Article:

Chudnovsky, M, Trotignon, N, Trunck, T et al. (2015) Coloring perfect graphs with no balanced skew-partitions. Journal of Combinatorial Theory, Series B, 115. pp. 26-65. ISSN: 0095-8956

https://doi.org/10.1016/j.jctb.2015.04.007

Coloring perfect graphs with no balanced skew-partitions

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April 21, 2015

Abstract

We present an $O(n^5)$ algorithm that computes a maximum stable set of any perfect graph with no balanced skew-partition. We present $O(n^7)$ time algorithm that colors them.

1 Introduction

A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. In the 1980's, Gröstchel, Lovász, and Schrijver [10] described a polynomial time algorithm that colors any perfect graph. A graph is *Berge* if none of its induced subgraphs, and none of the induced subgraphs of its complement, is an odd chordless cycle on at least five vertices. Berge [2] conjectured in the 1960s that a graph is Berge if and only if it is perfect. This

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The second and third authors are partially supported by ANR project Stint under reference ANR-13-BS02-0007 and by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program Investissements d'Avenir (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR). The three last authors are partially supported by PHC Pavle Savić grant 2010-2011, jointly awarded by EGIDE, an agency of the French Ministère des Affaires étrangères et européennes, and Serbian Ministry of Education and Science.

was proved in 2002 by Chudnovsky, Robertson, Seymour and Thomas [7]. Their proof relies on a *decomposition theorem*: every Berge graph is either in some simple basic class, or has some kind of decomposition. In 2002, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [6] described a polynomial time algorithm that decides whether any input graph is Berge. The method used in [10] to color perfect graphs (or equivalently by [7], Berge graphs) is based on the ellipsoid method, and so far no purely combinatorial method is known. In particular, it is not known whether the decomposition theorem from [7] may be used to color Berge graphs in polynomial time.

This question contains several potentially easier questions. Since the decomposition theorem has several outcomes, one may wonder separately for each of them whether it is helpful for coloring. The basic graphs are all easily colorable, so the problem is with the decompositions. One of them, namely the balanced skew-partition, seems to be hopeless. The other ones (namely the 2-join, the complement 2-join and the homogeneous pair) seem to be more useful for coloring and we now explain the first step in this direction. Chudnovsky [5, 4] proved a decomposition theorem for Berge graphs that is more precise than the theorem from [7]. Based on this theorem, Trotignon [19] proved an even more precise decomposition theorem, that was used by Trotignon and Vušković [20] to devise a polynomial algorithm that colors Berge graphs with no balanced skew-partition, homogeneous pair nor complement 2-join. This algorithm focuses on the 2-join decompositions. Here, we strengthen this result by constructing a polynomial time algorithm that colors Berge graphs with no balanced skew-partition.

Our algorithm is based directly on [5, 4], and a few results from [19] and [20] are used. It should be pointed out that the method presented here is significantly simpler and shorter than [20], while proving a more general result. This improvement is mainly due to the use of *trigraphs*, that are graphs where some edges are left "undecided". This notion introduced by Chudnovsky [5, 4] helps a lot to handle inductions, especially when several kinds of decompositions appear in an arbitrary order.

It is well known that an $O(n^k)$ algorithm that computes a maximum weighted stable set for a class of perfect graphs closed under complementation, yields an $O(n^{k+2})$ algorithm that computes an optimal coloring. See for instance [13], [18] or Section 8 below. This method, due to Gröstchel, Lovász, and Schrijver, is quite effective and combinatorial. Hence, from here on we just focus on an algorithm that computes a maximum weighted stable set. Also, in what follows, in order to keep the paper as readable as possible, we construct an algorithm that computes the weight of a maximum weighted stable set, but does not output a set. However, all our methods are clearly constructive, so our algorithm may easily be turned into an algorithm that actually computes the desired stable set.

Our algorithm may easily be turned into a *robust* algorithm, that is an algorithm that takes any graph as an input, and outputs either a stable set on k vertices and a partition of $V(G)$ into k cliques of G (so, a coloring of the complement), or some polynomial size certificate proving that G is not "Berge-with-no-balanced-skew-partition". In the first case, we know by the duality principle that the stable set is a maximum one, and the clique cover is an optimal one, even if the input graph is not in the class "Berge-with-no-balanced-skew-partition". This feature is interesting, because our algorithm is faster than the fastest one (so far) for recognizing Berge graphs (Berge graphs can be recognized in $O(n^9)$ time [6], and determining whether a Berge graph has a balanced skew partition can be done in $O(n^5)$ time [19, 3]). However, it should be pointed out that the certificate is not just and odd hole or antihole, or a balanced skew-partition. If the algorithm fails to find a stable set at some point, the certificate is a decomposition tree, one leaf of which satisfies none of the outputs of the decomposition theorem for Berge graphs with no balanced skew-partitions; and if the algorithm fails to find a clique cover, the certificate is a matrix on $n + 1$ rows showing that the graph is not perfect (see Section 8). This really certifies that a graph is not in our class, but is maybe not as desirable as a hole, antihole, or balanced skew-partition.

In Section 2, we give all the definitions and state some known results. In Section 3, we define a new class of Berge trigraphs called $\mathcal{F}$, and we prove a decomposition theorems for trigraphs from $\mathcal{F}$. In Section 4, we define *blocks* of decomposition. In Section 5, we show how to recognize all basic trigraphs, and find maximum weighted stable sets for them. In Section 6, we describe blocks of decomposition that allow us to compute the maximum weight of a weighted stable set. In Section 7, we give the main algorithm for computing the maximum weight of a stable set in time $O(n^5)$.

Results in the next sections are not needed to prove our main result. We include them because they are of indepedent interest (while on the same subject). In Section 8, we describe the classical algorithm that colors a perfect graph with a stable set oracle. We include it because it is hard to extract it from the deeper material that surrounds it in [10] or [13]. In Section 9, we show that Berge trigraphs with no balanced skew partitions admit extreme decompositions, that are decompositions one block of decomposition of which is a basic trigraph. We do not need them here, but extreme decompositions are sometimes very useful, in particular to prove properties by induction. In Section 10 we give an algorithm for finding extreme decompositions in a trigraph (if any). In Section 11, we state several open questions about how this work could be generalized to larger classes of graphs.

We now state our main result (the formal definitions are given in the next section, and the proof at the end of Section 8). In complexity of algorithms, n stands for the number of the vertices of the input graph.

1.1 *There is an $O(n^7)$ time algorithm that colors any Berge graph with no balanced skew-partition.*

2 Trigraphs

For a set X, we denote by $\binom{X}{2}$ the set of all subsets of X of size 2. For brevity of notation an element $\{u, v\}$ of $\binom{X}{2}$ is also denoted by uv or vu. A *trigraph* T consists of a finite set $V(T)$, called the *vertex set* of T, and a map $\theta : \binom{V(T)}{2} \longrightarrow \{-1, 0, 1\}$, called the *adjacency function*.

Two distinct vertices of T are said to be *strongly adjacent* if $\theta(uv) = 1$, *strongly antiadjacent* if $\theta(uv) = -1$, and *semiadjacent* if $\theta(uv) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent, or semiadjacent; and *antiadjacent* if they are either strongly antiadjacent, or semiadjacent. An *edge* (*antiedge*) is a pair of adjacent (antiadjacent) vertices. If u and v are adjacent (antiadjacent), we also say that u is *adjacent (antiadjacent) to* v, or that u is a *neighbor (antineighbor)* of v. Similarly, if u and v are strongly adjacent (strongly antiadjacent), then u is a *strong neighbor (strong antineighbor)* of v. Let $\eta(T)$ be the set of all strongly adjacent pairs of T, $\nu(T)$ the set of all strongly antiadjacent pairs of T, and $\sigma(T)$ the set of all semiadjacent pairs of T. Thus, a trigraph T is a graph if $\sigma(T)$ is empty. A pair $\{u, v\} \subseteq V(T)$ of distinct vertices is a *switchable pair* if $\theta(uv) = 0$, a *strong edge* if $\theta(uv) = 1$ and a *strong antiedge* if $\theta(uv) = -1$. An edge uv (antiedge, strong edge, strong antiedge, switchable pair) is *between* two sets $A \subseteq V(T)$ and $B \subseteq V(T)$ if $u \in A$ and $v \in B$ or if $u \in B$ and $v \in A$.

Let T be a trigraph. The *complement* $\overline{T}$ of T is a trigraph with the same vertex set as T, and adjacency function $\overline{\theta} = -\theta$. For $v \in V(T)$, let $N(v)$ denote the set of all vertices in $V(T) \setminus \{v\}$ that are adjacent to v. Let $A \subset V(T)$ and $b \in V(T) \setminus A$. We say that b is *strongly complete* to A if b is strongly adjacent to every vertex of A; b is *strongly anticomplete* to A if b is strongly antiadjacent to every vertex of A; b is *complete* to A if b is adjacent to every vertex of A; and b is *anticomplete* to A if b is antiadjacent to every vertex of A. For two disjoint subsets A, B of $V(T)$, B is *strongly complete (strongly anticomplete, complete, anticomplete)* to A if every vertex of B is strongly complete (strongly anticomplete, complete, anticomplete) to A. A set of vertices $X \subseteq V(T)$ *dominates (strongly dominates)* T if for all $v \in V(T) \setminus X$, there exists $u \in X$ such that v is adjacent (strongly adjacent) to u.

