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DYADIC EXPANSIONS AND MULTIVARIABLE

FEEDBACK DESIGN

by

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Dyadic Expansions and Multivariable Feedback Design

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Synopsis

The lecture will review and motivate the use of dyadic expansions in the analysis and design of unity negative feedback control configurations for the control of linear, time-invariant systems denoted by an mxm transfer function matrix G(s). The approach is formulated as a systematic attempt to reduce the structural complexity of the design problem by equivalence transformation of G(s) and hence to replace the problem by one of greater simplicity. The potential of the transformation technique is illustrated by application to the class of systems described by dyadic transfer function matrices. Finally the ideas are combined with the important concept of diagonal dominance to generate a general dyadic decomposition suitable for the systematic manipulation and compensation of system characteristic loci. Some relationships with the concepts of multivariable root-loci are also outlined.

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Background Material

This lecture is concerned with the feedback control analysis of strictly proper, linear time-invariant systems $S(A_1,B_1,C_1)$

$$\dot{x}_1(t) = A_1x_1(t) + B_1u(t) \quad , \quad x_1(t) \in R^{n_1}$$

$$y(t) = C_1x_1(t) \quad , \quad y(t) \in R^m \quad , \quad u(t) \in R^m \tag{1}$$

using the associated mxm transfer function matrix (TFM)

$$G(s) \triangleq C_1(sI_{n_1} - A_1)^{-1}B_1 \tag{2}$$

The forward path control system $S(A_2,B_2,C_2,D_2)$

$$\dot{x}_2(t) = A_2x_2(t) + B_2e(t) \quad , \quad x_2(t) \in R^{n_2} \tag{3}$$

$$u(t) = C_2x_2(t) + D_2e(t)$$

is proper, has mxm TFM

$$K(s) \triangleq C_2(sI_{n_2} - A_2)^{-1}B_2 + D_2 \tag{4}$$

and has 'output' $u(t)$ and 'input' $e(t)$ where

$$e(t) = r(t) - y(t) \in R^m \tag{5}$$

is the error vector and $r(t)$ is the vector of demand signals. This configuration is illustrated schematically in Fig. 1 and is conventionally termed a unity negative feedback system.

The use of the augmented state vector $x(t) = [x_1^T(t), x_2^T(t)]^T$ generates a state vector model of the forward path system $S(A,B,C)$

$$\dot{x}(t) = Ax(t) + Be(t) \quad , \quad x(t) \in R^n$$

$$y(t) = Cx(t) \quad , \quad n = n_1 + n_2 \tag{6}$$

where

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix}$$

$$C = [C_1 \quad , \quad 0] \tag{7}$$

which has TFM

$$Q(s) \triangleq C(sI_n - A)^{-1}B \equiv G(s)K(s) \tag{8}$$

and a characteristic polynomial

$$\rho_o(s) \triangleq |sI_n - A| \equiv |sI_{n_1} - A_1| . |sI_{n_2} - A_2| \tag{9}$$

simply equal to the product of the characteristic polynomials of the plant and forward paths controller. $\rho_o(s)$ is conventionally termed the 'open-loop characteristic polynomial'.

The closed-loop system is obtained by substitution of equation (5) into equation (6) to yield the system S(A-BC,B,C)

$$\dot{x}(t) = (A - BC)x(t) + Br(t)$$
$$y(t) = Cx(t) \tag{10}$$

with 'closed-loop characteristic polynomial'

$$\rho_c(s) \triangleq |sI_n - A + BC| \tag{11}$$

and 'closed-loop TFM'

$$H_c(s) \triangleq C(sI_n - A + BC)^{-1}B \equiv \{I_m + Q(s)\}^{-1} Q(s) \tag{12}$$

In general terms, analysis of $H_c(s)$ (and hence, implicitly, Q(s)) yields information on the transient behaviour of the plant outputs to demand signals whilst analysis of $\rho_c(s)$ yields information on the stability of the closed-loop system . Hence, equations (11) and (12) form the foundation of design theory, although it is more convenient to replace equation (11) by the identity,

$$\frac{\rho_c(s)}{\rho_o(s)} \equiv |I_m + Q(s)| = |T(s)| \tag{13}$$

where $T(s) = I_m + Q(s)$ is the 'matrix return-difference'

The above material can be regarded as the minimum necessary background material required for the understanding of multivariable feedback theory. Further details can be found elsewhere in this course, in the textbooks[1-5] and the forthcoming special issue[6] of the IEE Control and Science Record.

2. Some Basic Concepts

2.1 A General Design Philosophy:

Consider the special case of the diagonal plant,

$$G(s) = \text{diag}\,\{g_1(s), g_2(s), \ldots, g_m(s)\}$$

$$= \begin{bmatrix} g_1(s) & 0 & \cdots & \cdots & 0 \\ 0 & g_2(s) & & & \vdots \\ \vdots & & & & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & g_m(s) \end{bmatrix} \tag{14}$$

consisting of m non-interacting single-input/single-output systems with transfer functions $g_j(s)$, $1 \leqslant j \leqslant m$. It is obvious that each of these systems can be separately controlled! In multivariable notation we set

$$K(s) = \text{diag}\,\{k_1(s), k_2(s), \ldots, k_m(s)\} \tag{15}$$

where $k_j(s), 1 \leqslant j \leqslant m$, are scalar controller transfer functions and note that

$$|T(s)| = \prod_{j=1}^{m} (1+g_j(s)k_j(s)) \tag{16}$$

$$H_c(s) = \text{diag}\,\left\{\frac{g_1(s)k_1(s)}{1+g_1(s)k_1(s)}, \ldots, \frac{g_m(s)k_m(s)}{1+g_m(s)k_m(s)}\right\} \tag{17}$$

so that the closed-loop system is stable if, and only if, the scalar feedback systems

$$\frac{g_j(s)k_j(s)}{1+g_j(s)k_j(s)}, \qquad 1 \leqslant j \leqslant m \tag{18}$$

are stable. Moreover, the closed-loop system is non-interacting and its transient performance in response to unit step demands is 'as good as' the responses of the scalar systems (18).

In multivariable terms, the above problem is trivial but it does indicate that the presence of off-diagonal/interaction terms in the plant are a major source of design problems. These problems are not mathematical in nature but are due to the existence of a human decision maker in the design process and the general observation that the human being finds it impossible to consider more than one loop at a time. Contrast this with the intuitive expectation that interaction effects will, in general, produce complex interdependences between the control loops.

For reasons such as the ones outlined above it is generally true that multivariable design techniques based on frequency response/laplace transform techniques all attempt to reduce or eliminate (in some sense) the need to consider interactions effects in the design process and hence to reduce the design to a sequence of independent scalar design processes.

2.2 Eigenvector Techniques[4-9]

The natural approach to the diagonalization of a square matrix is the use of eigenvalue/eigenvector methods. Consider, for example, the forward path TFM $Q(s)$ to have frequency dependent eigenvector matrix $W(s)$ (with inverse $V(s)$) and corresponding eigenvalues $q_1(s), q_2(s), \ldots, q_m(s)$ i.e.

$$Q(s) = W(s) \text{ diag } \{q_1(s), q_2(s), \ldots, q_m(s)\} V(s) \tag{19}$$

It follows that

$$\frac{\rho_c(s)}{\rho_o(s)} = |I_m + Q(s)| = \prod_{j=1}^{m} (1 + q_j(s)) \tag{20}$$

and

$$H_c(s) = \{I_m + Q(s)\}^{-1} Q(s)$$

$$= W(s) \text{ diag } \left\{\frac{q_1(s)}{1+q_1(s)}, \ldots, \frac{q_m(s)}{1+q_m(s)}\right\} V(s) \tag{21}$$

so that the stability and transient performance of the closed-loop system can be described in terms of the 'characteristic transfer functions' $q_j(s)$, $1 \leqslant j \leqslant m$, subjected to unity negative feedback, and the eigenvector matrix $W(s)$.

