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IDENTIFICATION OF FACTORABLE VOLTERRA SYSTEMS

by

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ABSTRACT

Identification of systems which can be represented as a finite sum of factorable Volterra kernels each composed of individual linear dynamic subsystems connected in parallel with outputs multiplied in the time domain is considered. Both a multilevel testing and a sequential single test procedure are derived to isolate each factorable kernel and algorithms which provide estimates of the individual linear subsystems associated with each kernel are formulated using correlation techniques based on either Gaussian or pseudorandom excitation.

1. INTRODUCTION

Identification of the general class of non-linear systems which can be represented by a Volterra series expansion has been studied by several authors. A method of measuring the kernels in the related Wiener G-functional expansion under white Gaussian excitation was studied by Lee and Schetzen[1]. Later researchers investigated the identification of isolated Volterra kernels using correlation analysis based on pseudorandom signal excitation[2,3,4,5,6]. Whilst this approach is necessary for general non-linear systems it is often preferable to identify a system in terms of the component subsystems[7] when the system is composed of interconnected networks of linear dynamic and static non-linear elements. Identification in terms of the individual system components can often be achieved using simple extensions of linear techniques and provides a concise description of the process which maintains the original system structure.

In the present study identification of systems which can be represented as a finite sum of factorable Volterra kernels is considered. The factorable kernel of order k can be realized as a system composed of k linear dynamic subsystems connected in parallel with outputs multiplied in the time domain. Concepts of reachability and observability for this class of systems were introduced by Harper and Rugh[8] who developed an identification scheme based on the steady-state response of a finite number of two-tone sinusoidal inputs. The identification algorithm derived in the present study represents an extension of previous results derived for cascade[9,10,11,12], feedback[13] and feedforward[14] connections of linear dynamic and no-memory non-linear elements. Isolation of the kernels associated with factorable Volterra systems is considered using both multilevel testing techniques and sequential analysis based on single test excitation. Identification of the individual linear subsystems associated with the k'th order factorable kernel is achieved using correlation analysis based on both Gaussian and pseudorandom excitation. Selection of the pseudorandom input sequence is considered and simulated examples are included to demonstrate the feasibility of the algorithm.

2. FACTORABLE VOLTERRA SYSTEMS

A large class of non-linear systems can be represented by the Volterra series expansion

$$y(t) = \int_{-\infty}^{\infty} g_1(t_1)u(t-t_1)dt_1 + \iint_{-\infty}^{\infty} g_2(t_1,t_2)u(t-t_1)u(t-t_2)dt_1dt_2$$

$$+ \ldots + \int\ldots\int g_{\ell\ell}(t_1,t_2\ldots t_{\ell\ell})u(t-t_1) \;\ldots\; u(t-t_{\ell\ell})dt_1\ldots dt_{\ell\ell} \qquad (1)$$

where the function $g_{\ell\ell}(t_1,t_2,\ldots t_{\ell\ell})$ is termed the Volterra kernel of order $\ell\ell$. Volterra kernels are symmetrical, continuous in their arguments for all $t_i>0$, and for a non-anticipative system are zero for any $t_i<0$. A kernel is defined as stable if

$$\int\ldots\int |g_k(t_1,t_2\ldots t_k)|dt_1dt_2 \;\ldots\; dt_k < \infty \qquad (2)$$

and factorable if $g_k(t_1,t_2\ldots t_k)$ can be factored into the product of a function of t_1 only, a function of t_2 only etc, as

$$g_k(t_1,t_2\ldots t_k) = h_1(t_1)h_2(t_2) \;\ldots\; h_k(t_k) \qquad (3)$$

The kth order factorable kernel can be realized as k linear subsystems connected in parallel with outputs multiplied in the time domain as illustrated in Fig.1. This includes the class of systems illustrated in Fig.2.

The state-space representation for the system in Fig.1 is particularly simple and can be written as

$$\dot{x}(t) = Bx(t) + Cu(t) \tag{4}$$

$$y(t) = \prod_{i=1}^{k} d_i \, x_i(t) \tag{5}$$

where $x(t) = [x_1(t) \ldots x_k(t)]^T$, $x_i(t) = [x_{i,1}(t)x_{i,2}(t) \ldots x_{i,n_i}(t)]$, $B = \mathrm{diag}[b_k]$, $C = [c_1 \ldots c_k]^T$ and the vector x(t) is of dimension $n_1+n_2+..+n_k$ where n_i is the order of the i'th subsystem.

Extending the analysis above by summing together the outputs of factorable kernels up to order $\ell\ell$ defines the factorable Volterra system illustrated in Fig.3. Initially identification of the isolated kth order factorable Volterra kernel will be considered. The results will then be extended in section 4 to include the identification of general factorable Volterra systems.

3. IDENTIFICATION OF ISOLATED FACTORABLE VOLTERRA KERNELS

The identification problem can be formulated as identification of each linear subsystem $h_1(t), h_2(t) .., h_k(t)$ associated with the isolated k'th order factorable kernel, from measurements of the input u(t) and output $z_k(t)$ where

$$z_k(t) = \int_{-\infty}^{\infty} \ldots \int h_{1,k}(t_1) \ldots h_{k,k}(t_k) u(t-t_1) \ldots u(t-t_k) dt_1 \ldots dt_k \tag{6}$$

The identification algorithm considered involves a two-stage procedure and consists of identification of the isolated kernel using correlation analysis and decomposition of the kernel to provide estimates of the individual linear subsystems.

3.1 Correlation analysis with Gaussian white noise excitation

Assuming that the output of each kernel $z_1(t), z_2(t) \ldots z_k(t)$ has been isolated using the results of Section 4.1 the kernels can be identified in turn using correlation analysis based on a zero mean white Gaussian input signal u(t).

For the first order kernel

$$z_1(t) = \int_{-\infty}^{\infty} h_{1,1}(t_1)u(t-t_1)dt_1 \tag{7}$$

and $h_{1,1}(t_1)$ can be estimated by computing the cross-correlation function

$$\phi_{uz_1'}(\sigma_1) = \overline{(z_1(t)-\overline{z_1(t)})u(t-\sigma_1)} = \int_{-\infty}^{\infty} h_{1,1}(t_1)\overline{(u(t-t_1)-\overline{u(t-t_1)})u(t-\sigma_1)}dt_1$$

$$= \beta h_1(\sigma_1) \tag{8}$$

where $\beta = \int_{-\infty}^{\infty} \phi_{uu}(t)dt$ and the superscript ' is used throughout to indicate a zero mean process.

The second order kernel $g_2(t_1,t_2) = h_{1,2}(t_1)h_{2,2}(t_2)$ can be estimated by computing the second order correlation function defined as

$$\phi_{uuz_2'}(\sigma_1,\sigma) = \overline{(z_2(t)-\overline{z_2(t)})u(t-\sigma_1)u(t-\sigma)} \quad (9)$$

$$= \iint_{-\infty}^{\infty} h_{1,2}(t_1)h_{2,2}(t_2)\overline{(u(t-t_1)u(t-t_2)-\overline{u(t-t_1)u(t-t_2)})}$$

$$\overline{u(t-\sigma_1)u(t-\sigma)}\,dt_1dt_2$$

$$= \beta^2\{h_{1,2}(\sigma_1)h_{2,2}(\sigma) + h_{1,2}(\sigma)h_{2,2}(\sigma_1)\} \quad (10)$$

Thus

$$\sum_{i=1}^{2}\{h_{i,2}(\sigma_1)\prod_{\substack{j=1\\ j\neq i}}^{2} h_{j,2}(\sigma)\} = 1/\beta^2\overline{(z_2(t)-\overline{z_2(t)})u(t-\sigma_1)u(t-\sigma)}$$

$$= \psi_2(\sigma_1,\sigma) \quad (11)$$

The third order kernel $g_3(t_1,t_2,t_3) = h_{1,3}(t_1)h_{2,3}(t_2)h_{3,3}(t_3)$ can be evaluated by computing

$$\overline{(z_3(t)-\overline{z_3(t)})u(t-\sigma_1)u^2(t-\sigma)} - \overline{\overline{(z_3(t)-\overline{z_3(t)})u(t-\sigma_1)}u^2(t-\sigma)}$$

$$-2\overline{\overline{(z_3(t)-\overline{z_3(t)})u(t-\sigma)}u(t-\sigma_1)u(t-\sigma)} \quad (12)$$

$$= \phi_{uu^2z_3'}(\sigma_1,\sigma) - \phi_\alpha - \phi_\beta$$

to give

$$\sum_{i=1}^{3}\{h_{i,3}(\sigma_1)\prod_{\substack{j=1\\ j\neq i}}^{3} h_{j,3}(\sigma)\} = \frac{1}{2\beta^3}\{\phi_{uu^2z_3'}(\sigma_1,\sigma)-\phi_\alpha-\phi_\beta\} = \psi_3(\sigma_1,\sigma) \quad (13)$$

Extending the procedure outlined above the following general results can be derived to estimate higher order kernels

$$\psi_k(\sigma_1,\sigma) = \sum_{i=1}^{k} \{h_{i,k}(\sigma_1) \prod_{\substack{j=1 \\ j\neq i}}^{k} h_{j,k}(\sigma)\} \qquad (14)$$
(k even)

$$= \frac{1}{(k-1)!\beta^k} \{\overline{(z_k(t)-\overline{z_k(t)}).u(t-\sigma_1) u^{k-1}(t-\sigma)}$$

$$- \sum_{i=1}^{(k-2)/2} \overline{\overline{(z_k(t)-\overline{z_k(t)}).u(t-\sigma_1)u^{2i-1}(t-\sigma)}.u^{k-2i}(t-\sigma)} \frac{(k-1)!}{(k-2i+1)!(2i-1)!}$$

$$- \sum_{i=1}^{(k-2)/2} \overline{\overline{(z_k(t)-\overline{z_k(t)})u^{2i}(t-\sigma)}.u^{k-2i}(t-\sigma)u(t-\sigma_1)} \frac{(k-1)!}{(k-2i-1)!(2i)!}\} \qquad (15)$$

for k = 4,6 ..., and

$$\psi_k(\sigma_1,\sigma) = \sum_{i=1}^{k} \{h_{i,k}(\sigma_1) \prod_{\substack{j=1 \\ j\neq i}}^{k} h_{j,k}(\sigma)\} \qquad (16)$$
(k odd)

$$= \frac{1}{(k-1)!\beta^k} \{\overline{(z_k(t)-\overline{z_k(t)})u(t-\sigma_1)u^{k-1}(t-\sigma)}$$

$$- \sum_{i=0}^{(k-3)/2} \overline{\overline{(z_k(t)-\overline{z_k(t)})u(t-\sigma_1)u^{2i}(t-\sigma)}.u^{k-2i-1}(t-\sigma)} \frac{(k-1)!}{(k-2i-1)!(2i)!}$$

$$- \sum_{i=0}^{(k-3)/2} \overline{\overline{(z_k(t)-\overline{z_k(t)})u^{k-2i}(t-\sigma)}u^{k-2i-2}(t-\sigma)u(t-\sigma_1)} \frac{(k-1)!}{(k-2i-2)!(2i+1)!} \qquad (17)$$

for k = 5,7,...

