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SECOND QUANTISATION FOR SKEW CONVOLUTION PRODUCTS OF INFINITELY DIVISIBLE MEASURES

DAVID APPLEBAUM AND JAN VAN NEERVEN

ABSTRACT.

1. INTRODUCTION

Let $E_i, i = 1, 2$ be Banach spaces equipped with Radon probability measures μ_1 and μ_2 , respectively. A Borel measurable mapping $T : E_1 \rightarrow E_2$ is called a *skew map* for the pair (μ_1, μ_2) if there exists a Radon probability measure ρ on E_2 so that μ_2 is the convolution of ρ with the image of μ_1 under the action of T . In this case we obtain a linear contraction $P_T : L^p(E_2, \mu_2) \rightarrow L^p(E_1, \mu_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y) \rho(dy).$$

Such constructions arise naturally in the study of Mehler semigroups, linear stochastic partial differential equations driven by additive Lévy noise and operator self-decomposable measures (see [2]). In this context, the problem of “second quantisation” is to find a functorial manner of expressing P_T in terms of T . The reason for this name is that the first work on this subject [3], within the context of Gaussian measures, exploited constructions that were similar to those that are encountered in the construction of the free quantum field from one-particle space (see e.g. [7]) wherein the n th chaos spanned by multiple Wiener-Itô integrals corresponds to the n -particle space within the Fock space decomposition. In our previous paper [2] we implemented this programme and constructed P_T as the second quantisation of T in the two cases where for $i = 1, 2, \mu_i$ are Gaussian (generalising [3] and [6]), and are infinitely divisible measures of pure jump type (generalising [8]). In this article, we complete the programme by dealing with the case where the μ_i 's are general infinitely divisible measures, and so are convolutions of the cases previously considered.

2. BACKGROUND

Let μ be an infinitely divisible Radon probability measure defined on a (separable) Banach space E . It is well-known that the generic such measure may be written as the convolution $\mu = \mu_G * \mu_P$ where μ_G is a Gaussian measure (see e.g. [5, 4]). In fact, it follows from the Lévy-Itô decomposition of [9] that μ may always be realised as the law of an E -valued random variable X defined on some probability space (Ω, \mathcal{F}, P) for which $X = X_1 + X_2$, where the summands X_1 and X_2 are independent. Here X_1 is Gaussian and has law μ_G , while X_2 is controlled by a Poisson random measure on \mathbb{E} whose intensity measure is a Lévy measure ν , and X_2 has law μ_P . From [2], we know that we can effectively realise the second

quantisation of twist maps of μ_G in the symmetric Fock space $\Gamma(H)$ of the reproducing kernel space H of μ_G which is naturally isomorphic to $L^2(E, \mu_G)$. To second quantise twist maps of μ_P , we use $L^2(E, \mu_P) \simeq \Gamma(L^2(E, \nu))$. To unify these two approaches we make use of the following:

$$\begin{aligned} L^2(E, \mu) &= L^2(E, \mu_G * \mu_P) \hookrightarrow L^2(E, \mu_G) \otimes L^2(E, \mu_P) \\ &\simeq \Gamma(H) \otimes \Gamma(L^2(E, \nu)) \simeq \Gamma(H \oplus L^2(E, \nu)). \end{aligned}$$

We give a more detailed account of these embeddings and isomorphisms in the sequel.

3. MAIN RESULT

Suppose μ is an infinitely divisible measure, say

$$\mu = \gamma * \Pi$$

with γ centred Gaussian and Π as in [2]¹. For a function $f \in L^2(\mu)$ let

$$F_f(x, y) := f(x + y).$$

Using the fact that $L^2(\gamma) \widehat{\otimes} L^2(\Pi) = L^2(\gamma \times \Pi)$ isometrically (with $\widehat{\otimes}$ indicating the Hilbert space tensor product) it is immediate to verify that

$$\|f\|_{L^2(\mu)}^2 = \int_E \int_E |f(x + y)|^2 d\gamma(x) d\Pi(y) = \|F_f\|_{L^2(\gamma) \widehat{\otimes} L^2(\Pi)}^2.$$

As a result the mapping $f \mapsto F_f$ is an isometry from $L^2(\mu)$ into $L^2(\gamma) \widehat{\otimes} L^2(\Pi)$. This brings us to the setting with independence structure as discussed in [1]. Following that reference, on the algebraic tensor product $L^2(\gamma) \otimes L^2(\Pi)$ we define

$$D := D_\gamma \otimes I + I \otimes D_\Pi,$$

where we denote the ‘Gaussian’ and the ‘pure jump’ derivatives with subscripts γ and Π , respectively.