A *clique* in T is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strongly stable set* is a set of vertices all pairwise strongly antiadjacent. For $X \subset V(T)$ the trigraph *induced by* T *on* X (denoted by $T|X$) has vertex set X, and adjacency function that is the restriction of θ to $\binom{X}{2}$. Isomorphism between trigraphs is defined in the natural way, and for two trigraphs T and H we say that H is an *induced subtrigraph* of T (or T *contains* H *as an induced subtrigraph*) if H is isomorphic to $T|X$ for some $X \subseteq V(T)$. Since in this paper we are only concerned with the induced subtrigraph containment relation, we say that T *contains* H if T contains H as an induced subtrigraph. We denote by $T \setminus X$ the trigraph $T|(V(T) \setminus X)$.

Let T be a trigraph. A *path* P of T is a sequence of distinct vertices $p_1, \ldots, p_k$ such that either $k = 1$, or for $i, j \in \{1, \ldots, k\}$, p_i is adjacent to p_j if $|i - j| = 1$ and p_i is antiadjacent to p_j if $|i - j| > 1$. Under these circumstances, $V(P) = \{p_1, \ldots, p_k\}$ and we say that P is a path *from* p_1 *to* p_k, its *interior* is the set $P^* = V(P) \setminus \{p_1, p_k\}$, and the *length* of P is $k - 1$. We also say that P is a $(k-1)$*-edge-path*. Sometimes, we denote P by p_1-$\cdots$-p_k. Observe that, since a graph is also a trigraph, it follows that a path in a graph, the way we have defined it, is what is sometimes in literature called a chordless path.

A *hole* in a trigraph T is an induced subtrigraph H of T with vertices $h_1, \ldots, h_k$ such that $k \geq 4$, and for $i, j \in \{1, \ldots, k\}$, h_i is adjacent to h_j if $|i-j| = 1$ or $|i-j| = k-1$; and h_i is antiadjacent to h_j if $1 < |i-j| < k-1$. The *length* of a hole is the number of vertices in it. Sometimes we denote H by h_1-$\cdots$-h_k-h_1. An *antipath* (*antihole*) in T is an induced subtrigraph of T whose complement is a path (hole) in $\overline{T}$.

A *semirealization* of a trigraph T is any trigraph T' with vertex set $V(T)$ that satisfies the following: for all $uv \in \binom{V(T)}{2}$, if $uv \in \eta(T)$ then $uv \in \eta(T')$, and if $uv \in \nu(T)$ then $uv \in \nu(T')$. Sometimes we will describe a semirealization of T as an *assignment of values* to switchable pairs of T, with three possible values: "strong edge", "strong antiedge" and "switchable pair". A *realization* of T is any graph that is semirealization of T (so, any semirealization where all switchable pairs are assigned the value "strong edge" or "strong antiedge"). For $S \subseteq \sigma(T)$, we denote by G^T_S the realization of T with edge set $\eta(T) \cup S$, so in G^T_S the switchable pairs in S are assigned the value "edge", and those in $\sigma(T) \setminus S$ the value "antiedge". The realization $G^T_{\sigma(T)}$ is called the *full realization* of T.

Let T be a trigraph. For $X \subseteq V(T)$, we say that X and $T|X$ are *connected* (*anticonnected*) if the graph $G^{T|X}_{\sigma(T|X)}$ ($\overline{G^{T|X}_{\emptyset}}$) is connected. A *connected component* (or simply *component*) of X is a maximal connected subset of X, and an *anticonnected component* (or simply *anticomponent*) of X is a maximal anticonnected subset of X.

A trigraph T is *Berge* if it contains no odd hole and no odd antihole. Therefore, a trigraph is Berge if and only if its complement is. We observe that T is Berge if and only if every realization (semirealization) of T is Berge.

2.1 Basic trigraphs

A trigraph T is *bipartite* if its vertex set can be partitioned into two strongly stable sets. Every realization of a bipartite trigraph is a bipartite graph, and hence every bipartite trigraph is Berge, and so is the complement of a bipartite trigraph.

A trigraph T is a *line trigraph* if the full realization of T is the line graph of a bipartite graph and every clique of size at least 3 in T is a strong clique. The following is an easy fact about line trigraphs.

2.1 *If T is a line trigraph, then every realization of T is a line graph of a bipartite graph. Moreover, every semirealization of T is a line trigraph.*

Proof. From the definition, the full realization G of T is a line graph of a bipartite graph R. Let $S \subseteq \sigma(T)$. Define R_S as follows. For every $xy \in \sigma(T) \setminus S$, let v_{xy} be the common end of x and y in R. Then v_{xy} has degree 2 in R because every clique of size at least 3 in T is a strong clique. Let a_{xy} and b_{xy} be its neighbors. Now remove v_{xy} from R, and replace it by two new vertices, u_{xy}, w_{xy} such that u_{xy} is only adjacent to a_{xy}, and w_{xy} to b_{xy}. Then R_S is bipartite and G^T_S is the line graph of R_S. Hence, the first statement holds and the second follows (because the full realization of a semirealization is a realization). ▮

Note that this implies that every line trigraph is Berge and so is the complement of a line trigraph. Let us now define the trigraph analogue of the double split graph (first defined in [7]), namely the *doubled trigraph*. A *good partition* of a trigraph T is a partition (X, Y) of $V(T)$ (possibly, $X = \emptyset$ or $Y = \emptyset$) such that:

- Every component of $T|X$ has at most two vertices, and every anticomponent of $T|Y$ has at most two vertices.

- No switchable pair of T meets both X and Y.
- For every component C_X of $T|X$, every anticomponent C_Y of $T|Y$, and every vertex v in $C_X \cup C_Y$, there exists at most one strong edge and at most one strong antiedge between C_X and C_Y that is incident to v.

A trigraph is *doubled* if it has a good partition. Doubled trigraphs could also be defined as induced subtrigraphs of double split trigraphs (see [4] for a definition of double split trigraphs, we do not need it here). Note that doubled trigraphs are closed under taking induced subtrigraphs and complements (because (X, Y) is a good partition of some trigraph T if and only if (Y, X) is a good partition of $\overline{T}$). A *doubled graph* is any realization of a doubled trigraph. We now show that:

2.2 *If T is a doubled trigraph, then every realization of T is a doubled graph. Moreover, every semirealization of T is a doubled trigraph.*

Proof. The statement about realizations is clear from the definition. Let T be a doubled trigraph, and (X, Y) a good partition of T. Let T' be a semirealization of T. It is easy to see that (X, Y) is also a good partition for T' (for instance, if a switchable pair ab of $T|X$ is assigned value "antiedge", then $\{a\}$ and $\{b\}$ become components of $T'|X$, but they still satisfy the requirement in the definition of a good partition). This proves the statement about semirealizations. ∎

Note that this implies that every doubled trigraph is Berge, because every doubled graph is Berge. Note that doubled graphs could be defined equivalently as induced subgraphs of *double split graphs* (see [7] for a definition of double split graphs, we do not need the definition here).

A trigraph is *basic* if it is either a bipartite trigraph, the complement of a bipartite trigraph, a line trigraph, the complement of a line trigraph or a doubled trigraph. The following sums up the results of this subsection.

2.3 *Basic trigraphs are Berge, and are closed under taking induced subtrigraphs, semirealizations, realizations and complementation.*

2.2 Decompositions

We now describe the decompositions that we need to state the decomposition theorem. First, a *2-join* in a trigraph T is a partition (X_1, X_2) of $V(T)$ such that there exist disjoint sets $A_1, B_1, C_1, A_2, B_2, C_2 \subseteq V(T)$ satisfying:

- $X_1 = A_1 \cup B_1 \cup C_1$ and $X_2 = A_2 \cup B_2 \cup C_2$;
- A_1, A_2, B_1 and B_2 are non-empty;
- no switchable pair meets both X_1 and X_2;
- every vertex of A_1 is strongly adjacent to every vertex of A_2, and every vertex of B_1 is strongly adjacent to every vertex of B_2;
- there are no other strong edges between X_1 and X_2;
- for $i = 1, 2$ $|X_i| \geq 3$; and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$, then the full realization of $T|X_i$ is not a path of length two joining the members of A_i and B_i.

In these circumstances, we say that $(A_1, B_1, C_1, A_2, B_2, C_2)$ is a *split* of (X_1, X_2). The 2-join is *proper* if for $i = 1, 2$, every component of $T|X_i$ meets both A_i and B_i. Note that the fact that a 2-join is proper does not depend on the particular split that is chosen. A *complement* 2*-join* of a trigraph T is a 2-join of $\overline{T}$. More specifically, a *complement* 2*-join* of a trigraph T is a partition (X_1, X_2) of $V(T)$ such that (X_1, X_2) is a 2-join of $\overline{T}$; and $(A_1, B_1, C_1, A_2, B_2, C_2)$ is a *split* of this complement 2-join if it is a split of the respective 2-join in the complement, i.e. A_1 is strongly complete to $B_2 \cup C_2$ and strongly anticomplete to A_2, C_1 is strongly complete to X_2, and B_1 is strongly complete to $A_2 \cup C_2$ and strongly anticomplete to B_2.