Thus each kernel can be sequentially identified by computing $\psi_n(\sigma_1,\sigma)$, n = 1,2..k as outlined above where $\psi_n(\sigma_1,\sigma)$ is essentially a second order correlation function for all n.

3.2 Correlation analysis using compound pseudorandom inputs

As an alternative to the algorithm of the previous section an identification procedure which can be applied when the input excitation is a compound pseudorandom signal is derived below.

If the input to the isolated k'th order factorable Volterra kernel is a compound input u(t) defined as

$$u(t) = x_1(t) + x_2(t) + .. + x_k(t) \tag{18}$$

then from equation (6) the kernel output can be expressed as

$$z_k(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_{1,k}(t_1) \dots h_{k,k}(t_k)\{\prod_{j=1}^{k} \sum_{i=1}^{k} x_i(t-t_j)\} \, dt_1 \dots dt_k \tag{19}$$

The output correlation function for compound inputs is defined as

$$\phi_{x_1 \dots x_k z_k'}(\sigma_1,\sigma_2,\dots\sigma_k) = \overline{\{z_k(t)-\overline{z_k(t)}\}x_1(t-\sigma_1)\dots \dots x_k(t-\sigma_k)} \tag{20}$$

For the special case when $\sigma_2 = \sigma_3 = \dots = \sigma_k = \sigma$

$$\phi_{x_1 \dots x_k z_k'}(\sigma_1,\sigma\dots\sigma) = \int_{-\infty}^{\infty} \dots \int h_{2,k}(t_1)\dots h_{k,k}(t_k) \overline{\{\prod_{j=1}^{k} \sum_{i=1}^{k} x_i(t-t_j) - \overline{\prod_{j=1}^{k} \sum_{i=1}^{k} x_i(t-t_j)}\} x_1(t-\sigma_1) \prod_{n=2}^{k} x_n(t-\sigma)} \, dt_1 \dots dt_k \tag{21}$$

If $x_1(t),\ldots x_k(t)$ are independent zero mean processes $\phi_{x_i x_j}(\lambda) = 0 \; \forall \; i \neq j$, and have an autocorrelation function $\phi_{x_i x_i}(\lambda) = \beta_i \delta(\lambda)$, $i = 1,2\ldots k$ then equation (21) reduces to

$$\phi_{x_1\ldots x_k z_k'}(\sigma_1,\sigma,\ldots\sigma) = (k-1)!\left(\prod_{n=1}^{k} \beta_n\right) \sum_{i=1}^{k} \{h_{i,k}(\sigma_i) \prod_{\substack{j=1\\ j\neq i}}^{k} h_{j,k}(\sigma)\} \tag{22}$$

Comparison with equations (14) and (16) gives

$$\psi_k(\sigma_1,\sigma) = \phi_{x_1\ldots x_k z_k'}(\sigma_1,\sigma\ldots\sigma) \cdot \frac{1}{(k-1)!\left(\prod_{n=1}^{k} \beta_n\right)}$$

$$= \sum_{i=1}^{k} \{h_{i,k}(\sigma_i) \prod_{\substack{j=1\\ j\neq i}}^{k} h_{j,k}(\sigma)\} \tag{23}$$

The above result could be realised by using a compound input consisting of independent zero mean white Gaussian processes. However it would be preferable to use pseudorandom sequences since this would reduce the computational burden and simplify the correlation procedure. Providing independent zero mean pseudorandom sequences can be synthesised the terms involving anomalies[4,15] associated with multidimensional autocorrelation functions of pseudorandom sequences in equation (21) are eliminated and the result of equation (23) holds.

The selection of independent pseudorandom sequences has been studied previously by Briggs and Godfrey[16] in the context of multivariable system identification. Noting that a fundamental requirement for equation (23) to hold is that the individual inputs should each have zero mean, an obvious choice of input would be a compound ternary sequence[17]. Unfortunately, the difficulty of generating independent ternary sequences without incurring long integration times[16] precludes their use except for low order systems. They should not however be discounted and may well be preferable to Gaussian inputs in many applications.

Another obvious choice of input signal for the present application is the inverse repeat or antisymmetric pseudorandom binary sequence[18,19] defined as

$$C_r = (-1)^r e_r \tag{24}$$

where e_r is a prbs of period N with elements ±1. Whilst this signal has zero mean the autocorrelation function is not ideal and contains a ripple of amplitude ±1/N in addition to the alternating positive and negative impulse spikes of unit amplitude at lags Omod2N and Nmod2N respectively. Independent inverse repeat sequences can be generated by multiplying the fundamental sequence with the rows of a Hadamard matrix[16]. Although the ripple in the autocorrelation function will become small as N becomes large this will introduce a small bias into the results.

However by careful choice of correlation function it is possible to use the much more convenient binary maximal length sequences in this application. As in the case of inverse repeat sequences independent inputs can be generated by multiplying by the rows of a Hadamard matrix or correlating over the product of sequence lengths[16].

If the input to the isolated k'th order factorable Volterra kernel is the compound input $u(t) = \sum_{i=1}^{k} x_i(t)$ where $x_i(t)$ are independent pseudorandom maximal length binary sequences of amplitude $\pm a_i$, order n_i, clock interval Δt_i, and length $N_i = 2^{n_i}-1$, the individual kernels can be identified as detailed below.

From equation (7) the input/output correlation function for the first order kernel is defined as

$$\phi_{x_1'z_1'}(\sigma_1) = \overline{\int_{-\infty}^{\infty} h_{1,1}(t_1)(x_1(t-t_1)-\overline{x_1(t-t_1)})(x_1(t-\sigma_1)-\overline{x_1(t-\sigma_1)})\,dt_1} \qquad (25)$$

$$= \frac{(N_1+1)}{N_1}\, a_1^2\{\Delta t_1 h_{1,1}(\sigma_1)\} - \frac{a_1^2}{N_1}\int h_{1,1}(t_1)\,dt_1 \qquad (26)$$

Providing the system $h_1(t)$ is stable, bounded inputs bounded outputs, the last term on the rhs of equation (26) is a constant bias which can be readily removed to yield an estimate of $h_{1,1}(t)$.

The second order kernel can be estimated by computing the second order correlation function defined as

$$\phi_{x_1'x_2'z_2'}(\sigma_1,\sigma) = \overline{\iint_{-\infty}^{\infty} h_{1,2}(t_1)h_{2,2}(t_2)\{(x_1(t-t_1)+x_2(t-t_1))(x_1(t-t_2)+x_2(t-t_2))}$$

$$\overline{-\overline{(x_1(t-t_1)+x_2(t-t_1))(x_1(t-t_2)+x_2(t-t_2))}\}(x_1(t-\sigma_1)-\overline{x_1(t-\sigma_1)})(x_2(t-\sigma)}$$

$$\overline{-\overline{x_2(t-\sigma)})\,dt_1 dt_2} \qquad (27)$$

$$= \iint_{-\infty}^{\infty} h_{1,2}(t_1)h_{2,2}(t_2)\{\phi_{x_1x_1}(\sigma_1-t_1)\phi_{x_2x_2}(\sigma-t_2)$$

$$+\phi_{x_1x_1}(\sigma_1-t_2)\phi_{x_2x_2}(\sigma-t_1) - \bar{x}_1^2(\phi_{x_2x_2}(\sigma-t_2)+\phi_{x_2x_2}(\sigma-t_1))$$

$$-\bar{x}_2^{\,2}(\phi_{x_1x_1}(\sigma_1-t_1)+\phi_{x_1x_1}(\sigma_1-t_2))+2\bar{x}_1^{\,2}\bar{x}_2^{\,2}\}dt_1dt_2 \tag{28}$$

Equation (28) reduces to

$$\phi_{x_1'x_2'z_2'}(\sigma_1,\sigma) = \alpha_1\alpha_2\sum_{i=1}^{2}(h_{i,2}(\sigma_1)\prod_{\substack{j=1\\j\neq i}}^{2}h_{j,2}(\sigma))+e(\sigma_1,\sigma) \tag{29}$$

where

$$e(\sigma_1,\sigma) = 2\bar{x}_1\bar{x}_2\iint h_{1,2}(t_1)h_{2,2}(t_2)\{a_1a_2+\bar{x}_1a_2+\bar{x}_2a_1+\bar{x}_1\bar{x}_2\}dt_1dt_2$$

$$-(a_1\bar{x}_1\alpha_2+\bar{x}_1^{\,2}\alpha_2)\sum_{i=1}^{2}(h_{i,2}(\sigma)\prod_{\substack{j=1\\j\neq i}}^{2}\int h_{j,2}(t_j)dt_j)$$

$$-(a_2\bar{x}_2\alpha_1+\bar{x}_2^{\,2}\alpha_1)\sum_{i=1}^{2}(h_{i,2}(\sigma_1)\prod_{\substack{j=1\\j\neq i}}^{2}\int h_{j,2}(t_j)dt_j) \tag{30}$$

$$\alpha_i = \frac{a_i^{\,2}(N_i+1)\Delta t_i}{N_i} \tag{31}$$

$$\bar{x}_i = a_i/N_i \tag{32}$$

Providing the systems $h_{1,2}(t)$, $h_{2,2}(t)$ are stable bounded inputs bounded outputs the first term on the rhs of equation (30) is a constant bias that can readily be removed. The remaining terms on the rhs of equation (30) represent a time varying bias which tends to zero as N_1 and N_2 become large.