Consider the Hilbert spaces

$$\mathcal{H}_n := \bigoplus_{\substack{j, k \geq 0 \\ j+k=n}} H^{\otimes j} \widehat{\otimes} L^2(\nu)^{\otimes k}.$$

Then,

$$\begin{aligned} L^2(\gamma \times \Pi) &= L^2(\gamma) \widehat{\otimes} L^2(\Pi) = \left(\bigoplus_{j=0}^{\infty} H^{\otimes j} \right) \widehat{\otimes} \left(\bigoplus_{k=0}^{\infty} L^2(\nu)^{\otimes k} \right) \\ &= \bigoplus_{n=0}^{\infty} \left(\bigoplus_{\substack{j, k \geq 0 \\ j+k=n}} H^{\otimes j} \widehat{\otimes} L^2(\nu)^{\otimes k} \right) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \end{aligned}$$

may be viewed as the associated Wiener-Itô decomposition. We define the n -fold stochastic integral on $I_n : \mathcal{H}_n \rightarrow L^2(\Omega)$ by

$$I_n(f \otimes g) := I_{j, \gamma} f \otimes I_{k, \Pi} g$$

for $f \in H^{\otimes j}$ and $g \in L^2(\nu)^{\otimes k}$ with $j + k = n$, where we denote the ‘Gaussian’ and the ‘pure jump’ integrals with subscripts γ and Π , respectively.

¹Be more precise

In what follows, in order to tidy up the notation we will refrain from writing subscripts γ and Π ; expectations taken in the the left and right sides of tensor products refer to γ and Π , respectively.

Proposition 3.1. *For all $F \in L^2(\gamma \times \Pi)$,*

$$F = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E}(D^m F)).$$

Proof. Let $F = f \otimes g$ with $f \in H^{\otimes j}$ and $g \in L^2(\nu)^{\otimes k}$. By Leibniz's rule,

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{1}{m!} I_m \mathbb{E} D^m F &= \sum_{m=0}^{\infty} \frac{1}{m!} I_m \left(\mathbb{E} \sum_{\ell=0}^m \binom{m}{\ell} D^\ell f \otimes D^{m-\ell} g \right) \\ &= \sum_{m=0}^{\infty} \sum_{\ell=0}^m \frac{1}{\ell!(m-\ell)!} I_m \left(\mathbb{E}(D^\ell f \otimes D^{m-\ell} g) \right) \\ &= \sum_{m=0}^{\infty} \sum_{\ell=0}^m \frac{1}{\ell!(m-\ell)!} I_{\ell, \gamma}(\mathbb{E} D^\ell f) \otimes I_{m-\ell, \Pi}(D^{m-\ell} g) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} I_{j, \gamma}(\mathbb{E} D^j f) \otimes \sum_{k=0}^{\infty} \frac{1}{k!} I_{k, \Pi}(\mathbb{E} D^k g), \\ &= f \otimes g \\ &= F \end{aligned}$$

using the Last-Penrose type decompositions for γ and Π in the second last identity. \square

Suppose now that two measures μ_1 and μ_2 are given as above, on Banach spaces E_1 and E_2 , respectively, say $\mu_i = \gamma_i * \Pi_i$ for $i = 1, 2$. Let $T : E_1 \rightarrow E_2$ be a linear skew mapping with respect to both (γ_1, γ_2) and (Π_1, Π_2) with skew factors ρ_γ and ρ_Π . Recall that this means that $T\gamma_1 * \rho_\gamma = \gamma_2$ and $T\Pi_1 * \rho_\Pi = \Pi_2$.

Set $\rho := \rho_\gamma * \rho_\Pi$.

Lemma 3.2. *Under these assumptions, T is skew with respect to (μ_1, μ_2) with skew factor ρ .*

Proof. Since for any two measures on E_1 one has $T(\nu_1 * \nu_2) = (T\nu_1) * (T\nu_2)$, this follows from

$$T\mu_1 * (\rho_\gamma * \rho_\Pi) = (T\gamma_1 * T\Pi_1) * (\rho_\gamma * \rho_\Pi) = (T\gamma_1 * \rho_\gamma) * (T\Pi_1 * \rho_\Pi) = \gamma_2 * \Pi_2 = \mu_2. \quad \square$$

It follows from the lemma that we may define $P_T : L^2(E_2, \mu_2) \rightarrow L^2(E_1, \mu_1)$ by

$$P_T f(x) := \int_{E_2} f(Tx + y) d\rho(y), \quad x \in E_1,$$

where ρ is the skew factor on E_2 , i.e., $T\mu_1 * \rho = \mu_2$. Similarly we can define an operator $P_T \otimes P_T : L^2(\gamma_2) \otimes L^2(\Pi_2) \rightarrow L^2(\gamma_1) \otimes L^2(\Pi_1)$ in the obvious way (with an apology for the abuse of notation) and we then have:

Lemma 3.3. *Under the above assumptions, $F_{P_T f} = (P_T \otimes P_T)F_f$.*

Proof. For $(\gamma \times \Pi)$ -almost all $x, y \in E_2$ we have

$$\begin{aligned} (P_T \otimes P_T)(\phi \otimes \psi)(x, y) &= (P_T \phi \otimes P_T \psi)(x, y) \\ &= \int_{E_2} \phi(Tx + z) d\rho_\gamma(z) \int_{E_2} \psi(Ty + z) d\rho_\Pi(z) \\ &= \int_{E_2} \int_{E_2} (\phi \otimes \psi)(Tx + z_1, Ty + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2). \end{aligned}$$

Now suppose that $G_n = F_f$ in $L^2(\gamma \times \Pi)$, where each f_n belongs to the algebraic tensor product $L^2(\gamma) \otimes L^2(\Pi)$. By the above identity and linearity it follows, after passing to a subsequence if necessary, that for $(\gamma \times \Pi)$ -almost all $x, y \in E_2$ we have

$$\begin{aligned} (P_T \otimes P_T)F_f(x, y) &= \lim_{n \rightarrow \infty} (P_T \otimes P_T)G_n(x, y) \\ &= \lim_{n \rightarrow \infty} \int_{E_2} \int_{E_2} G_n(Tx + z_1, Ty + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2) \\ &= \int_{E_2} \int_{E_2} F_f(Tx + z_1, Ty + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2) \\ &= \int_{E_2} \int_{E_2} f(Tx + Ty + z_1 + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2) \\ &= \int_{E_2} f(Tx + Ty + z) d(\rho_\gamma * \rho_\Pi)(z) \\ &= \int_{E_2} f(Tx + Ty + z) d\rho(z) \\ &= P_T f(x + y) \\ &= F_{P_T}(x, y). \end{aligned}$$

□

For $h \in H$ and $y_1, \dots, y_n \in E$ and $h \in H$ we define

$$D_{h; y_1, \dots, y_n} := D_h \otimes I + I \otimes D_{y_1, \dots, y_n},$$

Lemma 3.4. For all $f \in L^2(E_2, \mu_2)$, $h \in H$, and $y_1, \dots, y_n \in E_1$,

$$(3.1) \quad \mathbb{E}_{\gamma_1 \times \Pi_1} D_{h; y_1, \dots, y_n}^n F_{P_T f} = \mathbb{E}_{\gamma_2 \times \Pi_2} D_{Th; Ty_1, \dots, Ty_n}^n F_f.$$

Proof. We approximate F_f by finite sums of elementary tensors as in the proof of the previous lemma. For such functions G_n the identity follows from the results in [2] for the Gaussian and Poissonian case.

Take care of details, closedness argument needed?

Our D 's are unbounded. □

Can we define a derivative D in $L^2(\mu)$ satisfying the requirement

$$E_\mu D^n f = E_{\gamma_1 \times \Pi_1} D^n F_f ?$$

(On the right, this is the D defined previously on $L^2(\gamma) \otimes L^2(\Pi)$, extended (by closability? check) to a closed operator on $L^2(\gamma \times \Pi)$). That would clean up the lemma as well as the commuting diagram.

For Hilbert spaces H and \underline{H} we note that

$$\Gamma(H, \oplus \underline{H}) = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{\substack{j,k \geq 0 \\ j+k=n}} H^{\otimes j} \widehat{\otimes} \underline{H}^{\otimes k} \right).$$

Theorem 3.5. *Putting everything together, under the above assumptions the following diagram commutes:*

$$\begin{array}{ccc} L^2(E_2, \mu_2) & \xrightarrow{P_T} & L^2(E_1, \mu_1) \\ f \mapsto F_f \downarrow & & \downarrow f \mapsto F_f \\ L^2(\gamma_2 \times \Pi_2) & \xrightarrow{P_T \otimes P_T} & L^2(\gamma_1 \times \Pi_1) \\ \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_{\gamma_2 \times \Pi_2} \tilde{D}^n \downarrow & & \downarrow \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_{\gamma_1 \times \Pi_1} \tilde{D}^n \\ \Gamma(L^2(E_2, \nu_2) \oplus H_2) & \xrightarrow{\bigoplus_{n=0}^{\infty} (T^*)^{\otimes n}} & \Gamma(L^2(E_1, \nu_1) \oplus H_1) \end{array}$$

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E-mail address: D.Applebaum@sheffield.ac.uk

SCHOOL OF MATHEMATICS AND STATISTICS,, UNIVERSITY OF SHEFFIELD,, SHEFFIELD S3 7RH, UNITED KINGDOM.

E-mail address: J.M.A.M.vanNeerven@tudelft.nl

DELFT INSTITUTE OF APPLIED MATHEMATICS,, DELFT UNIVERSITY OF TECHNOLOGY,, PO BOX 5031, 2600 GA DELFT,, THE NETHERLANDS.