2.4 *Let T be a Berge trigraph and $(A_1, B_1, C_1, A_2, B_2, C_2)$ a split of a proper 2-join of T. Then all paths with one end in A_i, one end in B_i and interior in C_i, for $i = 1, 2$, have lengths of the same parity.*

Proof. Otherwise, for $i = 1, 2$, let P_i be a path with one end in A_i, one end in B_i and interior in C_i, such that P_1 and P_2 have lengths of different parity. They form an odd hole, a contradiction. ∎

Our second decomposition is the balanced skew-partition. Let A, B be disjoint subsets of $V(T)$. We say the pair (A, B) is *balanced* if there is no odd path of length greater than 1 with ends in B and interior in A, and there is no odd antipath of length greater than 1 with ends in A and interior in B. A *skew-partition* is a partition (A, B) of $V(T)$ so that A is not connected and B is not anticonnected. A skew-partition (A, B) is *balanced* if the pair (A, B) is. Given a balanced skew-partition (A, B), (A_1, A_2, B_1, B_2) is a *split of* (A, B) if A_1, A_2, B_1 and B_2 are disjoint non-empty sets, $A_1 \cup A_2 = A$,

$B_1 \cup B_2 = B$, A_1 is strongly anticomplete to A_2, and B_1 is strongly complete to B_2. Note that for every balanced skew-partition, there exists at least one split.

The two decompositions we just described generalize some decompositions used in [7], and in addition all the "important" edges and non-edges in those graph decompositions are required to be strong edges and strong antiedges of the trigraph, respectively. We now state several technical lemmas.

A trigraph is called *monogamous* if every vertex of it belongs to at most one switchable pair. We are now ready to state the decomposition theorem for Berge monogamous trigraphs. This is Theorem 3.1 of [4].

2.5 *Let T be a monogamous Berge trigraph. Then one of the following holds:*

- *T is basic;*
- *T or $\overline{T}$ admits a proper 2-join; or*
- *T admits a balanced skew-partition.*

When (A, B) is a skew-partition of a trigraph T, we say that B is a *star cutset* of T if at least one anticomponent of B has size 1. The following is Theorem 5.9 from [5].

2.6 *If a Berge trigraph admits a star cutset, then it admits a balanced skew-partition.*

Let us say that X is a *homogeneous set* in a trigraph T if $1 < |X| < |V(T)|$, and every vertex of $V(T) \setminus X$ is either strongly complete or strongly anticomplete to X.

2.7 *Let T be a trigraph and let X be a homogeneous set in T, such that some vertex of $V(T) \setminus X$ is strongly complete to X, and some vertex of $V(T) \setminus X$ is strongly anticomplete to X. Then T admits a balanced skew-partition.*

Proof. Let A be the set of vertices of $V(T) \setminus X$ that are strongly anticomplete to X, and C the set of vertices of $V(T) \setminus X$ that are strongly complete to X. Let $x \in X$. Then $C \cup \{x\}$ is a star cutest of T (since A and $X \setminus \{x\}$ are non-empty and strongly anticomplete to each other), and so T admits a balanced skew-partition by 2.6. ▮

We also need the following (this is an immediate corollary of Theorem 5.13 in [5]):

2.8 *Let T be a Berge trigraph. Suppose that there is a partition of $V(T)$ into four nonempty sets X, Y, L, R, such that L is strongly anticomplete to R, and X is strongly complete to Y. If (L, Y) is balanced then T admits a balanced skew-partition.*

3 Decomposing trigraphs from $\mathcal{F}$

Let T be a trigraph, denote by $\Sigma(T)$ the graph with vertex set $V(T)$ and edge set $\sigma(T)$ (the switchable pairs of T). The connected components of $\Sigma(T)$ are called the *switchable components* of T. Let $\mathcal{F}$ be the class of Berge trigraphs such that the following hold:

- Every switchable component of T has at most two edges (and therefore no vertex has more than two neighbors in $\Sigma(T)$).
- Let $v \in V(T)$ have degree two in $\Sigma(T)$, denote its neighbors by x and y. Then either v is strongly complete to $V(T) \setminus \{v, x, y\}$ in T, and x is strongly adjacent to y in T (in this case we say that v and the switchable component that contains v are *heavy*), or v is strongly anticomplete to $V(T) \setminus \{v, x, y\}$ in T, and x is strongly antiadjacent to y in T (in this case we say that v and the switchable component that contains v are *light*).

Observe that $T \in \mathcal{F}$ if and only if $\overline{T} \in \mathcal{F}$; also v is light in T if and only if v is heavy in $\overline{T}$.

3.1 *Let T be a trigraph from $\mathcal{F}$ with no balanced skew-partition, and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of a 2-join (X_1, X_2) in T. Then the following hold:*

(i) (X_1, X_2) is a proper 2-join;

(ii) every vertex of X_i has a neighbor in X_i, $i = 1, 2$;

(iii) every vertex of A_i has an antineighbor in B_i, $i = 1, 2$;

(iv) every vertex of B_i has an antineighbor in A_i, $i = 1, 2$;

(v) every vertex of A_i has a neighbor in $C_i \cup B_i$, $i = 1, 2$;

(vi) *every vertex of* B_i *has a neighbor in* $C_i \cup A_i$*,* $i = 1, 2$*;*

(vii) *if* $C_i = \emptyset$*, then* $|A_i| \geq 2$ *and* $|B_i| \geq 2$*,* $i = 1, 2$*;*

(viii) $|X_i| \geq 4$*,* $i = 1, 2$*.*

Proof. Note that by 2.6, neither T nor $\overline{T}$ can have a star cutset.

To prove (i), we just have to prove that every component of $T|X_i$ meets both A_i and B_i, $i = 1, 2$. Suppose for a contradiction that some connected component C of $T|X_1$ does not meet B_1 (the other cases are symmetric). If there is a vertex $c \in C \setminus A_1$ then for any vertex $u \in A_2$, we have that $\{u\} \cup A_1$ is a star cutset that separates c from B_1, a contradiction. So, $C \subseteq A_1$. If $|A_1| \geq 2$ then pick any vertex $c \in C$ and $c' \neq c$ in A_1. Then $\{c'\} \cup A_2$ is a star cutset that separates c from B_1. So, $C = A_1 = \{c\}$. Hence, there exists some component of $T|X_1$ that does not meet A_1, so by the same argument as above we deduce $|B_1| = 1$ and the unique vertex of B_1 has no neighbor in X_1. Since $|X_1| \geq 3$, there is a vertex u in C_1. Now $\{c, a_2\}$ where $a_2 \in A_2$ is a star cutset that separates u from B_1, a contradiction.

To prove (ii), just notice that if some vertex in X_i has no neighbor in X_i, then it forms a component of $T|X_i$ that does not meet one of A_i, B_i. This is a contradiction to (i).

To prove (iii) and (iv), consider a vertex $a \in A_1$ strongly complete to B_1 (the other cases are symmetric). If $A_1 \cup C_1 \neq \{a\}$ then $B_1 \cup A_2 \cup \{a\}$ is a star cutset that separates $(A_1 \cup C_1) \setminus \{a\}$ from B_2. So $A_1 \cup C_1 = \{a\}$ and $|B_1| \geq 2$ because $|X_1| \geq 3$. But now B_1 is a homogeneous set, strongly complete to A_1 and strongly anticomplete to A_2, and so T admits a balanced skew-partition by 2.7, a contradiction.

To prove (v) and (vi), consider a vertex $a \in A_1$ strongly anticomplete to $C_1 \cup B_1$ (the other cases are symmetric). By (ii), a has a neighbor in A_1, and so $A_1 \neq \{a\}$. But now $\{a\} \cup B_1 \cup C_1 \cup B_2 \cup C_2$ is a star cutset in $\overline{T}$, a contradiction.

To prove (vii), suppose that $C_1 = \emptyset$ and $|A_1| = 1$ (the other cases are symmetric). By (iv) and (vi), and since $C_1 = \emptyset$, A_1 is both complete and anticomplete to B_1. This implies that the unique vertex of A_1 is semiadjacent to every vertex of B_1, and therefore, since $T \in \mathcal{F}, |B_1| \leq 2$. Since $|X_1| \geq 3$, we deduce that $|B_1| = 2$, and, since $T \in \mathcal{F}$, the unique vertex of A_1 is either strongly complete or strongly anticomplete to $V(T) \setminus (A_1 \cup B_1)$, which is a contradiction because A_1 is strongly complete to A_2 and strongly anticomplete to B_2.