Thus if the constant bias is removed and N_1 and N_2 are large

$$\psi_2(\sigma_1,\sigma) = \frac{\phi_{x_1'x_2'z_2'}(\sigma_1,\sigma)}{\alpha_1\alpha_2} \simeq \sum_{i=1}^{2}(h_{i,2}(\sigma_1)\prod_{\substack{j=1\\j\neq i}}^{2}h_{j,2}(\sigma)) \tag{33}$$

Higher order kernels can be identified by computing $\psi_n(\sigma_1,\sigma)$, $n = 3,4..k$ using the procedure above.

3.3 Estimation of the linear subsystems

Once the system kernels

$$\psi_k(\sigma_1,\sigma) = \sum_{i=1}^{k} (h_{i,k}(\sigma_1) \prod_{\substack{j=1 \\ j\neq i}}^{k} h_{j,k}(\sigma)) \tag{34}$$

have been identified, using either Gaussian white noise (section 3.1) or compound pseudorandom sequences, estimates of the individual subsystems $h_{i,k}(t)$ can be obtained by decomposing equation (34).

If $\psi_k(\sigma_1,\sigma)$ in equation (34) is computed for fixed $\sigma = \gamma_1$ and $\sigma_1 = \gamma_1,\gamma_2...\gamma_m$ the following set of equations results

$$\begin{aligned} \psi_k(\gamma_1,\gamma_1) &= \sum_{i=1}^{k} (h_{i,k}(\gamma_1) \prod_{\substack{j=1 \\ j\neq i}}^{k} h_{j,k}(\gamma_1)) \\ &\;\;\vdots \\ \psi_k(\gamma_m,\gamma_1) &= \sum_{i=1}^{k} (h_{i,k}(\gamma_m) \prod_{\substack{j=1 \\ j\neq i}}^{k} h_{j,k}(\gamma_1)) \end{aligned} \tag{35}$$

Continuing this procedure by fixing $\sigma = \gamma_j$, $j = 1,2,..m$ and in each case evaluating $\psi_k(\sigma_1,\sigma)$ for all $\sigma_1 = \gamma_1,\gamma_2..\gamma_m$ yields m^2 equations in k.m unknowns $h_{j,k}(\gamma_i)$, $j = 1,2..k$, $i = 1,2..m$

$$\psi_k(\gamma_p,\gamma_q) = \sum_{i=1}^{k} (h_{i,k}(\gamma_p) \prod_{\substack{j=1 \\ j\neq i}}^{k} h_{j,k}(\gamma_q)) \tag{36}$$

for $q = 1,2..m$, $p = 1,2..m$ for each value of q, and $\psi_k(\gamma_p,\gamma_q) \neq \psi_k(\gamma_\ell,\gamma_f)$ $p \neq \ell$, $q \neq f$.

Notice that for k = 2 equation (36) produces two identical equations. This redundancy can be excluded by modifying equation (36) for the special case of k = 2 to give

$$\psi_2(\gamma_p,\gamma_q) = \sum_{i=1}^{2} (h_{i,2}(\gamma_p) \prod_{\substack{j=1 \\ j\neq i}}^{2} h_{j,2}(\gamma_q)) \tag{37}$$

for q = 1,2..m and p = q,q+1,..m for each value of q.

Equation (36), or (37) for k = 2, can be solved for the km unknowns $h_{j,k}(\gamma_i)$, j = 1,2..k, i = 1,2..m by minimising the cost function

$$J(\underline{h}) = \sum_{i=1}^{m} \sum_{j=1}^{m} \{\psi_k(\gamma_j,\gamma_i)-\hat{\psi}_k(\gamma_j,\gamma_i)\}^2 \tag{38}$$

$$\underline{h} = [h_{1,k}(\gamma_1) \ldots h_{1,k}(\gamma_m),h_{2,k}(\gamma_1)\ldots h_{2,k}(\gamma_m),\ldots h_{k,k}(\gamma_m)]^T$$

using the modified Marquardt algorithm[20,21]. Although $h_{j,k}(\gamma_i)$, j = 1,2..k, i = 1,2..m can only be estimated to within constant scale factors this does not jeopardise the final identification results.

Once estimates of $h_{j,k}(\gamma_i)$, j = 1,2..k, i = 1,2..m are available the following matrix equation can be formulated

$$\begin{pmatrix} \psi_k(t,\gamma_1) \\ \vdots \\ \psi_k(t,\gamma_m) \end{pmatrix} = \begin{pmatrix} \prod_{i=2}^{k} \hat{h}_{i,k}(\gamma_1), & \prod_{\substack{i=1\\i\neq 2}}^{k} \hat{h}_{i,k}(\gamma_1) \ldots & \prod_{\substack{i=1\\i\neq k}}^{k} \hat{h}_{i,k}(\gamma_1) \\ \vdots & \vdots & \vdots \\ \prod_{i=2}^{k} \hat{h}_{i,k}(\gamma_m), & \prod_{\substack{i=1\\i\neq 2}}^{k} \hat{h}_{i,k}(\gamma_m) \ldots & \prod_{\substack{i=1\\i\neq k}}^{k} \hat{h}_{i,k}(\gamma_m) \end{pmatrix} \begin{pmatrix} \hat{h}_{1,k}(t) \\ \hat{h}_{2,k}(t) \\ \vdots \\ \hat{h}_{k,k}(t) \end{pmatrix}$$

$$\underline{F}_t = \underline{\theta} \cdot \underline{h}_t \tag{39}$$

Estimates of the individual linear subsystems $h_{i,k}(t)$, $i = 1,2..k$ can then be obtained by solving

$$\underline{h}_t = \underline{\theta}^{-1} \underline{F}_t \qquad \text{for } m = k \tag{40}$$

$$\text{or} \quad \underline{h}_t = (\underline{\theta}^T \underline{\theta})^{-1} \underline{\theta}^T \underline{F}_t \quad \text{for } m>k \tag{41}$$

for a range of t from zero to the system settling time.

If two columns of the matrix $\underline{\theta}$ are linearly dependent this implies that two of the linear subsystems are equal except for a scale factor, $h_{i,k}(t) = \nu h_{j,k}(t)$. In this case the matrix $\underline{\theta}$ should be modified by deleting one of the offending columns and multiplying the other by $(1+\nu)$. The corresponding column in $\underline{h}_t$ should also be removed.

Although a non-linear Marquardt algorithm must be implemented to isolate the linear subsystems associated with the k'th order kernel only k^2 parameters must be estimated in this way for $k>2$ and only six parameters when $k=2$.

4. IDENTIFICATION OF NON-LINEAR SYSTEMS WITH FACTORABLE VOLTERRA KERNELS

In the general case when a system can be represented as a factorable Volterra system, as illustrated in Fig.3, it is necessary to isolate the output of each kernel before the results of previous sections can be applied to identify the individual linear subsystems associated with each kernel. Two methods of achieving this objective are described below.

4.1 Multilevel testing

Inspection of the output

$$y(t) = \sum_{k=1}^{\ell\ell} z_k(t) = \sum_{k=1}^{\ell\ell} \int \ldots \int_{-\infty}^{\infty} h_{1,k}(t_1)\ldots h_{k,k}(t_k)u(t-t_1) \ldots u(t-t_k)\,dt_1\ldots dt_k \qquad (42)$$

of the factorable Volterra system illustrated in Fig.3 shows that for a given functional form of the input $u(t)$, the form of the k'th kernel output $z_k(t)$ is fixed but its amplitude is proportional to the k'th power of the amplitude of $u(t)$. Thus for a series of experiments with inputs $\omega_i u(t)$, and corresponding system outputs $y_{\omega_i}(t)$, the outputs of the individual kernels can be calculated from[22]

$$\begin{pmatrix} y_{\omega_1}(t) \\ \vdots \\ y_{\omega_{\ell\ell}}(t) \end{pmatrix} = \begin{pmatrix} \omega_1 & 0 & \ldots & 0 \\ 0 & \omega_2 & \ldots & \cdot \\ \vdots & & & \\ 0 & & & \omega_{\ell\ell} \end{pmatrix} \begin{pmatrix} 1 & \omega_1 & \ldots & \omega_1^{\ell\ell-1} \\ 1 & \omega_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \omega_{\ell\ell} & \ldots & \omega_{\ell\ell}^{\ell\ell-1} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_{\ell\ell}(t) \end{pmatrix} \qquad (43)$$

where the diagonal matrix on the rhs is nonsingular providing $\omega_i \neq 0 \forall i$ and the second matrix is the transpose of the Vandermonde matrix which is non-singular for $\omega_i \neq \omega_j$.

Alternatively, to obtain more accurate estimates in the case of noise corrupted data the procedure outlined below can be used. Testing the system with an input $\omega_i u(t)$ the system output can be expressed as

$$y_{\omega_i}(t) = \sum_{j=1}^{\ell\ell} \omega_i^{\,j} z_j(t) \tag{44}$$

and for an input $(-\omega_i u(t))$ by

$$y_{-\omega_i}(t) = \sum_{j=1}^{\ell\ell} (-1)^j \omega_i^{\,j} z_j(t) \tag{45}$$

Adding equations (44) and (45) gives the summation of all the even order kernel outputs for the input $\omega_i u(t)$

$$y_{e\omega_i}(t) = \tfrac{1}{2}(y_{\omega_i}(t) + y_{-\omega_i}(t)) \tag{46}$$

$$= \sum_{j=1}^{r} \omega_i^{\,2j} z_{2j}(t) \tag{47}$$

where $r = \ell\ell/2$ for $\ell\ell$ even, $r = (\ell\ell-1)/2$ for $\ell\ell$ odd. Similarly, for the odd order kernel outputs

$$y_{o\omega_i}(t) = \tfrac{1}{2}(y_{\omega_i}(t) - y_{-\omega_i}(t)) \tag{48}$$

$$= \sum_{j=1}^{s} \omega_i^{\,(2j-1)} z_{2j-1}(t) \tag{49}$$

where s = ℓℓ/2 for ℓℓ even, s = (ℓℓ+1)/2 for ℓℓ odd. By testing the system with the inputs $\pm\omega_i u(t)$ the outputs of the individual kernels can be calculated from

$$\begin{bmatrix} y_{e\omega_1}(t) \\ \cdot \\ \cdot \\ \cdot \\ y_{e\omega_r}(t) \end{bmatrix} = \begin{bmatrix} \omega_1^2 & \omega_1^4 & \cdots & \omega_1^{2r} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \omega_r^2 & \omega_r^4 & \cdots & \omega_r^{2r} \end{bmatrix} \begin{bmatrix} z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ z_{2r}(t) \end{bmatrix} \tag{50}$$

and

$$\begin{bmatrix} y_{O\omega_1}(t) \\ \cdot \\ \cdot \\ \cdot \\ y_{O\omega_s}(t) \end{bmatrix} = \begin{bmatrix} \omega_1 & \omega_1^3 & \cdots & \omega_1^{2s-1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \omega_s & \omega_s^3 & \cdots & \omega_s^{2s-1} \end{bmatrix} \begin{bmatrix} z_1(t) \\ \cdot \\ \cdot \\ \cdot \\ z_{2s-1}(t) \end{bmatrix} \tag{51}$$

Once the kernel outputs $z_i(t)$, i = 1,2..ℓℓ have been computed, the linear subsystems associated with each kernel can be identified using either the results of section 3.1, or the algorithm in section 3.2 for zero mean compound inputs, in conjunction with the results of section 3.3.