The pseudo-classical nature of this analysis is deceptive in its simplicity as it provides few explicit algorithms for the choice of controller $K(s)$. It does however establish the use of the eigenvalues and eigenvectors of $Q(s)$ as fundamental design quantities and suggests that transformation techniques will play important roles in design theory providing design rules and a necessary simplification of the design procedure.

3. Transformation of the Control Problem[4,6]

Although eigenvector methods are vital to analysis and transformation of Q and H_c, they are not the natural tool in the transformation of the plant G for design purposes as both the eigenvalues and eigenvectors are not invariant under such simple operations as interchange of inputs or outputs or even elementary changes in the physical units. There is hence a need to go deeper invoking the following concepts of dyadic expansion or, in other approaches[7], ideas such as the Principle of Alignment.

For conceptual and numerical simplicity, the transformations used here[10]

(a) introduce no extra dynamics into the system.

(b) have a meaningful physical interpretation.

The simplest class of transformations satisfying these requirements is the class of transformations of the form $G(s) \rightarrow H(s)$ defined by

$$G(s) = P_1 \, H(s) \, P_2 \tag{22}$$

where P_1 and P_2 are square nonsingular constant matrices. These transformations certainly introduce no new dynamics and can be interpreted physically as a decomposition of G into three separate factors as shown in Fig. 2. Alternatively, introducing transformed inputs $\hat{u}$ and outputs $\hat{y}$ by the relations (see Fig. 2)

$$\hat{u}(t) = P_2 u(t) \quad , \quad y(t) = P_1 \hat{y}(t) \quad , \tag{23}$$

then the relations $y(s) = G(s)u(s)$ indicate that

$$\hat{y}(s) = H(s)\,\hat{u}(s) \tag{24}$$

so that H is simply the TFM from $\hat{u}$ to $\hat{y}$.

It is a natural step to demand that the transformation pair (P_1, P_2) be real. There are however good theoretical and physical reasons[10] for allowing permissible transformations to have complex elements of a certain form, just as it is necessary to allow complex conjugate pair eigenvectors in spectral analysis of oscillatory linear systems.

Definition[10]

The mxm nonsingular complex matrices

$$P_1 = [\alpha_1, \alpha_2, \ldots, \alpha_m] \quad , \quad P_2 = \begin{pmatrix} \beta_1^T \\ \cdot \\ \cdot \\ \beta_m^T \end{pmatrix} \tag{25}$$

are said to be <u>permissible</u> if the <u>columns</u> (resp. rows) of P_1 (resp. P_2) are real or exist in complex conjugate pairs and if $\overline{\beta}_j = \beta_{\ell(j)}$ whenever $\overline{\alpha}_j = \alpha_{\ell(j)}$, $1 \leqslant j \leqslant m$.

Exercise 1

Show that $P_1 P_2$ is real and nonsingular if (P_1, P_2) is permissible and that $G(\overline{s}) = \overline{G(s)}$ (i.e. G. has elements with real coefficients) if, and only if,

$$\overline{H_{jk}(s)} \equiv H_{\ell(j)\ell(k)}(\overline{s}) \qquad 1 \leqslant j \leqslant m \tag{26}$$

Suppose now that the permissible transformation (P_1, P_2) is chosen such that the design of the forward path controller $\breve{K}(s)$ (see Fig. 3) for H(s) is 'simpler' than the design of K(s) for G(s) (see Fig. 1).

Noting that

$$\frac{\rho_c(s)}{\rho_o(s)} = |I_m + G(s)\ K(s)| = |I_m + P_1H(s)P_2K(s)|$$

$$= |I_m + H(s)\ P_2K(s)\ P_1| \tag{27}$$

it follows that the stability of the two configurations is identical if we choose

$$\tilde{K}(s) \equiv P_2K(s)\ P_1 \tag{28}$$

It is then easily verified that

$$H_c(s) = \{I_m + G(s)K(s)\}^{-1}G(s)K(s)$$

$$\equiv P_1\left(\{I_m + H(s)\tilde{K}(s)\}^{-1} H(s)\tilde{K}(s)\right)P_1^{-1} \tag{29}$$

and hence that the closed-loop TFM's of the two configurations are related by the similarity transformation P_1.

In general terms, the design tecniques described in the following sections replace the design of the configuration shown in Fig. 1 by the (assumed simpler) design of the configuration shown in Fig. 3. The controller K(s) can then be computed from equation (28). The stability of the closed-loop system is then guaranteed and the closed-loop performance assessed from equation (29).

The potential of the transformation can be illustrated by the examples,

$$G(s) = \frac{1}{(s+1)^2}\begin{pmatrix} 1-s & 2-s \\ \frac{1}{3}-s & 1-s \end{pmatrix}$$

$$\equiv \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{(s+1)} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 2/3 & 1 \end{pmatrix} \tag{30}$$

$$G(s) = \frac{1}{s(s+2)}\begin{pmatrix} s+1 & 1 \\ 1 & s+1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{(s+2)} \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \tag{31}$$

$$G(s) = \frac{1}{s(s+2)} \begin{pmatrix} s+1 & -1 \\ 1 & s+1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{s+1+i}{s(s+2)} & 0 \\ 0 & \frac{s+1-i}{s(s+2)} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ \frac{1}{2}i & \frac{1}{2} \end{pmatrix} \tag{32}$$

with the natural identifications of P_1, $H(s)$ and P_2. In each case, $H(s)$ is a diagonal 'quasi-classical' system making the design of $\tilde{K}(s)$ a straightforward classical problem.

Exercise 2

Show that $K(\bar{s}) = \overline{K(s)}$ if, and only if,

$$\tilde{K}_{jk}(\bar{s}) \equiv \overline{\tilde{K}_{\ell(j)\ell(k)}(s)} \qquad 1 \leq j,\ k \leq m$$

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4. Dyadic Transfer Function Matrices: An Important Special Case.

To provide insight into the potential benefits of the above transformations and to suggest design rules, we examine the following special case. Its importance lies in the fact that it is amenable to analysis and the fact that it does lead to a firm and natural basis for generalisation. It does also have direct application in several practical situations.(4,11,12)

Although there are several(10) equivalent definitions and interpretations, we will use the following:

Definition(4,10-12)

An mxm TFM $G(s)$ is said to be dyadic or a dyadic TFM (DTFM) if there exists a permissible pair (P_1,P_2) and non-zero transfer functions

$g_j(s)$, $1 \leqslant j \leqslant m$, such that

$$G(s) = P_1 \text{ diag } \{g_1(s),\ldots,g_m(s)\} P_2 \tag{33}$$

In effect, $G(s)$ is a DTFM if it is possible to choose P_1, P_2 such that

$$H(s) = \text{diag } \{g_1(s),\ldots,g_m(s)\} \tag{34}$$

has a quasi-classical non-interacting diagonal form (see, for example, equations (30)-(32)). It is therefore natural to choose

$$\tilde{K}(s) = \text{diag } \{k_1(s),\ldots,k_m(s)\} \tag{35}$$

and hence (equation (28)) the controller

$$K(s) = P_2^{-1} \text{diag } \{k_1(s),\ldots, k_m(s)\} P_1^{-1} \tag{36}$$

In particular, using equations (27) and (29), we obtain

$$\frac{\rho_c(s)}{\rho_o(s)} = \prod_{j=1}^{m} (1+g_j(s)k_j(s)) \tag{37}$$

$$H_c(s) = P_1 \text{ diag } \left\{\frac{g_1(s)k_1(s)}{1+g_1(s)k_1(s)},\ldots,\frac{g_m(s)k_m(s)}{1+g_m(s)k_m(s)}\right\} P_1^{-1} \tag{38}$$

indicating that the stability and transient performance of the closed-loop system are governed by the stability and performance of the independent scalar feedback systems

$$h_j(s) \triangleq \frac{g_j(s)k_j(s)}{1+g_j(s)k_j(s)} \quad , \quad 1 \leqslant j \leqslant m \tag{39}$$

and, of course, the matrix P_1.

