., MACFARLANE, A.G.J.: 'Geometric approach to analysis and synthesis of system zeros', Int. J. Control, 1976, 23, 149-182.