4.2 Sequential analysis

As an alternative to multilevel testing which involves a long experimentation time the output correlation functions $\phi_{x_1..x_k z_k'}(\sigma_1,\sigma..\sigma)$ can be evaluated sequentially when the compound input method is used.

Consider a factorable Volterra system which is composed of factorable kernels up to order ℓℓ. When the system is excited by the compound input $u(t) = \sum_{j=1}^{\ell\ell} x_j(t)$ the system response can be expressed as

$$y(t) = \sum_{j=1}^{\ell\ell} z_j(t) = \sum_{j=1}^{\ell\ell} \int_{-\infty}^{\infty} \cdots \int h_{1,\ell\ell}(t_1) \cdots h_{j,\ell\ell}(t_j) \left(\prod_{i=1}^{j} \sum_{k}^{\ell\ell} x_k(t-t_i) \right) dt_1 \cdots dt_j \tag{52}$$

If the individual inputs $x_j(t)$, $j = 1,2..\ell\ell$ are zero mean independent processes with autocorrelation functions $\phi_{x_j x_j}(\tau) = \beta_j \delta(\tau)$, $j = 1,2..\ell\ell$ then the system output correlation function defined as

$$\overline{y'(t)x_1(t-\sigma_1) \prod_{i=2}^{\ell\ell} x_i(t-\sigma)} = \sum_{j=1}^{\ell\ell-1} \int_{-\infty}^{\infty} \cdots \int h_{1,j}(t_1) \cdots h_{j,j}(t_j)$$

$$\overline{\left(\prod_{i=1}^{j} \sum_{k=1}^{\ell\ell} x_k(t-t_i) \right) . x_1(t-\sigma_1) \prod_{m=2}^{\ell\ell} x_m(t-\sigma)} \, dt_1 \ldots dt_j$$

$$+ \int_{-\infty}^{\infty} \cdots \int h_{1,\ell\ell}(t_1) \cdots h_{\ell\ell,\ell\ell}(t_{\ell\ell}) \overline{\left(\prod_{i=1}^{\ell\ell} \sum_{k=1}^{\ell\ell} x_k(t-t_i) \right) .}$$

$$\overline{x_1(t-\sigma_1) \prod_{m=2}^{\ell\ell} x_m(t-\sigma)} \, dt_1 \ldots dt_{\ell\ell} \tag{53}$$

reduces to the output correlation function for the $\ell\ell$'th kernel

$$\overline{y'(t)x_1(t-\sigma_1) \prod_{i=2}^{\ell\ell} x_i(t-\sigma)} = \phi_{x_1 .. x_{\ell\ell} y'}(\sigma_1, \sigma \ldots \sigma)$$

$$= \phi_{x_1 .. x_{\ell\ell} z_{\ell\ell}'}(\sigma_1, \sigma \ldots \sigma) \tag{54}$$

Thus by computing $\phi_{x_1 \ldots x_{\ell\ell} y'}(\sigma_1, \sigma \ldots \sigma)$ we have isolated the correlation function associated with the $\ell\ell$'th kernel. When a Gaussian white compound input is used $\psi_{\ell\ell}(\sigma_1, \sigma)$ can be determined directly from equation (23). Similarly for a compound ternary sequence. If a compound prbs sequence is used then $\phi_{x_1' \ldots x_{\ell\ell}' y'}(\sigma_1, \sigma) = \phi_{x_1' \ldots x_{\ell\ell}' z_{\ell\ell}'}(\sigma_1, \sigma)$ exactly and $\psi_{\ell\ell}(\sigma_1, \sigma)$ can be evaluated following the results of section 3.2.

Once the linear subsystems associated with the $\ell\ell$'th kernel have been estimated using the algorithm of section 3.3. the output $\hat{z}_{\ell\ell}(t)$ can be computed

$$\hat{z}_{\ell\ell}(t) = \int_{-\infty}^{\infty} \ldots \int \hat{h}_{1,\ell\ell}(t_1) \ldots \hat{h}_{\ell\ell,\ell\ell}(t_{\ell\ell})$$

$$\left(\prod_{j=1}^{\ell\ell} \sum_{i=1}^{\ell\ell} x_i(t - t_j) \right) dt_1 \ldots dt_{\ell\ell} \tag{55}$$

and a reduced system output $yy_{\ell-1}(t)$ can be defined

$$\hat{yy}'_{\ell-1}(t) = y'(t) - \hat{z}'_{\ell\ell}(t) \tag{56}$$

Continuing the above procedure the $(\ell\ell-1)$'th kernel can be identified by computing the $(\ell\ell-1)$'th system output correlation function

$$\overline{yy'_{(\ell\ell-1)}(t) x_1(t-\sigma_1) \prod_{i=2}^{\ell\ell-1} x_i(t-\sigma)}$$

$$= \sum_{j=1}^{\ell\ell-1} \int_{-\infty}^{\infty} \ldots \int h_{1,j}(t_1) \ldots h_{j,j}(t_j)$$

$$\overline{\left(\prod_{i=1}^{j}\sum_{k=1}^{\ell\ell} x_k(t-t_i)\right)x_i(t-\sigma_1)\prod_{m=2}^{\ell\ell-1} x_m(t-\sigma)\,dt_1\ldots dt_j} \qquad (57)$$

which reduces to

$$\phi_{x_1\ldots x_{\ell\ell-1}yy'_{\ell\ell-1}}(\sigma_1,\sigma\ldots\sigma) = \phi_{x_1\ldots x_{\ell\ell-1}z'_{\ell\ell-1}}(\sigma_1,\sigma\ldots\sigma) \qquad (58)$$

the output correlation function of the ($\ell\ell$-1)'th kernel.

Similarly, the k'th order kernel can be identified sequentially by computing

$$\hat{yy}'_k(t) = y'(t) - \sum_{i=k+1}^{\ell\ell} \hat{z}'_i(t) \qquad (59)$$

$$\phi_{x_1\ldots x_k yy'_k}(\sigma_1,\sigma\ldots\sigma) = \phi_{x_1\ldots x_k z'_k}(\sigma_1,\sigma\ldots\sigma) \qquad (60)$$

It can readily be shown that providing any noise corrupting the system output $y(t)$ is independent of the input process this tends to zero in the analysis and unbiased estimates are obtained for all the algorithms described above.

5. SIMULATION RESULTS

The identification results presented above have been used to identify both isolated factorable kernels and a factorable Volterra system. All the models were simulated on an ICL 1906S digital computer using a compound input $u(t) = x_1(t)+x_2(t)$ where $x_1(t)$ and $x_2(t)$ can be defined as either

(i) pseudorandom binary sequences which satisfy the difference equations

$$(I\oplus_2 D^3 \oplus_2 D^7)x = 0$$

$$(I\oplus_2 D^1 \oplus_2 D^6)x = 0$$

where D is the delay operator and $\oplus_2$ denotes modulo 2 addition. These two sequences are independent when the correlation is performed over $N_1N_2 = (2^7-1)(2^6-1) = 8001$ points.

(ii) ternary sequences which satisfy the difference equations

$$(I\oplus_3 D^3 \oplus_3 2D^4)x = 0$$

$$(I\oplus_3 2D^4 \oplus_3 D^5)x = 0$$

where $\oplus_3$ denotes modulo 3 addition. These sequences are independent when the correlation is performed over $(3^4-1)(3^5-1)/2 = 9680$ points.

Initially the isolated second order kernel composed of pulse transfer functions

$$H^1_{1,2}(z^{-1}) = Z\{h_{2,1}(t)\} = \frac{0.2z^{-1}}{1-1.58z^{-1}+0.67z^{-2}} \tag{61}$$

$$H^1_{2,2}(z^{-1}) = Z\{h_{2,2}(t)\} = \frac{0.2z^{-1}}{1-1.7z^{-1}+0.72z^{-2}} \tag{62}$$

was simulated using both ternary and prbs compound inputs defined in (i) and (ii) above. A comparison of the estimated and theoretical weighting sequences for $h_{2,1}(k)$ and $h_{2,2}(k)$ are illustrated in Figs. 4(a) and (b) respectively for prbs inputs. The small bias in $\hat{h}_{2,1}(k)$ and $\hat{h}_{2,2}(k)$ is due to the error defined in equation (30). Estimates of the parameters in the pulse transfer function equations are summarised in Table 1.

A factorable second order kernel with pulse transfer functions

$$H^{2}_{1,2}(z^{-1}) = \frac{0.2z^{-1}}{1-0.8z^{-1}} \tag{63}$$

$$H^{2}_{2,2}(z^{-1}) = \frac{0.2z^{-1}}{1-1.45z^{-1}+0.5z^{-1}} \tag{64}$$

was simulated using the compound ternary input sequence. Inspection of Fig.5(a) and (b) shows the close correspondence between the estimated and theoretical impulse responses. Estimates of the system parameters are summarised in Table 1.