Exercise 3

Use exercise 2 to prove that $K(s)$ has elements with real coefficients if, and only if,

$$k_j(\bar{s}) \equiv \overline{k_{\ell(j)}(s)} \quad , \quad 1 \leqslant j \leqslant m$$

The analysis suggests the following design techniques(4):

STEP 1: Compute[4] P_1, P_2 and the $g_j(s)$, $1 \leq j \leq m$

STEP 2: Choose compensation elements $k_j(s)$, $1 \leq j \leq m$ (see exercise 3) to ensure satisfactory stability and transient performance properties of the subsystems $h_j(s)$, $1 \leq j \leq m$.

STEP 3: Compute $K(s)$ and simulate the resulting closed-loop system. If unsatisfactory return to STEP 2. If satisfactory, stop!

Before continuing to illustrate these ideas with an example it is worthwhile pausing to consider the interaction properties of the closed-loop system $H_c(s)$ in response to unit step demands. It is possible[4] to deduce the following working rules:

(a) Closed-loop interaction willbe small if P_1 is 'nearly diagonal' when (equations (38) and (39) $H_c(s) \simeq \text{diag}\,\{h_1(s), h_2(s), \ldots, h_m(s)\}$.

(b) Closed-loop interaction will be small if the step responses of the scalar feedback systems are 'similar' when (symbolically) $h_j(s) \simeq h_1(s)$, $1 \leq j \leq m$, and (equations (38) and (39) $H_c(s) \simeq h_1(s) I_m$.

The analysis can also be extended[4] to consider the stability of the closed-loop system in the presence of component failures.

5. Level Control of a Two-vessel liquid Storage System

Consider the liquid-level system[4] illustrated schematically in Fig. 4 and the problem of manipulation of inlet flows to regulate the liquid levels and (rather artificially perhaps!) to ensure that the closed-loop system possesses small interaction effects in response to unit step demands in liquid level in either vessel.

Using the data $a_1 = 1$, $a_2 = 2$, $\beta = 2$ it can be deduced[4] that the system TFM takes the form

$$G(s) = \frac{1}{s(s+3)} \begin{pmatrix} s+1 & 1 \\ 1 & \frac{1}{2}(s+2) \end{pmatrix} \tag{40}$$

It can easily be verified that the system is dyadic with the data

$$P_1 = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \quad , \quad P_2 = \begin{pmatrix} {}^1/_3 & {}^1/_3 \\ -{}^1/_3 & {}^1/_6 \end{pmatrix} \tag{41}$$

$$G_1(s) = \frac{1}{s} \qquad g_2(s) = \frac{1}{s+3}$$

Given this decomposition the next step in the design procedure is to design scalar compensation networks $k_1(s)$ and $k_2(s)$ for the scalar systems $g_1(s)$ and $g_2(s)$ respectively. Noting their simple first order lag structure we will suppose that proportional controllers $k_1(s)=k_1, k_2(s)=k_2$ are to be used i.e. the closed-loop scalar feedback systems take the form

$$h_1(s) = \frac{k_1}{s+k_1} \quad , \quad h_2(s) = \frac{k_2}{s+3+k_2} \tag{42}$$

In particular, the closed-loop system is stable if, and only if,

$$k_1 > 0 \quad , \quad 3 + k_2 > 0 \tag{43}$$

Noting that P_1 is certainly not 'approximately diagonal' (in any reasonable sense that is!) we will attempt to reduce closed-loop interaction by ensuring that the step responses of the systems in equation (41) are similar. This can be achieved by equalizing their time constants by setting

$$k \triangleq k_1 = k_2 + 3 \tag{44}$$

to achieve similar response speeds and to choose

$$k >> 0 \tag{45}$$

to ensure similar steady state characteristics. The interested reader can see the undesirable effects of ignoring these considerations in ref.(4).

Substituting back into equations (36) and (38) yields the forward path controller

$$K(s) = \begin{pmatrix} k-2 & 2 \\ 2 & 2k-2 \end{pmatrix} \tag{46}$$

and the closed-loop TFM

$$H_c(s) = \frac{k'}{s+k} = \begin{pmatrix} 1 - \frac{2}{k} & \frac{2}{k} \\ \frac{1}{k} & 1 - \frac{1}{k} \end{pmatrix} \tag{47}$$

Consider the response of the closed-loop system to a unit step demand in level one

$$y(t) = \mathcal{L}^{-1}\, H_c(s)\, \frac{1}{s} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1-e^{-kt}) \begin{pmatrix} 1 - \frac{2}{k} \\ \frac{1}{k} \end{pmatrix} \tag{48}$$

and the response to a unit step demand in level two

$$y(t) = \mathcal{L}^{-1}\, H_c(s)\, \frac{1}{s} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1-e^{-kt}) \begin{pmatrix} \frac{2}{k} \\ 1 - \frac{1}{k} \end{pmatrix} \tag{49}$$

It follows directly that both steady state errors and transient interaction effects are less than 0.1 (say) in magnitude if, and only if, $k \geqslant 20$. The responses to a unit step demand in level two with $k = 20$ is shown in Fig. 5. Overall the design is successful and can be improved by the introduction of integral action into $k_2(s)$ and/or an increase in control system gains.

6. Approximation and Feedback Design[4,10]: An introduction

The example of DTFM's is a particular case[4] satisfying the general design philosophy of section 2.1 (i.e. by suitable choice of controller, the design procedure reduces to a sequence of independent design processes). It is also a particular case of the general transformation procedure of section 3, suggesting that a fruitful theoretical approach may be to consider permissible transformations $G(s) \rightarrow H(s)$ such that, for suitable choice of transformed controller $\tilde{K}(s)$, the off-diagonal terms of $H(s)$ can be neglected in the analysis of the stability of the closed-loop system.

More precisely, we consider conditions under which the dyadic approximation (a DTFM!)

$$G_A(s) \triangleq P_1 \operatorname{diag}\{g_1(s),\ldots,g_m(s)\} P_2^{-1}$$

$$g_j(s) \triangleq H_{jj}(s) \quad , \quad 1 \leqslant j \leqslant m \tag{50}$$

to the dynamics of the plant $G(s)$ controlled by the dyadic controller (see section 4)

$$K(s) = P_2^{-1} \operatorname{diag}\{k_1(s),k_2(s),\ldots,k_m(s)\} P_1^{-1} \tag{51}$$

'induced' by G_A can be used as the basis for assessing the stability of the closed-loop system. There is a related problem using the idea of inverse dyadic approximation[4,10] but this is not considered here.