To prove $(viii)$, we may assume by (vii) that $C_1 \neq \emptyset$, so suppose for a contradiction that $|A_1| = |C_1| = |B_1| = 1$. Let a, b, c be the vertices in

A_1, B_1, C_1 respectively. By (iii), ab is an antiedge. Also, c is adjacent to a, for otherwise, there is a star cutset centered at b that separates a from c. Similarly, c is adjacent to b. Since the full realization of $T|X_1$ is not a path of length 2 from a to b, we know that ab is a switchable pair. But this contradicts 2.4. ∎

Let b be a vertex of degree two in $\Sigma(T)$, and let a, c be the neighbors of b in $\Sigma(T)$. Assume that b is light. We call a vertex $w \in V(T) \setminus \{a, b, c\}$ an *a-appendage* of b if there exist $u, v \in V(T) \setminus \{a, b, c\}$ such that:

- a-u-v-w is a path;
- u is strongly anticomplete to $V(T) \setminus \{a, v\}$;
- v is strongly anticomplete to $V(T) \setminus \{u, w\}$; and
- w has no neighbors in $\Sigma(T)$ except possibly v (i.e. there is no switchable pair containing w in T except possibly vw).

A c-appendage is defined similarly. If b is a heavy vertex of T, then w is an a-appendage of b in T if and only if w is an a-appendage of b in $\overline{T}$.

The following is an analogue of 2.5 for trigraphs in $\mathcal{F}$. It can be easily deduced from [5], but for the reader's convenience we include a short proof, whose departure point is 2.5.

3.2 *Every trigraph in $\mathcal{F}$ is either basic, or admits a balanced skew-partition, a proper 2-join, or a proper 2-join in the complement.*

Proof. For $T \in \mathcal{F}$, let $\tau(T)$ be the number of vertices of degree two in $\Sigma(T)$. The proof is by induction on $\tau(T)$. If $\tau(T) = 0$, then the result follows from 2.5. Now let $T \in \mathcal{F}$ and let b be a vertex of degree two in $\Sigma(T)$. Let a, c be the two neighbors of b in $\Sigma(T)$. By passing to the complement if necessary, we may assume that b is light.

Let T' be the trigraph obtained from T by making a strongly adjacent to b. If b has no a-appendages, then no further changes are necessary; set $W = \emptyset$. Otherwise choose an a-appendage w of b, and let u, v be as in the definition of an a-appendage; set $V(T') = V(T) \setminus \{u, v\}$ and make a semiadjacent to w in T'; set $W = \{w\}$.

If $W = \emptyset$ then clearly $T' \in \mathcal{F}$ and $\tau(T) > \tau(T')$. Suppose that $W \neq \emptyset$. If $t \in V(T')$ is adjacent to both a and w, then a-u-v-w-t is an odd hole in T. Thus no vertex of T' is adjacent to both a and w. In particular, no antihole of length at least 7 of T' goes through a and w. Also, there is no odd hole that goes through a and w. Hence T' is in $\mathcal{F}$. Moreover, $\tau(T) > \tau(T')$ (we remind the reader that v is the only possible neighbor of w in $\Sigma(T)$).

Inductively, one of the outcomes of 3.2 holds for T'. We consider the following cases, and show that in each of them, one of the outcomes of 3.2 holds for T.

Case 1: T' is basic.

Suppose first that T' is bipartite. We claim that T is bipartite. Let $V(T') = X \cup Y$ where X and Y are disjoint strongly stable sets. The claim is clear if b has no a-appendage, so we may assume that $W = \{w\}$. We may assume that $a \in X$; then $w \in Y$. Then $X \cup \{v\}$ and $Y \cup \{u\}$ are strongly stable sets of T with union $V(T)$, and thus T is bipartite.

Suppose T' is a line trigraph. First observe that no clique of size at least three in T contains u, v or b. So, if $W = \emptyset$, then clearly T is a line trigraph. So assume that $W \neq \emptyset$. Note that the full realization of T is obtained from the full realization of T' by subdividing twice the edge aw. Since no vertex of T' is adjacent to both a and w, it follows that T is a line trigraph (because line graphs are closed under subdividing an edge whose ends have no common neighbors, and line graphs of bipartite graphs are closed under subdividing twice such an edge).

Suppose $\overline{T'}$ is bipartite, and let X, Y be a partition of $V(T)$ into two strong cliques of T'. We may assume that $a \in X$. Assume first that $b \in Y$. Since a is the unique strong neighbor of b in T', it follows that $Y = \{b\}$, so X contains a and c, a contradiction. Thus we may assume that $b \in X$. Since a is the unique strong neighbor of b in T', it follows that $X = \{a, b\}$, and b is strongly anticomplete to $Y \setminus \{c\}$. Let N be the set of strong neighbors of a in $Y \setminus \{c\}$, and M the set of strong antineighbors of a in $Y \setminus \{c\}$. Since $T \in \mathcal{F}$, it follows that $Y = N \cup M \cup W \cup \{c\}$. If either $|N| > 1$ or $|M| > 1$, then T admits a balanced skew-partition by 2.7, so we may assume that $|N| \leq 1$ and $|M| \leq 1$. Since no vertex of T' is adjacent to both a and w, it follows that $|N \cup W| \leq 1$. Now if $M = \emptyset$ or $N \cup W = \emptyset$ then T' is bipartite and we proceed as above, otherwise $N \cup W \cup \{c\}$ is a clique cutset of T' of size 2, which is a star cutset in T, and hence T admits a balanced skew-partition by 2.6.

Next assume that $\overline{T'}$ is a line trigraph. Since bc is a switchable pair in T' and b is strongly anticomplete to $V(T') \setminus \{a, b, c\}$, it follows that c is strongly complete to $V(T') \setminus \{a, b, c\}$ else there would be in $\overline{T'}$ a clique of size 3 with a switchable pair. Since $\overline{T'}$ is a line trigraph, it follows that for every triangle S of T' and a vertex $v \in V(T') \setminus S$, v has at least one strong neighbor in S. If $x, y \in V(T') \setminus \{a, b, c\}$ are adjacent, then $\{x, y, c\}$ is a triangle and b has no strong neighbor in it, and hence $V(T') \setminus \{a, b, c\}$ is a strongly stable set.

But now, $V(T') \setminus \{a,c\}, \{a,c\}$ form a partition of $V(T')$ into two strongly stable sets of T'. So T' is bipartite and we proceed as above.

Finally, suppose that T' is doubled and let (X,Y) be a good partition of T'. If $T'|Y$ is empty or has a unique anticomponent, then T' is bipartite. Hence, we may assume that Y contains two strongly adjacent vertices x and x'. If there exist $y \neq x$ and $y' \neq x'$ such that $\{x,y\}$ and $\{x',y'\}$ are anticomponents of $T'|Y$, then every vertex of T' has at least two strong neighbors, a contradiction because of b. It follows that $\{x\}$, say, is an anticomponent of $T'|Y$. If $T'|X$ has a single component or is empty, then T' is the complement of a bipartite trigraph. Hence we may assume that $T'|X$ has at least two components. Therefore, Y is a star cutset of T' centered at x. This is handled in the next case.

Case 2: T' admits a balanced skew-partition.

Let (A,B) be a balanced skew-partition of T'. If $W \neq \emptyset$, let $A' = A \cup \{u,v\}$; and if $W = \emptyset$, let $A' = A$. Then $T|A'$ is not connected. We claim that if some anticomponent Y of B is disjoint from $\{a,b\}$, then T admits a balanced skew-partition. Since a is complete to W in T', some component L of A is disjoint from $\{a\} \cup W$, and hence L is a component of A' as well. We may assume w.l.o.g. that Y is disjoint from W (this is clearly the case if $B \cap \{a,b\} \neq \emptyset$, and if $B \cap \{a,b\} = \emptyset$ we may assume w.l.o.g. that $Y \cap W = \emptyset$). Now, in T, Y is strongly complete to $B \setminus Y$, L is strongly anticomplete to $A' \setminus L$, and thus (A',B) is a skew-partition of T and $(L \cup Y) \cap (\{a,b\} \cup W \cup (A' \setminus A)) \subseteq \{b\}$. Since (A,B) is a balanced skew-partition of T', the pair (L,Y) is balanced in T; consequently 2.8 implies that T admits a balanced skew-partition. This proves the claim.

Thus we may assume that no such Y exists, and therefore $T'|B$ has exactly two anticomponents, B_1 and B_2, and $a \in B_1$ and $b \in B_2$. Since a is the unique strong neighbor of b in T', it follows that $B_1 = \{a\}$. Since a is anticomplete to $W \cup \{c\}$, we deduce that $W \cup \{c\} \subseteq A'$. Let A_1 be the component of $T|A'$ containing c and $A_2 = A' \setminus A_1$. Suppose that a does not have a strong neighbor in T. Then $B_2 = \{b\}$, and since $T \in \mathcal{F}$, a is strongly anticomplete to A'. We may assume that T is not bipartite, since otherwise T satisfies one of the outcomes of the theorem we are proving. Then T contains an odd cycle C, which must be in A_1 or A_2 (since $\{a,b\}$ is strongly anticomplete to A'). Since $T \in \mathcal{F}$, C must contain at least one strong edge, say xy. But then $\{x,y\}$ is a star cutest in T separating $\{a,b\}$ from a vertex of A_2. So by 2.6, T has a balanced skew-partition. Therefore we may assume that a has at least one strong neighbor in T.