Finally, identification of the factorable Volterra system illustrated in Fig.3 was considered using the prbs compound input, and the sequential analysis procedure of section 4.2 to isolate the kernel correlation functions. The system included both first and second order kernels and was defined as

$$H^{3}_{1,1}(z^{-1}) = \frac{0.2z^{-1}}{1-1.5z^{-1}+0.62z^{-2}} \tag{65}$$

$$H^{3}_{1,2}(z^{-1}) = \frac{0.2z^{-1}}{1-1.62z^{-1}+0.7z^{-2}} \tag{66}$$

$$H^{3}_{2,2}(z^{-1}) = \frac{0.2z^{-1}}{1-1.56z^{-1}+0.7z^{-2}} \tag{67}$$

The estimated and theoretical weighting functions are illustrated in Fig.6(a), (b) and (c). Although a bias is present in the estimate of $h_{1,1}(t)$ Fig.6(a) this is quite small and does not introduce significant errors in the system parameters in Table 1.

All the examples were simulated with m = 3, $\gamma_1 = 3$, $\gamma_2 = 5$, $\gamma_3 = 7$ in equations (37), (39), and convergence of the Marquardt algorithm was achieved in a maximum of seventeen iterations.

6. CONCLUSIONS

Identification of factorable Volterra systems in terms of the linear component subsystems has been investigated and a series of algorithms based on correlation analysis using both white Gaussian and pseudorandom input sequences have been derived. Methods of isolating the kernels associated with a general factorable Volterra system have been presented based on both a multilevel testing procedure and a sequential algorithm which isolates the kernel correlation functions. The excessive experimentation time associated with the former approach will often dictate the use of the simple sequential analysis algorithm.

Whilst isolated kernels can be identified using the algorithm of either section 3.1 or 3.2 with a Gaussian white input process the convenience of pseudorandom inputs suggests that the compound input method of section 3.2 would be more appropriate in many applications.

All the algorithms presented are relatively simple to implement and represent extensions of the correlation techniques associated with linear system identification methods.

Identification of factorable Volterra systems in terms of the individual linear subsystems overcomes many of the disadvantages associated with a black-box Volterra series description and provides valuable information for control.

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PARAMETERS			n_1	n_2	d_1	d_2
$H^1_{1,2}(z^{-1})$	Theoretical Values		0.2	0.0	-1.58	0.67
	Estimates	PRBS	0.203	-0.002	-1.583	0.676
		Ternary	0.198	0.0	-1.579	0.668
$H^1_{2,2}(z^{-1})$	Theoretical Values		0.2	0.0	-1.7	0.72
	Estimates	PRBS	0.179	0.019	-1.706	0.725
		Ternary	0.188	0.021	-1.686	0.707
$H^2_{1,2}(z^{-1})$	Theoretical Values		0.2	-	-0.8	-
	Estimates	Ternary	0.209	-	-0.799	-
$H^2_{2,2}(z^{-1})$	Theoretical Values		0.2	0.0	-1.45	0.5
	Estimates	Ternary	0.20	0.003	-1.450	0.500
$H^3_{1,1}(z^{-1})$	Theoretical Values		0.2	0.0	-1.5	0.62
	Estimates	PRBS	0.197	0.004	-1.499	0.620
$H^3_{1,2}(z^{-1})$	Theoretical Values		0.2	0.0	-1.62	0.7
	Estimates	PRBS	0.201	-0.003	-1.618	0.699
$H^3_{2,2}(z^{-1})$	Theoretical Values		0.2	0.0	-1.56	0.7
	Estimates	PRBS	0.198	0.004	-1.558	0.701