Intuitively the approximation G_A to G will be adequate if the off-diagonal terms of $H(s)$ are 'small' in some sense. The criterion for smallness used here is simply diagonal dominance[1,2,4,10,13] and the mathematical techniques have a strong relationship to the inverse Nyquist array design method[1,2,4,13,14]

Substituting the controller of equation (51) into the relation (27) yields the identity

$$\frac{\rho_c(s)}{\rho_o(s)} = |I_m + H(s) \operatorname{diag}\{k_1(s),k_2(s),\ldots,k_m(s)\}| \tag{52}$$

and, for example, application of the stability theorems of the direct Nyquist array yields the result

Theorem 1 (A stability theorem)

Let D be the usual Nyquist contour in the complex plane and let n_c (resp. n_o) be the number of closed (resp. open)-loop poles in the interior of D. Suppose that $I_m + H(s)\ \mathrm{diag}\ \{k_1(s),\ldots,k_m(s)\}$ is diagonally row or column dominant at each point s on D. Let the j^{th} diagonal term of $H(s)\ \mathrm{diag}\ \{k_1(s),\ldots,k_m(s)\}$ map D into the closed-contour Γ_j encircling the (-1,0) point of the complex plane n_j times in a clockwise manner, $1 \leqslant j \leqslant m$. Then

$$n_c - n_o = \sum_{j=1}^{m} n_j \tag{53}$$

Noting that the diagonal terms of $H(s)\ \mathrm{diag}\ \{k_j(s)\}_{1\leqslant j\leqslant m}$ are simply equal to $g_j(s)\ k_j(s)$, $1 \leqslant j \leqslant m$, then the theorem can be interpreted on stating that the dyadic approximation can be used for analysis of the stability of the closed-loop system provided that their Nyquist loci with superimposed Gershgorin circles do not contain the (-1,0) point of the complex plane. Moreover, if the Gershgorin circles are of 'small enough' radius it is expected intuitively that the DTFM

$$\{I_m + G_A(s)K(s)\}^{-1}G_A(s)K(s) \equiv P_1\ \mathrm{diag}\ \left\{\frac{g_j(s)k_j(s)}{1+g_j(s)k_j(s)}\right\}_{1\leqslant j\leqslant m} \tag{54}$$

obtained by replacing G(s) by $G_A(s)$ in $H_c(s)$ will be an adequate approximation, for design purposes, to the closed-loop dynamics.

To illustrate these ideas, consider the controllable and observable stable system

$$G(s) = \frac{1}{(s+1)(s+2)}\begin{pmatrix} 2s+6 & s+3 \\ 12s+20 & 9s+15 \end{pmatrix} \tag{55}$$

which, choosing the permissible transformation

$$P_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad , \quad P_2 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} , \qquad (56)$$

yeilds a transformed system of very simple and highly symmetric structure,

$$H(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{2}{s+1} \end{pmatrix} \qquad (57)$$

with $g_1(s) \equiv g_2(s) \equiv 2/(s+1)$. The Nyquist array of H(s) in the frequency interval $o \leq \omega < +\infty$ can be illustrated (due to the symmetry) by the single frequency response, together with Gershgorin circles, shown in Fig. 6. Choosing, for simplicity, the case of proportional controllers $k_1(s) \equiv k_1$ and $k_2(s) \equiv k_2$ and, due to the symmetry of H(s), $k_1 = k_2 = k$, it is easily seen that $I_2 + H(s)k$ is diagonally dominant on the whole of the D contour for all choices of $k > 0$. In this case, we have $n_1 = n_2 = 0$ and hence, noting that $n_o = 0$, theorem 1 indicates that the closed-loop system is stable for all choices of controller of the form

$$K(s) = k\, P_2^{-1} P_1^{-1} = \frac{k}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} , \quad k \geqslant 0 \qquad (58)$$

Exercise 4

Verify that the plant of equation (55) has the decomposition,

$$G(s) = \begin{pmatrix} \frac{(s+3)}{(s+1)(s+2)} & 0 \\ 0 & \frac{3s+5}{(s+1)(s+2)} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

and hence is a DTFM. Following the approach of section 4, design unity feedback system for this plant and show that the resulting closed-loop system is non-interacting.

---

7. Dyadic Approximation and Characteristic Loci[4,10,15,16]

The ideas outlined in section 6 suggest that an intuitively reasonable objective in the choice of permissible P_1 and P_2 is the diagonal dominance of H(s) on the Nyquist D contour. The transformation could be guessed on physical grounds,[4,11,12] on a trial and error basis as in the inverse Nyquist array technique or deduced from general theoretical considerations[4,10,15,16] as described below.

Theorem 2[4,10,15,16]

Let ω_1 be a real frequency such that $G(i\omega_1)$ is finite and non-singular. Then there exists a permissible transformation $(P_1(\omega_1)\ ,\ P_2(\omega_1))$ such that

$$H(s,\omega_1) \overset{\Delta}{=} P_1^{-1}(\omega_1)\ G(s)\ P_2^{-1}(\omega_1) \tag{59}$$

is diagonal at the point $s = i\omega_1$ if, and only if, the matrix

$$M(\omega_1) \overset{\Delta}{=} G(-i\omega_1)\ G^{-1}(i\omega_1) \tag{60}$$

has a complete set of eigenvectors. $P_1(\omega_1)$ (resp. $P_2^{-1}(\omega_1)$) is then an eigenvector matrix of $M(\omega_1)$ (resp. $N(\omega_1) \overset{\Delta}{=} G^{-1}(i\omega_1)\ G(-i\omega_1)$).

In essence the result indicates that, under mild (generic!) conditions, it is possible to choose a permissible transformation such that H is diagonal at the specified frequency point and hence (by continuity) diagonally dominant in the vicinity of that point. The dyadic approximation is hence exact at the specified point $s = i\omega_1$.

Although theorem 2 could be regarded simply as a basis for computing candidates for P_1 and P_2 in the context of section 6, it has important applications to the systematic manipulation and compensation of system characteristic loci[4,7,8,9,10,16].

The characteristic locus method regards the design objective, in particular, as the gain and phase compensation of the characteristic loci of the plant $G(s)$ by the choice of controller $K(s)$. Unfortunately, as there is no general formula giving the eigenvalues of the product of two matrices as a function of the eigenvalues of the individual matrices, the systematic design of $K(s)$ to produce required loci compensation is a major practical problem. This problem can be partially overcome by the use of ideas such as 'approximately commutative control'[4,7] or by the use of theorem 2[4,10,15,16] qs outlined below.

Consider the basic problem of the choice of $K(s)$ to produce desired gain and phase characteristics of the loci in the vicinity of the specified frequency point $s = i\omega_1$. Suppose also that the conditions of theorem 2 are satisfied and that the permissible transformation $(P_1(\omega_1), P_2(\omega_1))$ and $H(s,\omega_1)$ have been computed. Consider the use of the dyadic controller,

$$K(s,\omega_1) \triangleq P_2^{-1}(\omega_1)\ \mathrm{diag}\{k_1(s,\omega_1),\ldots,k_m(s,\omega_1)\}\ P_1^{-1}(\omega_1) \qquad (61)$$

yielding

$$G(s)K(s,\omega_1) \equiv P_1(\omega_1)\ H(s,\omega_1)\mathrm{diag}\ \{k_1(s,\omega_1),\ldots,k_m(s,\omega_1)\}\ P_1^{-1}(\omega_1) \qquad (62)$$

Applying Gershgorin's theorem to H diag $\{k_j\}_{1 \leq j \leq m}$, it follows that the <u>natural rational approximations</u>, $1 \leq j \leq m$, to the characteristic transfer functions

$$q_j(s) \simeq H_{jj}(s,\omega_1)k_j(s,\omega_1) \qquad (63)$$

(obtained by neglecting the off-diagonal terms) are exact at $s = i\omega_1$ and in error at other points to an extent defined by the Gershgorin circles. These ideas are represented schematically in Fig. 7. Note that[4,16] the relative magnitudes of the circles are independent of the compensation elements and are also small in the vicinity of $s = i\omega_1$. It follows that equation (63) can be used with confidence to design the required gain and phase characteristics of the characteristic loci in an open frequency interval containing $s = i\omega_1$.

The above methodology can be used to formulate a systematic design technique[4,16] with guaranteed and quantifiable accuracy in the analysis and design of compensation elements. In essence the approach has the structure:

STEP ONE: Choose an intermediate to high frequency ω_h at which gain and phase compensation is required, and apply the above procedure to design $K(s,\omega_h)$ to produce the required gain and phase compensation in the vicinity of $s = i\omega_h$. Check the success of the procedure by examination of the characteristic loci. If the system is stable and its steady state and transient characteristics are as required, stop! Otherwise, continue to step two.