Let $x \in A_2$. Let N be the set of strong neighbors of a in T. Then $(N \cup \{a\}) \setminus \{x\}$ is a star cutset in T separating b from x, unless x is the unique strong neighbor of a. In this case $\{a, x\}$ is a star cutset separating A_1 from $A_2 \setminus \{x\}$, unless $A_2 = \{x\}$. Now suppose that c has a neighbor y (that is in fact a strong neighbor since $T \in \mathcal{F}$). Then $\{c, y\}$ is a star cutset separating $A_1 \setminus \{c, y\}$ from x, unless $A_1 = \{c, y\}$, in which case T is bipartite. So we may assume that c has no neighbor in A_1. Now, either T is bipartite, or T has an odd cycle. But in this later case, the cycle is in A_1 and any strong edge of it (which exists since $T \in \mathcal{F}$) forms a star cutset separating c from the rest of the cycle. Therefore, by 2.6, T has a balanced skew-partition.

Case 3: T' admits a proper 2-join.

Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of a proper 2-join of T'. We may assume that $a \in A_1 \cup B_1 \cup C_1$. Then $W \subseteq A_1 \cup B_1 \cup C_1$. If $W \neq \emptyset$ let $C_1' = C_1 \cup \{u, v\}$, and otherwise let $C_1' = C_1$. We may assume that $(A_1, B_1, C_1', A_2, B_2, C_2)$ is not a proper 2-join of T, and hence w.l.o.g. $a \in A_1$ and $b \in A_2$. Then $c \in B_2 \cup C_2$. Since a is the unique strong neighbor of b in T', it follows that $A_1 = \{a\}$. By Case 2, we may assume that T' does not admit a balanced skew-partition, and therefore 3.1 implies that a is anticomplete to B_1. Note that since $T \in \mathcal{F}$, ab is the only switchable pair in T that involves a. Let N be the set of strong neighbors of a in C_1' in T. It follows from the definition of a proper 2-join that $N \neq \emptyset$. We may assume that T does not admit a balanced skew-partition, and hence by 3.1, every 2-join of T is proper. So either $(N, B_1, C_1' \setminus N, \{a\}, B_2, C_2 \cup A_2)$ is a split of a proper 2-join in T, or $|N| = |B_1| = 1$ and the full realization of $T|(C_1' \cup B_1)$ is a path of length two joining the members of N and B_1. Let this path be n-n'-b_1 where $n \in N$ and $b_1 \in B_1$. Since b_1 has no neighbor in $\Sigma(T)$ except possibly n', it follows that b_1 is an a-appendage of b. In particular, $W \neq \emptyset$. Since $W \subseteq B_1 \cup C_1$, it follows that $w = b_1$, $u = n$ and $v = n'$. But then $|A_1 \cup B_1 \cup C_1| = 2$, contrary to the fact that $(A_1, B_1, C_1, A_2, B_2, C_2)$ is a split of a proper 2-join of T'.

Case 4: $\overline{(T')}$ admits a proper 2-join.

Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of a proper 2-join in $\overline{T'}$. First suppose that $W \neq \emptyset$. Then we may assume that $a, w \in A_1 \cup B_1 \cup C_1$. Since no vertex of T' is adjacent to both a and w, it follows w.l.o.g. that $a \in A_1$, $w \in B_1$ and $C_2 = \emptyset$. Since a is the unique strong neighbor of b in T', it follows that $b \in B_2$ and $C_1 = \emptyset$. But now $(A_1, B_1, \emptyset, B_2, A_2, \emptyset)$ is a split of a 2-join in T'. By Case 2 we may assume that T' does not admit a balanced skew-partition, and hence this 2-join is proper by 3.1. But then we may proceed as in Case 3. Therefore we may assume that $W = \emptyset$.

We may assume that $(A_1, B_1, C_1, A_2, B_2, C_2)$ is not a split of a proper 2-join in $\overline{T}$, and therefore $a \in A_1 \cup B_1 \cup C_1$, and $b \in A_2 \cup B_2 \cup C_2$ (up to symmetry). Since a is the unique strong neighbor of b in T', and since A_1, B_1 are both non-empty, we deduce that $b \notin C_2$, and so we may assume that $b \in B_2$. Since $A_1 \neq \emptyset$, it follows that $C_1 = \emptyset$ and $A_1 = \{a\}$. Since $|A_1 \cup B_1 \cup C_1| \geq 3$, it follows that $|B_1| \geq 2$. Since c is strongly antiadjacent to a and semiadjacent to b in T, we deduce that $c \in A_2$. But now, if a has a neighbor $x \in B_1$ in T (which is therefore a strong neighbor), then $\{x, a\} \cup A_2 \cup C_2$ is a star cutset in T, and if a is strongly anticomplete to B_1 in T, then it follows from the definition of a proper 2-join that B_1 is a homogeneous set in T. In both cases, by 2.6 and 2.7, respectively, we deduce that T admits a balanced skew-partition. ∎

4 Blocks of decomposition

The way we use decompositions for computing stable sets in Section 6 requires building blocks of decomposition and asking several questions on the blocks. To do that we need to ensure that the blocks of decomposition are still in our class.

A set $X \subseteq V(T)$ is a *fragment* of a trigraph T if one of the following holds:

1. $(X, V(T) \setminus X)$ is a proper 2-join of T;
2. $(X, V(T) \setminus X)$ is a proper complement 2-join of T.

Note that a fragment of T is a fragment of $\overline{T}$. We now define the *blocks of decomposition* T_X with respect to some fragment X. A 2-join is *odd* or *even* according to the parity of the lengths of the paths described in 2.4.

If (X_1, X_2) is a proper odd 2-join and $X = X_1$, then let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). We build the block of decomposition $T_{X_1} = T_X$ as follows. We start with $T|(A_1 \cup B_1 \cup C_1)$. We then add two new *marker vertices* a and b such that a is strongly complete to A_1, b is strongly complete to B_1, ab is a switchable pair, and there are no other edges between $\{a, b\}$ and X_1. Note that $\{a, b\}$ is a switchable component of T_X. We call it the *marker component of* T_X.

If (X_1, X_2) is a proper even 2-join and $X = X_1$, then let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). We build the block of decomposition $T_{X_1} = T_X$ as follows. We start with $T|(A_1 \cup B_1 \cup C_1)$. We then add three new *marker vertices* a, b and c such that a is strongly complete to A_1, b is strongly complete to B_1, ac and cb are switchable pairs, and there are no other edges between $\{a, b, c\}$ and X_1. Again, $\{a, b, c\}$ is called the *marker component of* T_X.

If (X_1, X_2) is a proper odd complement 2-join and $X = X_1$, then let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). We build the block of decomposition $T_{X_1} = T_X$ as follows. We start with $T|(A_1 \cup B_1 \cup C_1)$. We then add two new *marker vertices* a and b such that a is strongly complete to $B_1 \cup C_1$, b is strongly complete to $A_1 \cup C_1$, ab is a switchable pair, and there are no other edges between $\{a, b\}$ and X_1. Again, $\{a, b\}$ is called the *marker component of* T_X.

If (X_1, X_2) is a proper even complement 2-join and $X = X_1$, then let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). We build the block of decomposition $T_{X_1} = T_X$ as follows. We start with $T|(A_1 \cup B_1 \cup C_1)$. We then add three new *marker vertices* a, b and c such that a is strongly complete to $B_1 \cup C_1$, b is strongly complete to $A_1 \cup C_1$, c is strongly complete to X_1, ac and cb are switchable pairs, ab is a strong edge, and there are no other edges between $\{a, b, c\}$ and X_1. Again, $\{a, b, c\}$ is called the *marker component of* T_X.

4.1 *If X is a fragment of a trigraph T from $\mathcal{F}$ with no balanced skew-partition, then T_X is a trigraph from $\mathcal{F}$.*

Proof. From the definition of T_X, it is clear that every vertex of T_X is in at most one switchable pair, or is heavy, or is light. So, to prove that $T_X \in \mathcal{F}$, it remains only to prove that T_X is Berge.

Let $X = X_1$ and (X_1, X_2) be a proper 2-join of T. Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). Let Z be the marker component of T_X.

Suppose first that T_X has an odd hole $H = h_1\text{-}\cdots\text{-}h_k\text{-}h_1$. Assume that the vertices of Z are consecutive in H, then $H \setminus Z$ is a path P with one end in A_1, the other one in B_1 and interior in C_1. A hole of T is obtained by adding to P a path with one end in A_2, the other one in B_2, and interior in C_2. By 2.4, this hole is odd, a contradiction. Thus the marker vertices are not consecutive in H, and since c has no neighbors in $V(T) \setminus \{a, b, c\}$, we deduce that $c \notin V(H)$. Now a hole of the same length as H is obtained in T by possibly replacing a and/or b by some vertices $a_2 \in A_2$ and $b_2 \in B_2$, chosen to be antiadjacent (this is possible by 3.1).