TABLE 1 Summary of the Identification Results

Figure Captions

Fig. 1 The k'th order factorable kernel

Fig. 2 Equivalent kernel

Fig. 3 A non-linear factorable Volterra system

(a) o o o theoretical response $h_{1,2}(k)$

--- estimated values $\hat{h}_{1,2}(k)$

(b) x x x theoretical response $h_{2,2}(k)$

— - — estimated values $\hat{h}_{2,2}(k)$

Fig. 4 A comparison of impulse responses

(a) o o o theoretical response $h_{1,2}(k)$

--- estimated values $\hat{h}_{1,2}(k)$

(b) x x x theoretical response $h_{2,2}(k)$

— - — estimated values $\hat{h}_{2,2}(k)$

Fig. 5 A comparison of impulse responses

(a) + + + theoretical response $h_{1,1}(k)$

--- estimated values $\hat{h}_{1,1}(k)$

(b) o o o theoretical response $h_{1,2}(k)$

— — estimated values $\hat{h}_{1,2}(k)$

(c) x x x theoretical response $h_{2,2}(k)$

— - — estimated values $\hat{h}_{2,2}(k)$

Fig. 6 A comparison of impulse responses

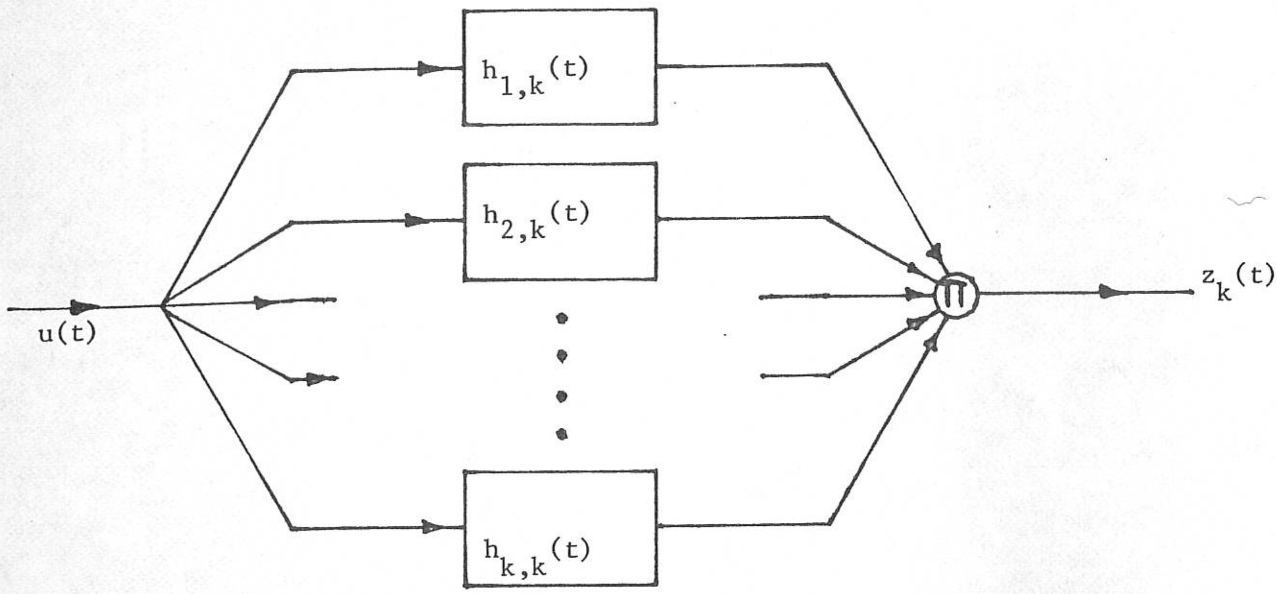

Fig. 1 The k'th order factorable kernel

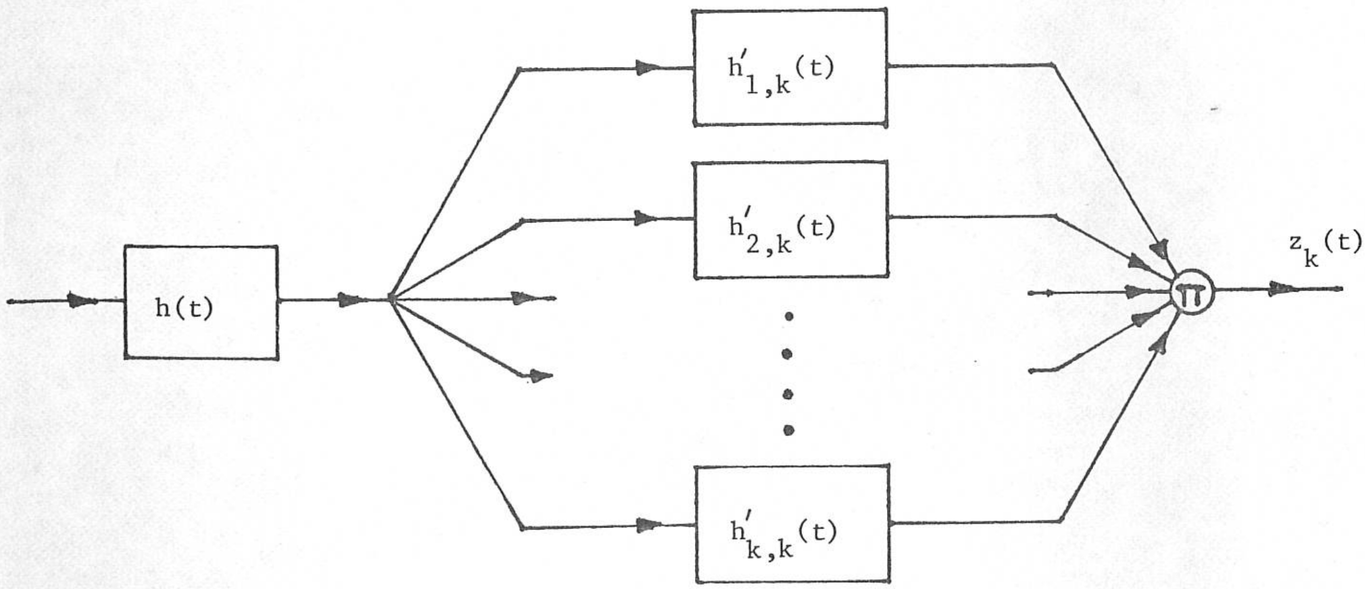

Fig. 2 Equivalent kernel

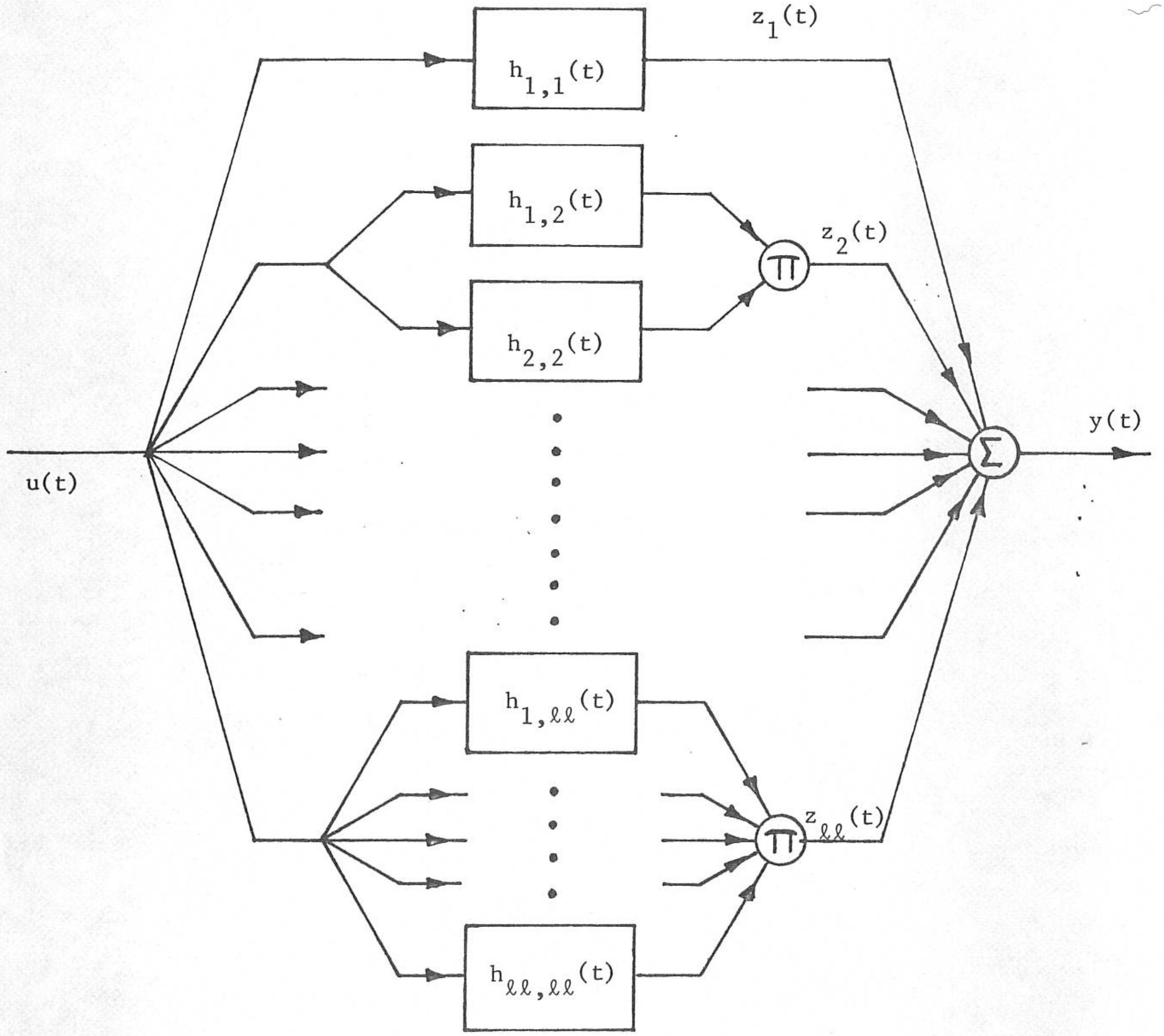

Fig. 3 A non-linear factorable Volterra system

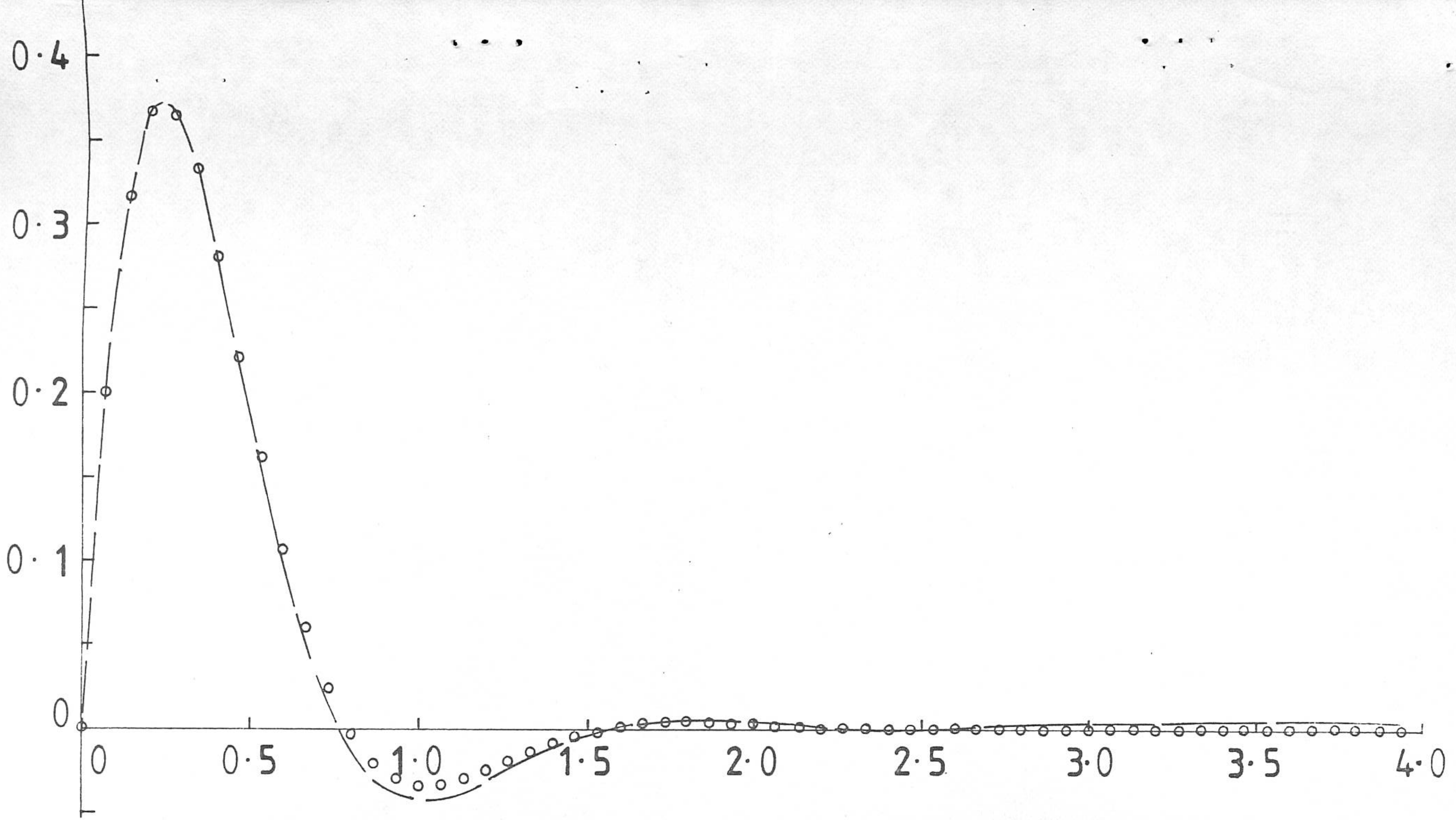

Fig. 4 (a) o o o theoretical response $h_{1,2}(k)$

- - - estimated values $\hat{h}_{1,2}(k)$

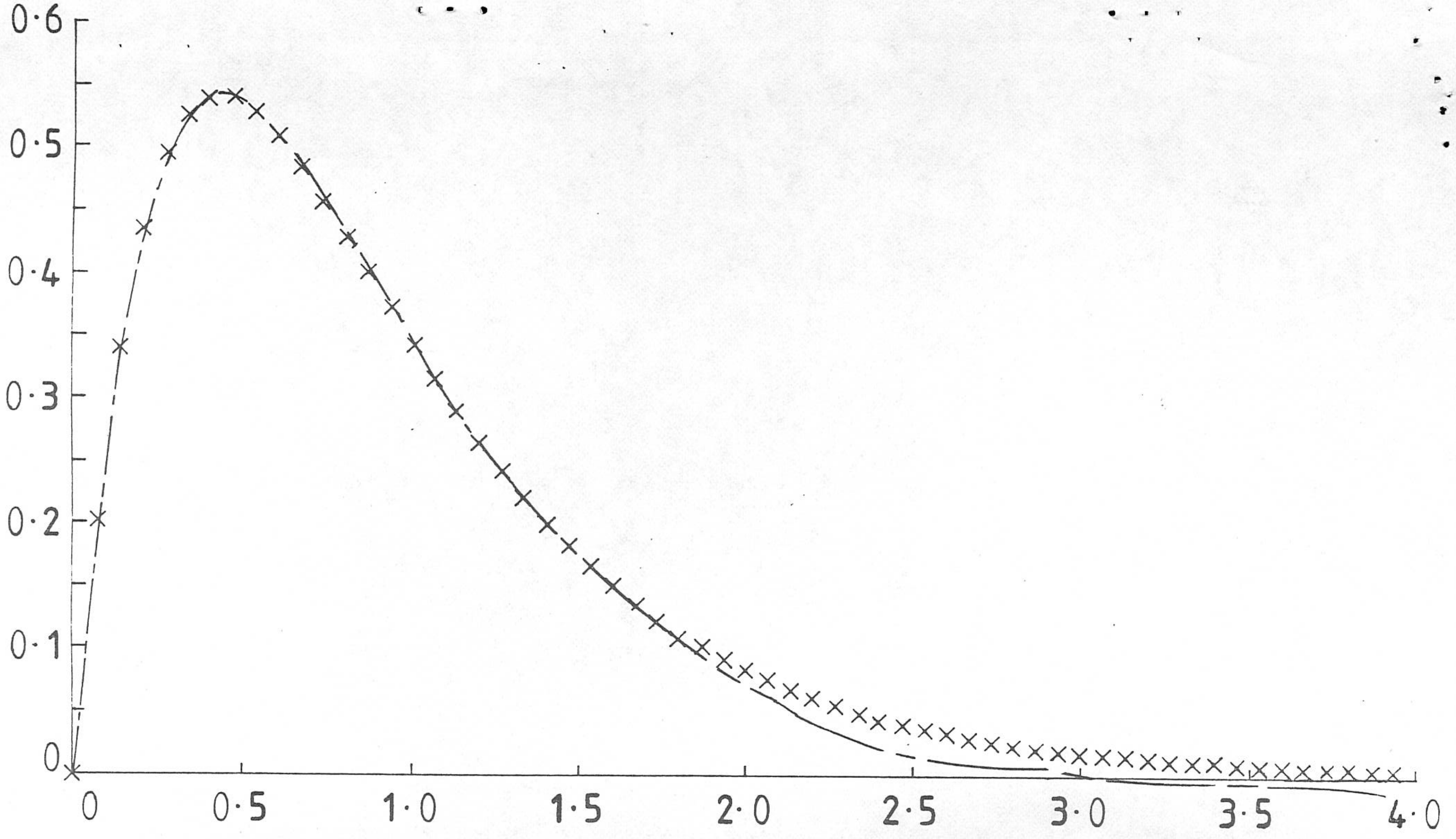

Fig. 4 (b) x x x theoretical response $h_{2;2}(k)$