STEP TWO: Choose a low frequency ω_ℓ and design $K(s,\omega_\ell)$ for the composite system $G(s)\ K(s,\omega_h)$ to produce the desired gain and phase compensation of the characteristic loci in the vicinity of $s = i\omega_\ell$. Check the exact characteristic loci.

STEP THREE: Set

$$K(s) = K(s,\omega_h)(I_m + K(s,\omega_\ell)) \tag{64}$$

when the desired compensation of the loci at both high and low frequencies will be essentially retained if

$$\lim_{|s|\to\infty} K(s,\omega_\ell) = 0 \tag{65}$$

e.g. if $K(s,\omega_\ell) = s^{-1}K(\omega_\ell)$ is a pure integrator.

For examples of the application of the technique see refs. (4), (5) and (16).

8. Summary and Comments

The concept of the dyadic representation of an mxm TFM $G(s)$ has its origins in the design of nuclear reactor spatial control systems[11,12] where the dyadic structure is a reflection of the modal structure of the reactor equations. There are many ways of generalizing these ideas[4,10] but perhaps the most succinct is obtained by introducing the idea of permissible transformation[10] of the control problem into an (assumed simpler) control problem with identical stability characteristics and closed-loop responses related by similarity transformations. The idea of a dyadic TFM then arises naturally as a special case possessing the property that suitable choice of forward path controller reduces the design procedure to m independent classical designs. The structure also arises naturally in applications[4,11,12].

A combination of these ideas with the concept of diagonal dominance[4,10] extends the scope of techniques to the larger class of systems that can be transformed into systems of 'almost diagonal' form. The techniques then have a similar structure to the direct or inverse Nyquist array methods.

The ideas can be extended[4,10,15,16] to cope with quite general TFM's by the use of permissible transformations and the construction of rational approximations to the characteristic transfer functions of GK. These approximations are exact at a chosen frequency of interest and in error at other frequencies to an extent defined by Gershgorin's theorem.

The analysis illustrates that dyadic representations of multivariable systems are fundamental. In particular, they can be used to motivate the definition and analysis of simple multivariable structures[4,17,18] possessing structural similarity to certain classical transfer functions. They also play a role in other design techniques[7]. More recently the idea of the root-locus of a multivariable feedback system has been introduced[4,8,18-25] and a theory of design and compensation is emerging[23-25]. Central to this design theory is the idea of dyadic TFM's in the sense that the required compensation of the root-locus can always be achieved using a dyadic controller TFM.

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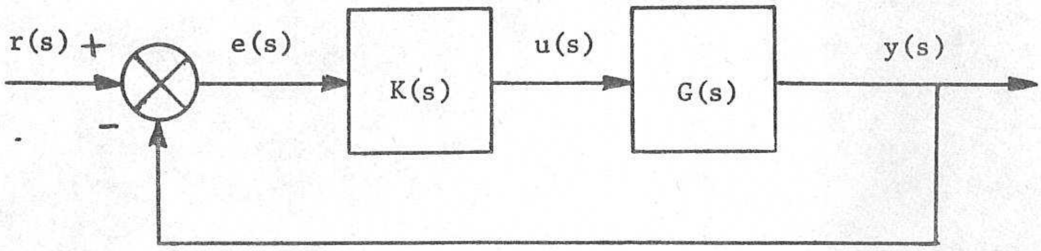


Fig. 1. Unity Negative Feedback System

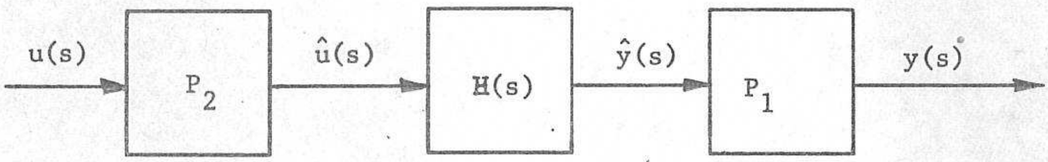


Fig. 2 Dyadic Decomposition of the Plant G(s)

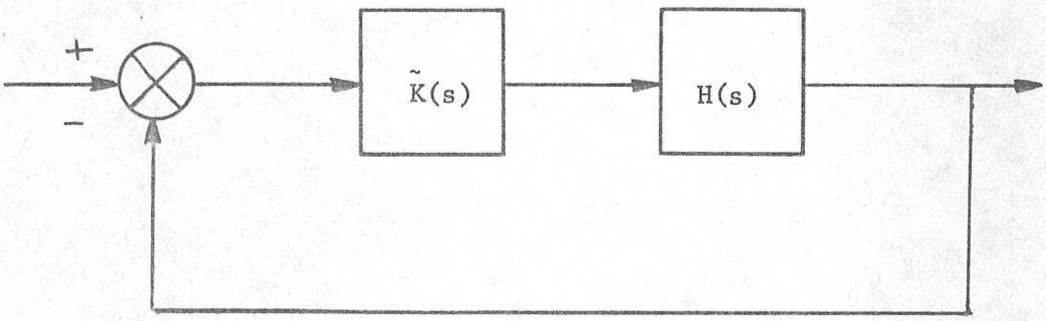
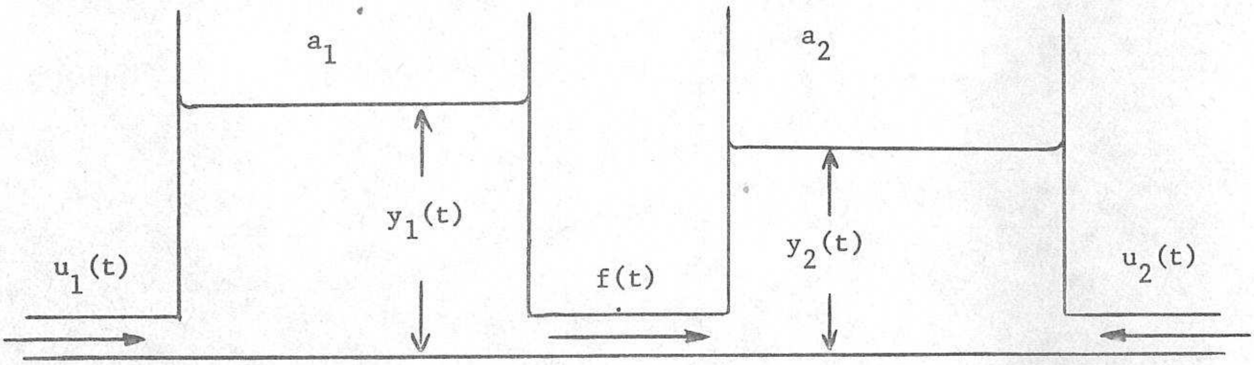