Suppose now that T_X has an odd antihole $H = h_1\text{-}\cdots\text{-}h_k\text{-}h_1$. Since an antihole of length 5 is also a hole, we may assume that H has length at least 7. So, in H, any pair of vertices has a common neighbor. It follows that at most one of a, b, c is in H, and because of its degree, c is not in H.

An antihole of same length as H is obtained in T by possibly replacing a or b by some vertices $a_2 \in A_2$ or $b_2 \in B_2$, a contradiction.

Note that the case when T has a complement 2-join follows by complementation.

∎

4.2 *If X is a fragment of a trigraph T from $\mathcal{F}$ with no balanced skew-partition, then the block of decomposition T_X has no balanced skew-partition.*

Proof. To prove this, we suppose that T_X has a balanced skew-partition (A', B') with a split (A'_1, A'_2, B'_1, B'_2). From this, we find a skew-partition in T. Then we use 2.8 to prove the existence of a *balanced* skew-partition in T. This gives a contradiction that proves the theorem.

Let $X = X_1$ and (X_1, X_2) be a proper 2-join of T. Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). Let Z be the marker component of T_X.

Since the marker vertices in T_X, a and b have no common strong neighbor and c has no strong neighbor, there are up to symmetry two cases: $a \in A'_1$ and $b \in A'_1$, or $a \in A'_1$ and $b \in B'_1$. Note that when (X_1, X_2) is even, the marker vertex c must be in A'_1 because it is adjacent to a and has no strong neighbor.

Assume first that a and b are both in A'_1. Then $(X_2 \cup A'_1 \setminus Z, A'_2, B'_1, B'_2)$ is a split of a skew-partition (A, B) in T. The pair (A'_2, B'_1) is balanced in T because it is balanced in T_X. Hence, by 2.8, T admits a balanced skew-partition, a contradiction.

Thus not both a and b are in A'_1, and so $a \in A'_1$ and $b \in B'_1$. In this case, $(A_2 \cup C_2 \cup A'_1 \setminus \{a, c\}, A'_2, B_2 \cup B'_1 \setminus \{b\}, B'_2)$ is a split of a skew-partition (A, B) in T. The pair (A'_2, B'_2) is balanced in T because it is balanced in T_X. Hence, by 2.8, T admits a balanced skew-partition, a contradiction.

The case when T has a complement 2-join follows by complementation ∎

5 Handling basic trigraphs

Our next goal is to compute maximum strong stable sets. We need to work in weighted trigraphs for the sake of induction. So, throughout the remainder of the paper, by "trigraph" we mean a trigraph with weights on the vertices. Weights are numbers from K where K is either the set $\mathbb{R}_+$ of non-negative real numbers or the set $\mathbb{N}_+$ of non negative integers. The statements of the theorems will be true for $K = \mathbb{R}_+$ but the algorithms are to be implemented with $K = \mathbb{N}_+$. Note that we view a trigraph where no weight is assigned to the vertices as a weighted trigraph all of whose vertices have weight 1. Observe that a set of vertices of a trigraph is a strong stable set if and only if it is a stable set of its full realization.

5.1 *There is an $O(n^4)$ algorithm whose input is a trigraph and whose output is either the true statement "T is not basic", or the name of a basic class in which T is and the maximum weight of a strong stable set of T.*

Proof. For each basic class, we provide an at most $O(n^4)$ time algorithm that decides whether a trigraph T belongs to the class, and if so, computes a maximum weighted strong stable set.

For bipartite trigraphs, we construct the full realization G of T. It is easy to see that T is bipartite if and only if G is bipartite, and deciding whether a graph is bipartite can be done in linear time by the classical Breadth First Search. If T is bipartite, a maximum weighted stable set of G (which is a maximum weighted strong stable set of T) can be computed in time $O(n^3)$, see [18].

For complements of bipartite trigraph, we proceed similarly: we first take the complement $\overline{T}$ of the input trigraph T, and then recognize whether the full realization of $\overline{T}$ is bipartite. We then compute the maximum weighted clique in $G^{\overline{T}}_{\emptyset}$. All this can clearly be done in $O(n^3)$ time.

For line trigraphs, we compute the full realization G, and test whether G is a line graph of a bipartite graph by a classical algorithm from [14] or [16]. Note that these algorithms also provide a graph R such that $G = L(R)$. In time $O(n^3)$ we can check that every clique of size at least 3 in T is a strong clique so we can decide whether T is a line trigraph. If so, a maximum stable set in G can be computed in time $O(n^3)$ by computing a maximum weighted matching (see [18]) in a bipartite graph R such that $G = L(R)$.

For complements of line trigraphs, we proceed similarly for the recognition except that we work with the full realization of $\overline{T}$. And computing a maximum weighted strong stable set is easy: compute the full realization G of T, then compute a bipartite graph R such that $G = \overline{L(R)}$ (this exists because by 2.1, line trigraphs are closed under taking realizations) and compute a maximum weighted stable set in G (note that such a set is an inclusion-wise maximal set of pairwise adjacent edges in R, and there are linearly many such sets). This is a maximum weighted strong stable set in T.

For doubled trigraphs, the situation is slightly more complicated, because we do not know how to rely on classical results. But for one who starts from scratch (with no knowledge of matching theory for instance), they are in fact the easiest basic graphs to handle. To decide whether a graph G is doubled, we may use the list of minimally non-doubled graphs described in [1]. This list is made of 44 graphs on at most 9 vertices, so it yields an $O(n^9)$ time recognition algorithm. We propose here something faster, and which also works for trigraphs.

If a partition (X, Y) of the vertices of a trigraph is given, deciding whether it is good can be done by a brute force checking of all items from the definition in time $O(n^2)$. And if an edge ab from $T|X$ is given, reconstructing the good partition is easy: all the vertices strongly antiadjacent to a and b go into X, and all the vertices strongly adjacent to at least one of a or b go into Y. So, by checking all edges uv, one can guess one that is in $T|X$, then reconstruct (X, Y), and therefore test in time $O(n^4)$ whether a trigraph T has a good partition (X, Y) such that X contains at least one edge. Similarly, one can test in time $O(n^4)$ whether a trigraph T has a good partition (X, Y) such that Y contains at least one antiedge. We are left with the recognition of doubled trigraphs such that all good partitions are made of one strong stable set and one strong clique. These are in fact graphs (there is no switchable pair), and are known as *split graphs* (in fact, double split graphs were named after split graphs). They can be recognized in linear time, see [12] where it it shown that by looking at the degrees, one can easily output a partition of a graph into a clique and a stable set, if any such partition exists.

Now, we know that T is a doubled graph, and we look for a maximum weighted strong stable set in T. To do so, we compute the full realization G of T. So, by 2.2 , G is a doubled graph, and in fact, (X, Y) is good partition for G. We then compute a maximum weighted stable set in $G|X$ (that is bipartite), in $G|Y$ (that is complement of bipartite), and all stable sets made of a vertex from Y together with its non-neighbors in X. One of these is a maximum weighted stable set of G, and so a strong one in T. ∎

6 Keeping track of α

In this section, we define several blocks of decompositions that allow us to compute maximum strong stable sets. From here on, $\alpha(T)$ denotes the weight of a maximum weighted strong stable set of T.

In what follows, T is a trigraph from $\mathcal{F}$ with no balanced skew-partition, X is an fragment of T and $Y = V(T) \setminus X$ (so Y is also fragment of T). To compute $\alpha(T)$, it is not enough to consider the blocks T_X and T_Y (as defined in Section 4) separately. Instead, we need to enlarge T_Y slightly, to encode information from X. In this section, we define four different kinds of gadgets, named $T_{Y,1}, \dots, T_{Y,4}$ and for $i = 1, \dots, 4$, we prove that $\alpha(T)$ may easily be computed from $\alpha(T_{Y,i})$. We sometimes have to define different gadgets for handling the same situation. This is because in Section 7 (namely to prove 7.1), we need that gadgets preserve being basic, and depending on the basic class under consideration, we need to use different gadgets. Note that the gadgets are not class-preserving (some of them introduce balanced skew-partitions). In this section, this is not a problem, but in the next section, this makes things a bit more complicated.

6.1 Complement 2-join

If (X, Y) is a proper complement 2-join of T then let $X_1 = X$, $X_2 = Y$, and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). We build the gadget $T_{Y,1}$ as follows. We start with $T|Y$. We then add two new *marker vertices* a, b, such that a is strongly complete to $B_2 \cup C_2$, b is strongly complete to $A_2 \cup C_2$ and ab is a strong edge. We give weights $\alpha_A = \alpha(T|A_1)$ and $\alpha_B = \alpha(T|B_1)$ to a and b respectively. We set $\alpha_X = \alpha(T|X)$.

6.1 *If (X, Y) is a proper complement 2-join of T, then $T_{Y,1}$ is Berge and $\alpha(T) = \max(\alpha(T_{Y,1}), \alpha_X)$.*

Proof. Since $T_{Y,1}$ is a semirealization of an induced subtrigraph of the block T_Y as defined in Section 4, it is clearly Berge by 4.1.