— - — estimated values $\hat{h}_{2,2}(k)$

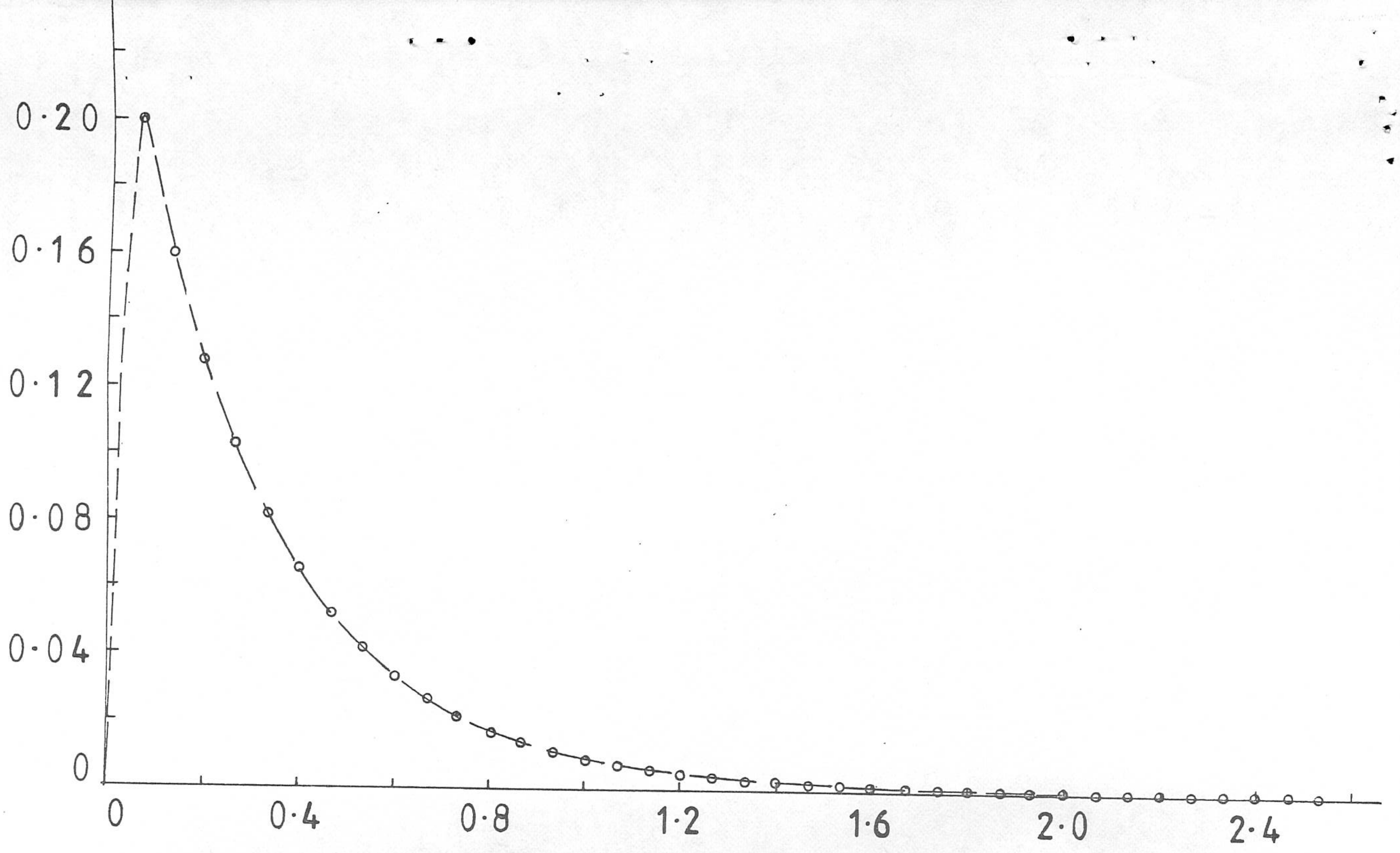

Fig. 5 (a) o o o theoretical response $h_{1,2}(k)$

--- estimated values $\hat{h}_{1,2}(k)$

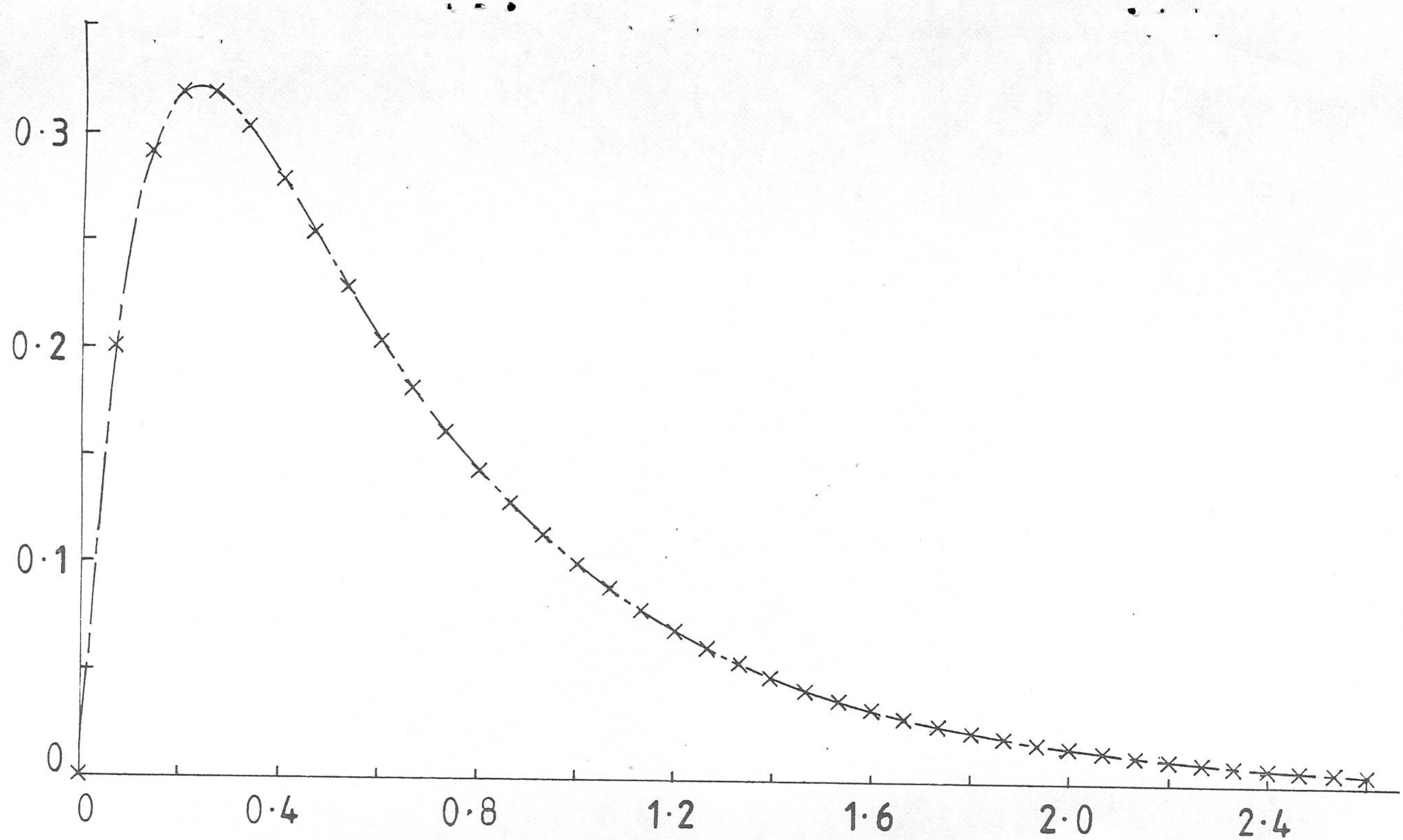

Fig. 5 (b) x x x theoretical response $h_{2,2}(k)$

— - — estimated values $\hat{h}_{2,2}(k)$

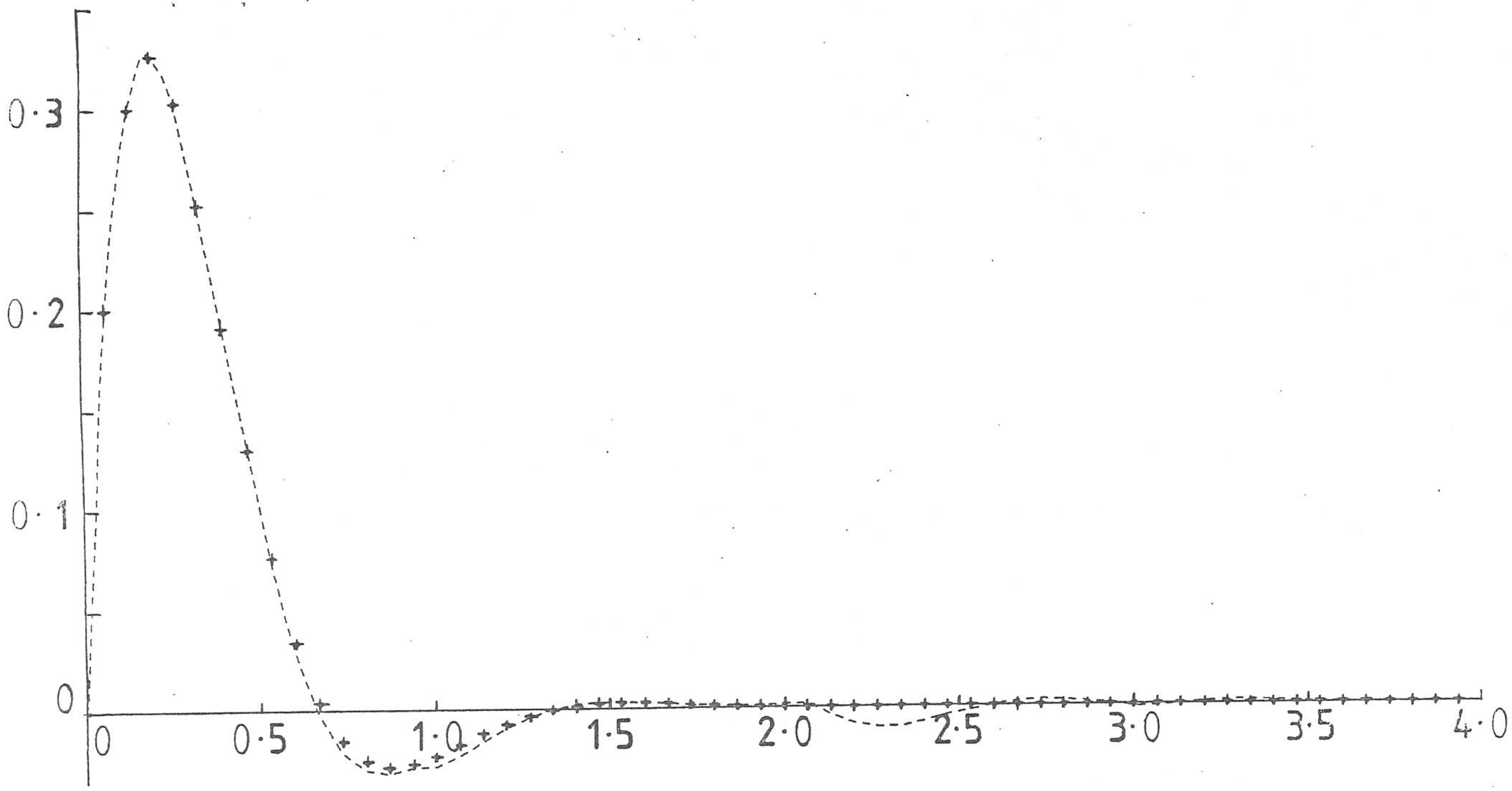

Fig. 6 (a) + + + theoretical response $h_{1,1}(k)$

- - - estimated values $\hat{h}_{1,1}(k)$

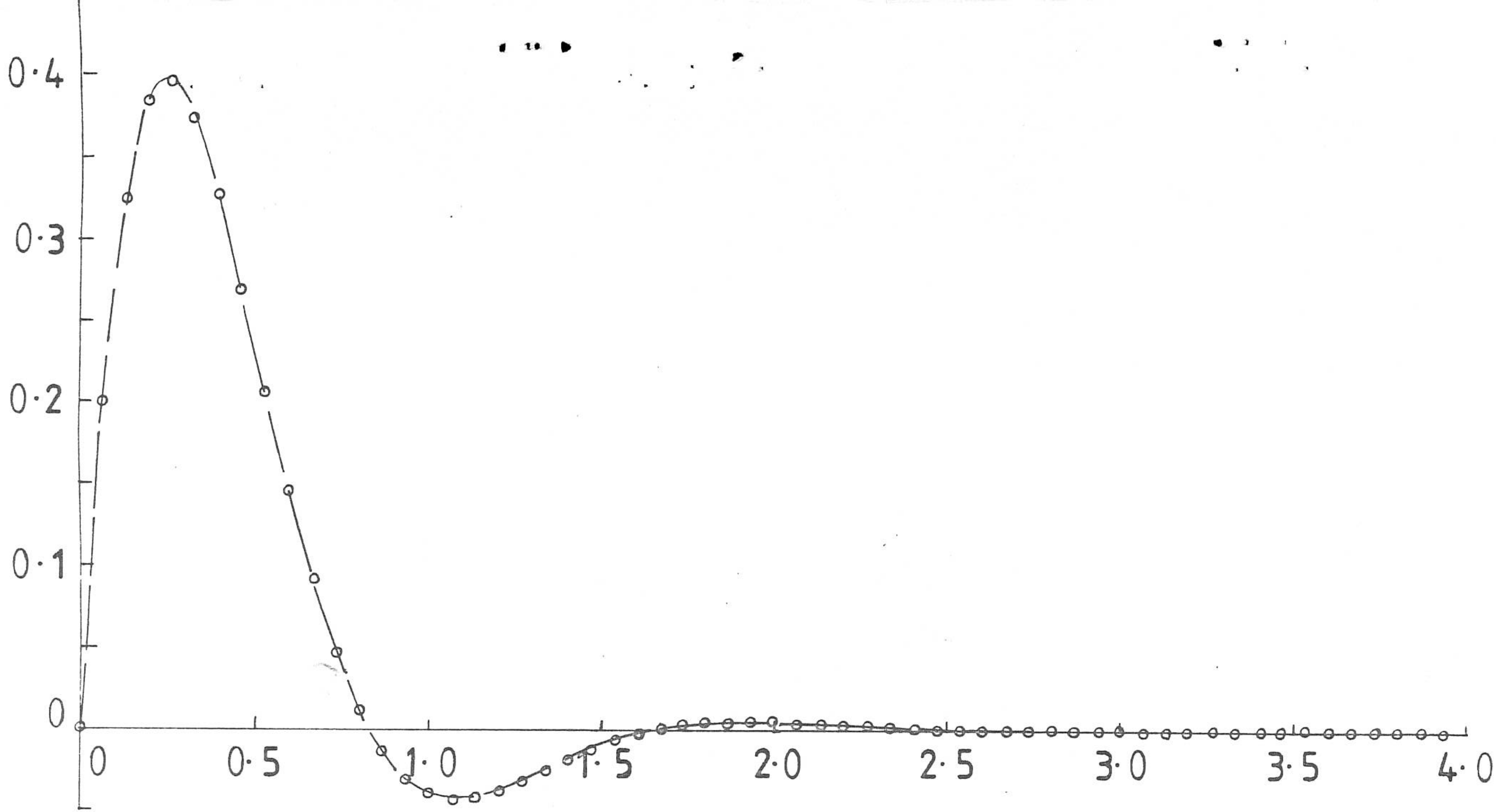

Fig. 6 (b) o o o theoretical response $h_{1,2}(k)$

— — estimated values $\hat{h}_{1,2}(k)$

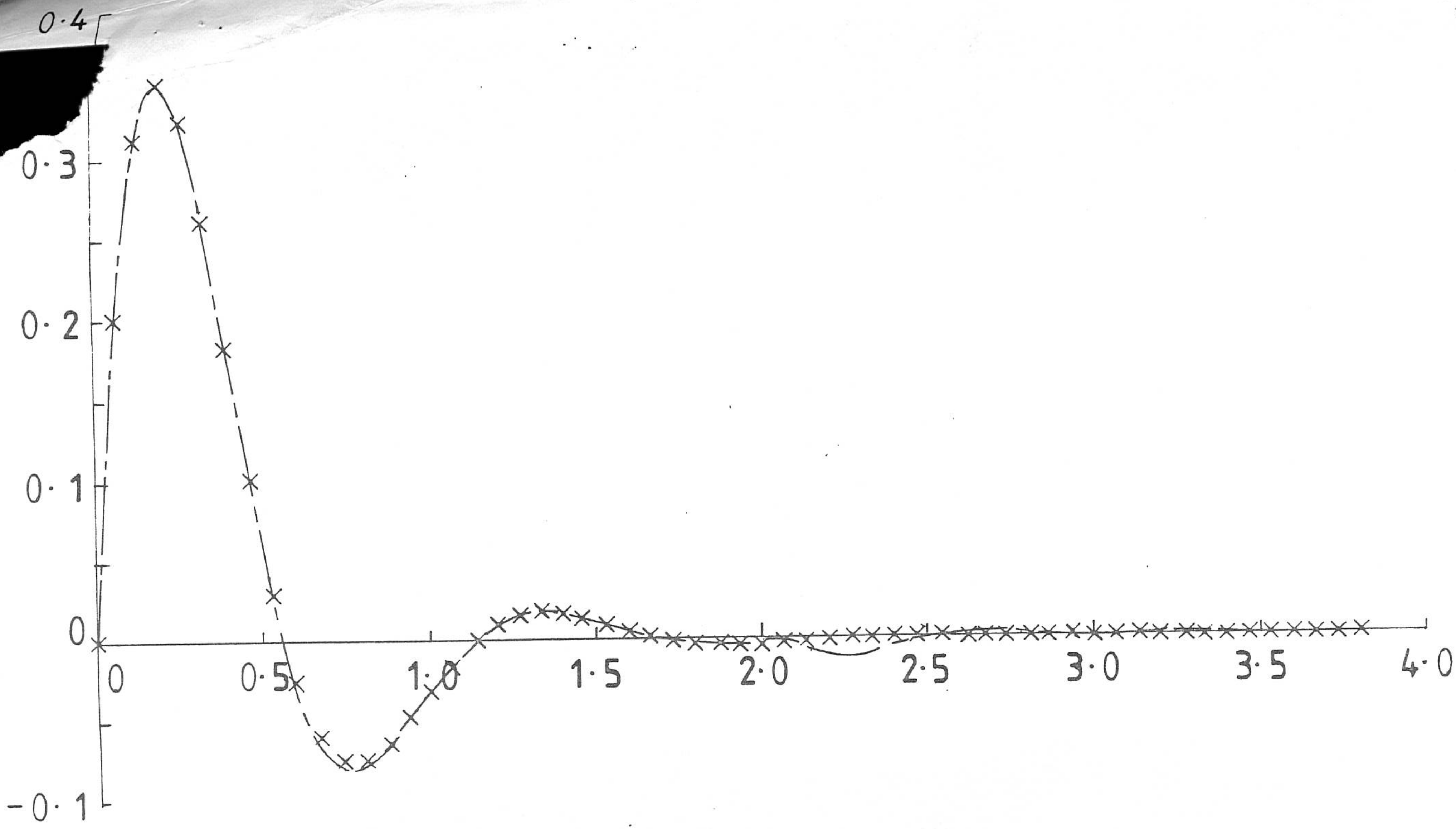

Fig. 6 (c) x x x theoretical response $h_{2,2}(k)$

— - — estimated values $\hat{h}_{2,2}(k)$