Fig. 3 The transformed control problem



a_i = area of vessel i

y_i = level in vessel i

u_i = input flow into vessel i

f = intervessel flow = $\beta(y_1 - y_2)$

Fig. 4 A liquid-level system

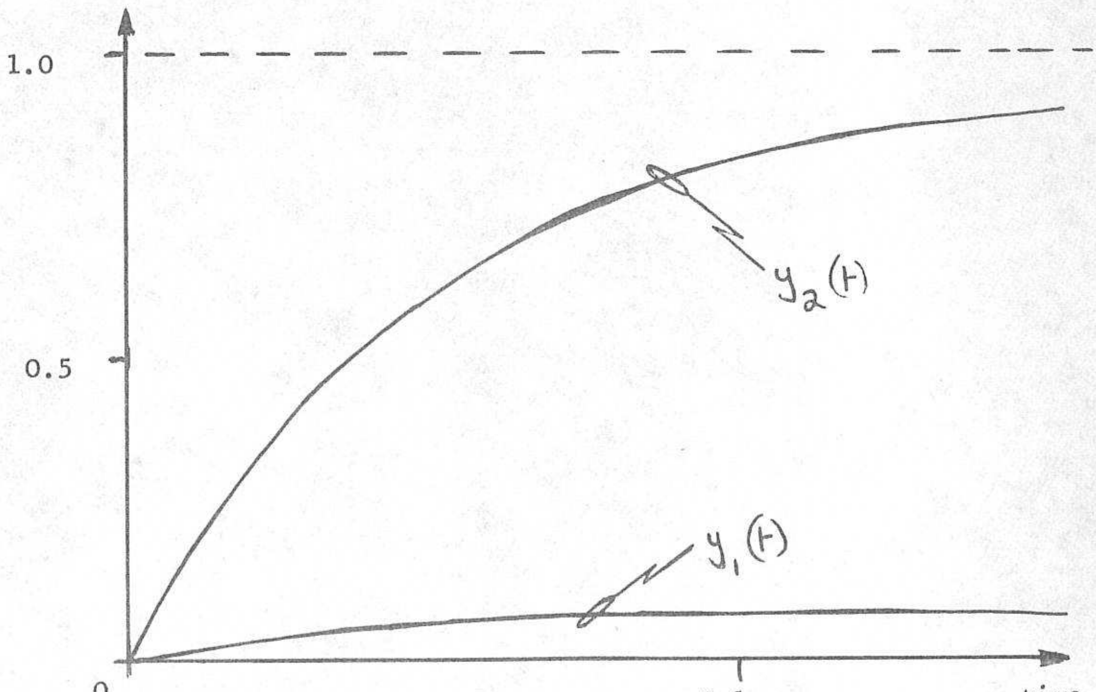


Fig. 5 Level responses to a unit demand in $y_2(t)$

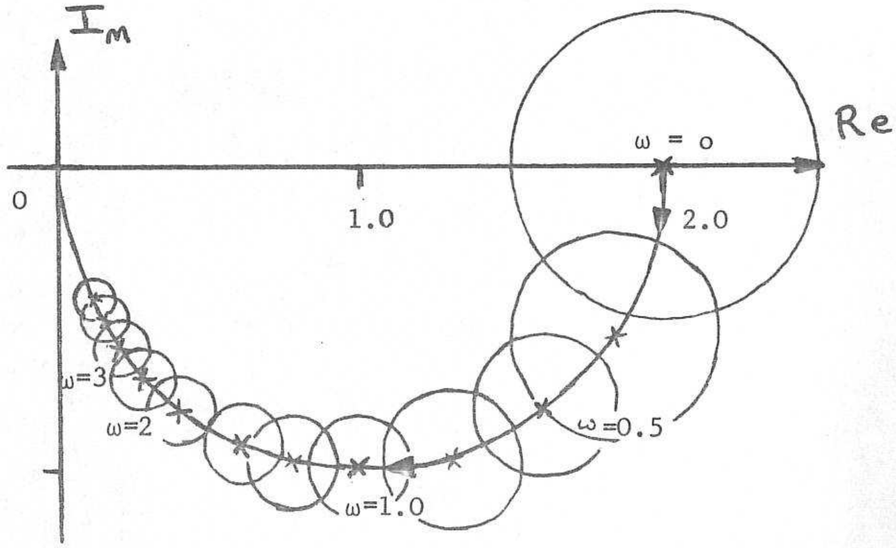


Fig. 6 Nyquist plot of $H_{11}(s)$, $o \leqslant \omega + \infty$, plus Gershgorin Circles

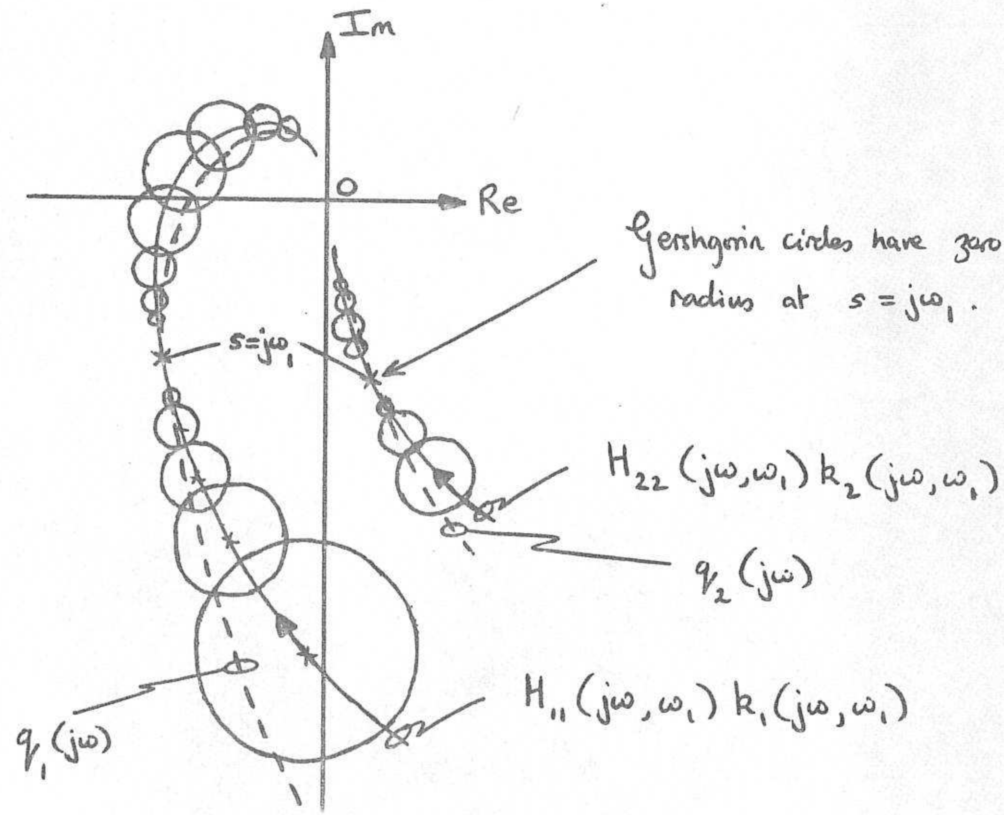


Fig. 7 Schematic Representation of Rational Approximation

PROBLEM ONE

Given that the mxm DTFM

$$G(s) = P_1 \text{ diag } \{g_1(s), g_2(s), \ldots, g_m(s)\} P_2 \quad , \quad |G(s)| \not\equiv 0$$

where the matrices

$$P_1 = [\alpha_1, \alpha_2, \ldots, \alpha_m] \quad , \quad P_2 = \begin{bmatrix} \beta_1^t \\ \vdots \\ \beta_m^t \end{bmatrix}$$

are permissible, show that

(a) $$g_j(s) \not\equiv 0 \quad , \quad 1 \leqslant j \leqslant m$$

(b) $$G(s) = \tilde{P}_1 \text{ diag } \{\tilde{g}_1(s), \ldots, \tilde{g}_m(s)\} \tilde{P}_2$$

where

$$\tilde{P}_1 = [\mu_1\alpha_1, \mu_2\alpha_2, \ldots, \mu_m\alpha_m]$$

$$\tilde{P}_2 = \begin{bmatrix} \gamma_1 & \beta_1^t \\ \vdots & \vdots \\ \gamma_m & \beta_m^t \end{bmatrix}$$

and $$\tilde{g}_j(s) = \frac{1}{\mu_j\gamma_j} \, g_j(s) \quad , \quad 1 \leqslant j \leqslant m$$

for any choice of non-zero complex constants μ_j, γ_j, $1 \leqslant j \leqslant m$.

(c) Show that $(\tilde{P}_1, \tilde{P}_2)$ is permissible if

$$\overline{\mu}_j = \mu_{\ell(j)} \quad , \quad \overline{\gamma}_j = \gamma_{\ell(j)} \qquad 1 \leqslant j \leqslant m.$$

Hence infer that the dyadic decomposition of G(s) is non-unique.