Let Z be a maximum weighted strong stable set in T. If $Z \cap X_1 = \emptyset$, then Z is also a strong stable set in $T_{Y,1}$, so $\alpha(T) \leq \alpha(T_{Y,1}) \leq \max(\alpha(T_{Y,1}), \alpha_X)$. If $Z \cap A_1 \neq \emptyset$ and $Z \cap (B_1 \cup C_1) = \emptyset$, then $\{a_1\} \cup (Z \cap X_2)$ is a strong stable set in $T_{Y,1}$ of weight $\alpha(T)$, so $\alpha(T) \leq \alpha(T_{Y,1}) \leq \max(\alpha(T_{Y,1}), \alpha_X)$. If $Z \cap B_1 \neq \emptyset$ and $Z \cap (A_1 \cup C_1) = \emptyset$, then $\{b_1\} \cup (Z \cap X_2)$ is a strong stable set in $T_{Y,1}$ of weight $\alpha(T)$, so $\alpha(T) \leq \alpha(T_{Y,1}) \leq \max(\alpha(T_{Y,1}), \alpha_X)$. If $Z \cap (A_1 \cup C_1) \neq \emptyset$ and $Z \cap (B_1 \cup C_1) \neq \emptyset$, then $\alpha(T) = \alpha_X$, so $\alpha(T) \leq \max(\alpha(T_{Y,1}), \alpha_X)$. In all cases, we proved that $\alpha(T) \leq \max(\alpha(T_{Y,1}), \alpha_X)$.

Conversely, let $\alpha = \max(\alpha(T_{Y,1}), \alpha_X)$. If $\alpha = \alpha_X$, then by considering any maximum strong stable set of $T|X_1$, we see that $\alpha = \alpha_X \leq \alpha(T)$. So we may assume that $\alpha = \alpha(T_{Y,1})$ and let Z be a maximum weighted strong stable set in $T_{Y,1}$. If $a \notin Z$ and $b \notin Z$, then Z is also a strong stable set in T, so $\alpha \leq \alpha(T)$. If $a \in Z$ and $b \notin Z$, then $Z' \cup Z \setminus \{a\}$, where Z' is a maximum weighted strong stable set in $T|A_1$, is also a strong stable set in T of same weight as Z, so $\alpha \leq \alpha(T)$. If $a \notin Z$ and $b \in Z$, then $Z' \cup Z \setminus \{b\}$ where Z'

is a maximum weighted stable set in $T|B_1$ is also a strong stable set in T of same weight as Z, so $\alpha \leq \alpha(T)$. In all cases, we proved that $\alpha \leq \alpha(T)$. ∎

6.2 2-join

In [20], an NP-hardness result is proved, that suggests that the 2-join is maybe not the most convenient tool to compute maximum stable sets. It seems that to use them, we really need to take advantage of Bergeness in some way. This is done here by proving several inequalities.

If (X, Y) is a 2-join of T then let $X_1 = X$, $X_2 = Y$ and let (A_1, B_1, C_1, A_2, B_2, C_2) be a split of (X_1, X_2). We define $\alpha_{AC} = \alpha(T|(A_1 \cup C_1))$, $\alpha_{BC} = \alpha(T|(B_1 \cup C_1))$, $\alpha_C = \alpha(T|C_1)$ and $\alpha_X = \alpha(T|X_1)$. Let w be the weight function on $V(T)$. When H is an induced subtrigraph of T, or a subset of $V(T)$, $w(H)$ denotes the sum of the weights of vertices in H.

6.2 *Let S be a maximum weighted strong stable set of T. Then exactly one of the following holds:*

1. *$S \cap A_1 \neq \emptyset$, $S \cap B_1 = \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $T|(A_1 \cup C_1)$ and $w(S \cap X_1) = \alpha_{AC}$;*
2. *$S \cap A_1 = \emptyset$, $S \cap B_1 \neq \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $T|(B_1 \cup C_1)$ and $w(S \cap X_1) = \alpha_{BC}$;*
3. *$S \cap A_1 = \emptyset$, $S \cap B_1 = \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $T|C_1$ and $w(S \cap X_1) = \alpha_C$;*
4. *$S \cap A_1 \neq \emptyset$, $S \cap B_1 \neq \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $T|X_1$ and $w(S \cap X_1) = \alpha_X$.*

Proof. Follows directly from the definition of a 2-join. ∎

We need several inequalities that say more about how strong stable sets and 2-joins overlap. These lemmas are proved in [20] in the context of graphs. The proofs are the same for trigraphs, but for the sake of completeness, we rewrite them.

6.3 $0 \leq \alpha_C \leq \alpha_{AC}, \alpha_{BC} \leq \alpha_X \leq \alpha_{AC} + \alpha_{BC}$.

Proof. The inequalities $0 \leq \alpha_C \leq \alpha_{AC}, \alpha_{BC} \leq \alpha_X$ are trivially true. Let D be a maximum weighted strong stable set of $T|X_1$. We have:

$$\alpha_X = w(D) = w(D \cap A_1) + w(D \cap (C_1 \cup B_1)) \leq \alpha_{AC} + \alpha_{BC}.$$

∎

6.4 *If* (X_1, X_2) *is an odd* 2*-join of* T*, then* $\alpha_C + \alpha_X \leq \alpha_{AC} + \alpha_{BC}$.

Proof. Let D be a strong stable set of $T|X_1$ of weight α_X and C a strong stable set of $T|C_1$ of weight α_C. In the bipartite trigraph $T|(C \cup D)$, we denote by Y_A (resp. Y_B) the set of those vertices of $C \cup D$ for which there exists a path in $T|(C \cup D)$ joining them to some vertex of $D \cap A_1$ (resp. $D \cap B_1$). Note that from the definition, $D \cap A_1 \subseteq Y_A$, $D \cap B_1 \subseteq Y_B$ and there are no edges between $Y_A \cup Y_B$ and $(C \cup D) \setminus (Y_A \cup Y_B)$. We claim that $Y_A \cap Y_B = \emptyset$, and Y_A is strongly anticomplete to Y_B. Suppose not. then there exists a path P in $T|(C \cup D)$ from a vertex of $D \cap A_1$ to a vertex of $D \cap B_1$. We may assume that P is minimal with respect to this property, and so the interior of P is in C_1; consequently P is of even length because $T|(C \cup D)$ is bipartite. This contradicts the assumption that (X_1, X_2) is odd. Now we set:

- $Z_A = (D \cap Y_A) \cup (C \cap Y_B) \cup (C \setminus (Y_A \cup Y_B))$;
- $Z_B = (D \cap Y_B) \cup (C \cap Y_A) \cup (D \setminus (Y_A \cup Y_B)$.

From all the definitions and properties above, Z_A and Z_B are strong stable sets and $Z_A \subseteq A_1 \cup C_1$ and $Z_B \subseteq B_1 \cup C_1$. So, $\alpha_C + \alpha_X = w(Z_A) + w(Z_B) \leq \alpha_{AC} + \alpha_{BC}$. ∎

6.5 *If* (X_1, X_2) *is an even* 2*-join of* T*, then* $\alpha_{AC} + \alpha_{BC} \leq \alpha_C + \alpha_X$.

Proof. Let A be a strong stable set of $T|(A_1 \cup C_1)$ of weight α_{AC} and B a strong stable set of $T|(B_1 \cup C_1)$ of weight α_{BC}. In the bipartite trigraph $T|(A \cup B)$, we denote by Y_A (resp. Y_B) the set of those vertices of $A \cup B$ for which there exists a path P in $T|(A \cup B)$ joining them to a vertex of $A \cap A_1$ (resp. $B \cap B_1$). Note that from the definition, $A \cap A_1 \subseteq Y_A$, $B \cap B_1 \subseteq Y_B$, and $Y_A \cup Y_B$ is strongly anticomplete to $(A \cup B) \setminus (Y_A \cup Y_B)$. We claim that $Y_A \cap Y_B = \emptyset$ and Y is strongly anticomplete to Y_B. Suppose not, then there is a path P in $T|(A \cup B)$ from a vertex of $A \cap A_1$ to a vertex of $B \cap B_1$. We may assume that P is minimal with respect to this property, and so the interior of P is in C_1; consequently it is of odd length because $T(A \cup B)$ is bipartite. This contradicts the assumption that (X_1, X_2) is even. Now we set:

- $Z_D = (A \cap Y_A) \cup (B \cap Y_B) \cup (A \setminus (Y_A \cup Y_B))$;
- $Z_C = (A \cap Y_B) \cup (B \cap Y_A) \cup (B \setminus (Y_A \cup Y_B))$.

From all the definitions and properties above, Z_D and Z_C are strong stable sets and $Z_D \subseteq X_1$ and $Z_C \subseteq C_1$. So, $\alpha_{AC} + \alpha_{BC} = w(Z_C) + w(Z_D) \leq \alpha_C + \alpha_X$. ∎

We are now ready to build the gadgets.