L5 Answer to Problem One:

(a) $0 \not\equiv |G(s)| \equiv |P_1 \text{ diag } \{g_1(s),\ldots,g_m(s)\} P_2|$

$$\equiv |P_1| \, g_1(s)g_2(s)\ldots g_m(s) \, |P_2|$$

from which $|P_i| \neq 0$, $i = 1,2$, and, in particular $g_j(s) \not\equiv 0$, $1 \leqslant j \leqslant m$.

(b) Simply write

$$\tilde{P}_1 = P_1 \text{ diag } \{\mu_1, \ldots, \mu_m\} \ , \ \tilde{P}_2 = \text{diag}\{\gamma_1, \ldots, \gamma_m\} \begin{pmatrix} \beta_1^t \\ \vdots \\ \beta_m^t \end{pmatrix}$$

and substitute the identity

$$\text{diag } \{g_j(s)\}_{1\leqslant j\leqslant m} = \text{diag } \{\mu_j\}_{1\leqslant j\leqslant m} \text{diag } \{\tilde{g}_j(s)\}_{1\leqslant j\leqslant m} \text{diag}\{\gamma_j\}_{1\leqslant j\leqslant m}$$

into the original dyadic decomposition of G.

(c) AS P_1 and P_2 are permissible we have

$$\overline{\alpha_j} = \alpha_{\ell(j)} \quad , \quad \overline{\beta_j} = \beta_{\ell(j)} \quad , \quad 1 \leqslant j \leqslant m.$$

If $\tilde{P}_1$ and $\tilde{P}_2$ are to be permissible we need , $1 \leqslant j \leqslant m$,

$$\overline{(\mu_j\alpha_j)} = \mu_{\ell(j)} \, \alpha_{\ell(j)} \quad , \quad \overline{(\gamma_j\beta_j)} = \gamma_{\ell(j)} \, \beta_{\ell(j)}$$

i.e. $\overline{\alpha_j} = \alpha_{\ell(j)}$, $\overline{\beta_j} = \beta_{\ell(j)}$, $1 \leqslant j \leqslant m$.

PROBLEM TWO

Given the mxm DTFM

$$G(s) = P_1 \text{ diag } \{g_1(s),\ldots,g_m(s)\}\ P_2 , \qquad |G(s)| \not\equiv 0$$

with P_1, P_2 permissible and, for simplicity, the $g_j(s)$, $1 \leqslant j \leqslant m$, are distinct show that a suitable decomposition is achieved by following the procedure outlined below:

STEP 1: Choose a real number $\hat{s}$ such that $G(\hat{s})$ is finite and nonsingular

STEP 2: Find, by trial and error, a real frequency point s_1 such that the eigenvalues of $G(s_1)\ G^{-1}(\hat{s})$ are distinct, and set P_1 to be equal to an eigenvector matrix.

STEP 3: Define $P_2 = P_1^{-1}\ G(\hat{s})$

STEP 4: Compute $g_j(s)$ from the formula,

$$\text{diag } \{g_1(s),\ldots,g_m(s)\} = P_1^{-1}\ G(s)\ P_2^{-1}$$

L5 Answer to Problem Two:

STEP 1: The assumption that $|G(s)| \neq 0$ guarantees the existence of such a point $\hat{s}$.

STEP 2: Noting that

$$G(s)\, G^{-1}(\hat{s}) = P_1 \operatorname{diag}\{\frac{g_1(s)}{g_1(\hat{s})}, \ldots, \frac{g_m(s)}{g_m(\hat{s})}\} P_1^{-1}$$

it follows that P_1 is an eigenvector matrix of $G(s)\, G^{-1}(\hat{s})$ for any choice of s. In the particular case of the $g_j(s)$ distinct, we can always choose s_1 such that

$$\frac{g_j(s_1)}{g_j(\hat{s})} \neq \frac{g_k(s_1)}{g_1(\hat{s})} \quad , \quad 1 \leqslant j , k \leqslant m$$

and hence P_1 is uniquely defined to within scaling and reordering of columns. Moreover, $G(s_1)\, G^{-1}(\hat{s})$ is real and hence we can always arrange that the columns of P_1 occur in complex conjugate pairs,

$$\bar{\alpha}_j = \alpha_{\ell(j)} \quad , \quad 1 \leqslant j \leqslant m$$

STEP 3: Using exercise 1 with $s = \hat{s}$ real, we see that

$$\overline{g_j(\hat{s})} = g_{\ell(j)}(\hat{s}) \quad , \quad 1 \leqslant j \leqslant m$$

In particular we can use problem 1(b) and 1(c) with $\mu_j = 1$, $1 \leqslant j \leqslant m$ and $\gamma_j = g_j(\hat{s})$, $1 \leqslant j \leqslant m$, to demonstrate that it is always possible to take

$$g_j(\hat{s}) = 1 \quad , \quad 1 \leqslant j \leqslant m$$

when $G(\hat{s}) = P_1 P_2$ yields the result with P_1 and P_2 permissible.

STEP 4: obvious.

L5

PROBLEM THREE

Given that the TFM

$$G(s) = \frac{1}{s(s+3)} \begin{bmatrix} s+2 & 2 \\ 2 & 2s+2 \end{bmatrix}$$

is a DTFM, design a unity-negative proportional output feedback controller for the system described by this TFM ensuring that

(a) the closed-loop system is asymptotically stable.

(b) steady-state errors in response to unit step demands in any loop are less than 0.1.

(c) transient interaction effects in response to unit step demands in any loop are less than 0.05 for all time.

L5 Answer to Problem Three:

Using the technique of problem 2 with $\hat{s} = 1.0$ $s_1 = 2.0$ we obtain

$$G(2.0)\, G^{-1}(1.0) = \frac{1}{10}\begin{bmatrix} 6 & -1 \\ -2 & 7 \end{bmatrix}$$

with a real eigenvector matrix

$$P_1 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

from which

$$P_2 = P_1^{-1} G(1.0) = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & 2/3 \end{bmatrix}$$

and $$P_1^{-1} G(s)\, P_2^{-1} = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{4}{s+3} \end{bmatrix}$$

i.e. $g_1(s) = \frac{1}{s}$, $g_2(s) = \frac{4}{s+3}$

The next step in the design is to choose <u>proportional</u> controllers k_1, k_2 such that the scalar feedback systems

$$h_1(s) \triangleq \frac{g_1(s)k_1}{1+g_1(s)k_1} = \frac{k_1}{s+k_1} \quad , \quad h_2(s) \triangleq \frac{g_2(s)k_2}{1+g_2(s)k_2} = \frac{4k_2}{s+3+4k_2}$$

are stable and have similar response characteristics. Stability is ensured if

$$k_1 > 0 \quad , \quad 3 + 4\, k_2 > 0$$

and similar transient response characteristics are achieved if we set

$$k \triangleq k_1 = 3 + 4\, k_2$$

(equalizing their time constants) and use a 'large enough' value of

L5 Answer to Problem Three (cont'd)

k or k_2 to ensure similar steady state characteristics. The resulting controller

$$K = P_2^{-1} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} P_1^{-1} = \begin{pmatrix} k-1 & 1 \\ 1 & {}^{k}/_{2-1} \end{pmatrix}$$

and it remains to choose a suitable value of $k > 0$. Evaluating the closed-loop TFM

$$H_c(s) = P_1 \operatorname{diag}\{h_1(s)\ ,\ h_2(s)\}\ P_1^{-1}$$

$$= \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{k}{s+k} & 0 \\ 0 & \frac{k-3}{s+k} \end{pmatrix} \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{k}{s+k} \begin{pmatrix} 1 - \frac{1}{k} & \frac{1}{k} \\ \frac{2}{k} & 1 - \frac{2}{k} \end{pmatrix}$$

it follows that the design requirement on steady state errors is satisfield for $k > 20$ and similarly for transient interaction effects.

L5

PROBLEM FOUR

Given the DTFM

$$G(s) = \frac{1}{(s+1)^2}\begin{bmatrix} 1-s & 2-s \\ {}^1/_3-s & 1-s \end{bmatrix}$$

$$\equiv \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{(s+1)} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix}\begin{bmatrix} -{}^1/_2 & -{}^1/_2 \\ {}^2/_3 & 1 \end{bmatrix}$$

consider the design of a unity-negative feedback system ensuring stability and high performance responses to unit step demands.

L5 Answer to Problem Four (cont'd):

when

$$h_2(s) = \frac{k_2}{s^2 + 10s + 9 + k_2}$$

Choosing $k_2 = 41.0$ to produce a damping ratio of $1/\sqrt{2}$. The subsystem now has a response speed of the order (-1) x 1/(intercept of root-locus) $= {}^1/5$ second.

(c) We can now attempt to equalize the response speeds of the two subsystems by setting

$$\frac{1}{k_1 + 1} = \frac{1}{5}$$

or $\qquad k_1 = 4$

and using the controller

$$K(s) = P_2^{-1} \begin{bmatrix} 4 & 0 \\ 0 & \frac{41(s+1)}{(s+9)} \end{bmatrix} P_1^{-1}$$

The resulting step responses are shown in the Figure. As can be seen we have quite large transient interaction effects but these are swiftly suppressed. They can be removed by the introduction of more phase advance into $k_2(s)$ but, ultimately, the system performance will be limited by the essentially different dynamic characters of $g_1(s)$ and $g_2(s)$.

L5 Answer to Problem Four:

We use the natural identification

$$P_1 = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \quad , \quad P_2 = \begin{pmatrix} -1/2 & -1/2 \\ 2/3 & 1 \end{pmatrix}$$

$$g_1(s) = \frac{1}{(s+1)} \quad , \quad g_2(s) = \frac{1}{(s+1)^2}$$

and move immediately to the design of the scalar feedback systems

$$h_1(s) = \frac{g_1(s)k_1(s)}{1+g_1(s)k_1(s)} \quad , \quad h_2(s) = \frac{g_2(s)k_2(s)}{1+g_2(s)k_2(s)}$$

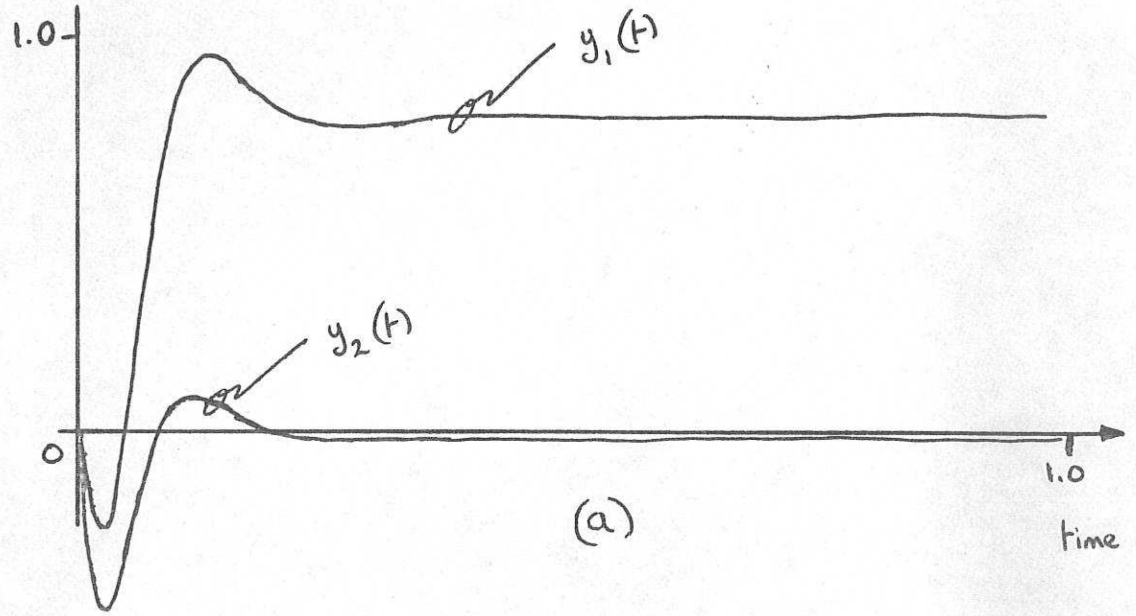
(a) The first order lag nature of $g_1(s)$ suggests the choice of proportional controllers $k_1(s) = k_1$ when we need $k_1 + 1 > 0$ for stability and $k_1 >> 1$ for small steady state errors i.e. its response speed is of the order of $(1+k_1)^{-1} << 1$ sec.

(b) If we restrict our attention to a proportional controller $k_2(s) = k_2$ we immediately hit problems as

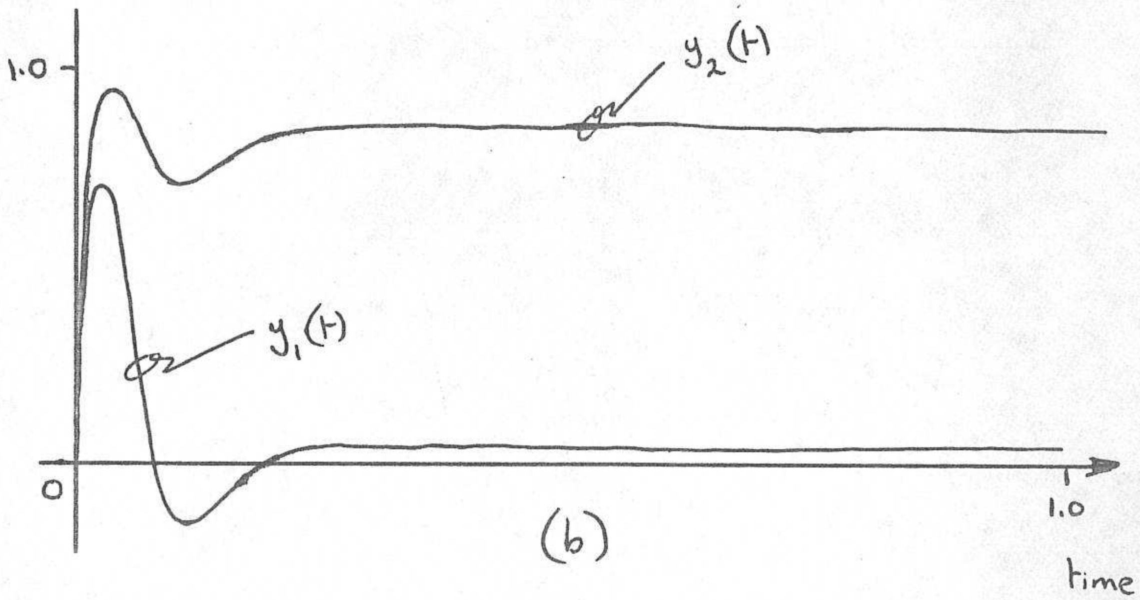
$$h_2(s) = \frac{k_2}{s^2 + 2s + 1 + k_2}$$

is a second order lag with a response speed of the order of 1 sec independent of the choice of 'reasonable' k_2 and if we increase k_2 to reduce steady state errors oscillations will set in. Equivalently it seems to be unlikely that we can make the responses of $h_1(s)$ and $h_2(s)$ similar with proportional control alone. In fact we need to increase the response speed of $h_2(s)$ by the introduction of phase compensation of the form (say)

$$k_2(s) = k_2 \frac{(s+1)}{(s+9)}$$



(a)



(b)

Fig. (a) Response to a unit step demand in output one.

(b) Response to a unit step demand in output two.