If (X_1, X_2) is a proper odd 2-join of T, then we build the gadget $T_{Y,2}$ as follows. We start with $T|Y$. We then add four new *marker vertices* a, a', b, b', such that a and a' are strongly complete to A_2, b and b' are strongly complete to B_2, and ab is a strong edge. We give weights $\alpha_{AC} + \alpha_{BC} - \alpha_C - \alpha_X$, $\alpha_X - \alpha_{BC}$, $\alpha_{AC} + \alpha_{BC} - \alpha_C - \alpha_X$ and $\alpha_X - \alpha_{AC}$ to a, a', b and b' respectively. Note that by 6.3 and 6.4, all the weights are non-negative.

We define another gadget of decomposition $T_{Y,3}$ for the same situation, as follows. We start with $T|Y$. We then add three new *marker vertices* a, a', b, such that a and a' are strongly complete to A_2, b is strongly complete to B_2, and $a'a$ and ab are strong edges. We give weights $\alpha_{AC} - \alpha_C$, $\alpha_X - \alpha_{BC}$ and $\alpha_{BC} - \alpha_C$ to a, a' and b respectively. Note that by 6.3, all the weights are non-negative.

6.6 *If (X, Y) is a proper odd 2-join of T, then $T_{Y,2}$ and $T_{Y,3}$ are Berge, and $\alpha(T) = \alpha(T_{Y,2}) + \alpha_C = \alpha(T_{Y,3}) + \alpha_C$.*

Proof. Suppose that $T_{Y,2}$ contains an odd hole H. Since an odd hole has no strongly dominated vertex, it contains at most one of a, a' and at most one of b, b'. Hence, H is an odd hole of some semirealization of the block T_Y (as defined in Section 4). This contradicts 4.1. Similarly, $T_{Y,2}$ contains no odd antihole, and therefore, it is Berge. The proof that $T_{Y,3}$ is Berge is similar.

Let Z be a strong stable set in T of weight $\alpha(T)$. We build a strong stable set in $T_{Y,2}$ by adding to $Z \cap X_2$ one the following (according to the outcome of 6.2): $\{a, a'\}$, $\{b, b'\}$, $\emptyset$, or $\{a, a', b'\}$. In each case, we obtain a strong stable set of $T_{Y,2}$ with weight $\alpha(T) - \alpha_C$. This proves that $\alpha(T) \leq \alpha(T_{Y,2}) + \alpha_C$.

Conversely, let Z be a stable set in $T_{Y,2}$ with weight $\alpha(T_{Y,2})$. We may assume that $Z \cap \{a, a', b, b'\}$ is one of $\{a, a'\}$, $\{b, b'\}$, $\emptyset$, or $\{a, a', b'\}$, and respectively to these cases, we construct a strong stable set of T by adding to $Z \cap X_2$ a maximum weighted strong stable set of the following: $T|(A_1 \cup C_1)$, $T|(B_1 \cup C_1)$, $T|C_1$, or $T|X_1$. We obtain a strong stable set in T with weight $\alpha(T_{Y,2}) + \alpha_C$, showing that $\alpha(T_{Y,2}) + \alpha_C \leq \alpha(T)$. This completes the proof for $T_{Y,2}$.

Let us now prove the equality for $T_{Y,3}$. Let Z be a strong stable set in T of weight $\alpha(T)$. We build a strong stable set in $T_{Y,3}$ by adding to $Z \cap X_2$ one the following (according to the outcome from 6.2): $\{a\}$, $\{b\}$, $\emptyset$, or $\{a', b\}$.

In each case, we obtain a strong stable set of $T_{Y,3}$ with weight $\alpha(T) - \alpha_C$. This proves that $\alpha(T) \leq \alpha(T_{Y,3}) + \alpha_C$.

Conversely, let Z be a stable set in $T_{Y,3}$ with weight $\alpha(T_{Y,3})$. By 6.4, $\alpha_{AC} - \alpha_C \geq \alpha_X - \alpha_{BC}$, so we may assume that $Z \cap \{a, a', b\}$ is one of $\{a\}$, $\{b\}$, $\emptyset$, or $\{a', b\}$, and respectively to these cases, we construct a strong stable set of T by adding to $Z \cap X_2$ a maximum weighted strong stable set of the following: $T|(A_1 \cup C_1)$, $T|(B_1 \cup C_1)$, $T|C_1$, or $T|X_1$. We obtain a strong stable set in T with weight $\alpha(T_{Y,3}) + \alpha_C$, showing that $\alpha(T_{Y,3}) + \alpha_C \leq \alpha(T)$. This completes the proof for $T_{Y,3}$. ∎

If (X_1, X_2) is a proper even 2-join of T and $X = X_1$, $Y = X_2$, then let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). We build the gadget $T_{Y,4}$ as follows. We start with $T|Y$. We then add three new *marker vertices* a, b, c such that a is strongly complete to A_2, b is strongly complete to B_2, and c is strongly adjacent to a, b and has no other neighbors. We give weights $\alpha_X - \alpha_{BC}$, $\alpha_X - \alpha_{AC}$, and $\alpha_X + \alpha_C - \alpha_{AC} - \alpha_{BC}$ to a, b and c respectively. Note that by 6.3 and 6.5, these weights are non-negative.

6.7 *If (X, Y) is a proper even 2-join of T, then $T_{Y,4}$ is Berge and $\alpha(T) = \alpha(T_{Y,4}) + \alpha_{AC} + \alpha_{BC} - \alpha_X$.*

Proof. Clearly, $T_{Y,4}$ is Berge, because it is a semirealization of the block T_Y as defined in Section 4, which is Berge by 4.1.

Let Z be a strong stable set in T of weight $\alpha(T)$. We build a strong stable set in $T_{Y,4}$ by adding to $Z \cap X_2$ one the following (according to the outcome of 6.2): $\{a\}$, $\{b\}$, $\{c\}$, or $\{a, b\}$. In each case, we obtain a strong stable set of $T_{Y,4}$ with weight $\alpha(T) - (\alpha_{AC} + \alpha_{BC} - \alpha_X)$. This proves that $\alpha(T) \leq \alpha(T_{Y,4}) + \alpha_{AC} + \alpha_{BC} - \alpha_X$.

Conversely, let Z be a strong stable set in $T_{Y,4}$ with weight $\alpha(T_{Y,4})$. We may assume that $Z \cap \{a, b, c\}$ is one of $\{a\}$, $\{b\}$, $\{c\}$, or $\{a, b\}$, and respectively to these cases, we construct a strong stable set of T by adding to $Z \cap X_2$ a maximum weighted strong stable set of the following: $T|(A_1 \cup C_1)$, $T|(B_1 \cup C_1)$, $T|C_1$, or $T|X_1$. We obtain a strong stable set in T with weight $\alpha(T_{Y,4}) + \alpha_{AC} + \alpha_{BC} - \alpha_X$, showing that $\alpha(T_{Y,4}) + \alpha_{AC} + \alpha_{BC} - \alpha_X \leq \alpha(T)$. ∎

7 Computing α

We are ready to describe our main algorithm, that computes a maximum weighted stable set. The main difficulty is that blocks of decompositions as defined in Section 4 have to be used in order to stay in the class, while gadgets as defined in Section 6 have to be used for computing α. Our idea is to use blocks in a first stage, and to replace them by gadgets in a second stage. To transform a block into a gadget (this operation is called an *expansion*), one needs to erase a switchable component, and to replace it by some vertices with the appropriate weights. Two kinds of information are needed. The first one is the type of decomposition that is originally used and the weights; this information is encoded into what we call a *prelabel*. The second one is the type of basic class in which the switchable component ends up (because not all gadgets preserve being a basic class); this information is encoded into what we call a *label*. Note that the prelabel is known right after decomposing a trigraph, while the label becomes known much later, when the decomposition is fully processed. Let us make all this formal.

Let S be a switchable component of a trigraph T from $\mathcal{F}$. A *prelabel* for S is one of the following:

- ("Complement odd 2-join", $\alpha_A, \alpha_B, \alpha_X$) where α_A, α_B and α_X are integers, if S is a switchable pair.
- ("Odd 2-join", $\alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X$) where α_{AC}, α_{BC}, α_C and α_X are integers, if S is switchable pair and no vertex of T is complete to S.
- ("Complement even 2-join", $\alpha_A, \alpha_B, \alpha_X$) where α_A, α_B and α_X are integers, if S is a heavy component.
- ("Even 2-join", $\alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X$) where α_{AC}, α_{BC}, α_C and α_X are integers, if S is a light component.

We remark that certain types of switchable components are "eligible" for both the first and second type of prelabel.

A prelabel should be thought of as "the decomposition from which the switchable component has been built". When T is a trigraph and $\mathcal{S}$ is a set of switchable components of T, a *prelabeling* for $(T, \mathcal{S})$ is a function that associates to each $S \in \mathcal{S}$ a prelabel. It is important to notice that $\mathcal{S}$ is just a set of switchable component, so that some switchable components may have no prelabel.

What follows is slightly ambiguous when we talk about "the basic class containing the trigraph ", because some trigraphs may be members of several basic classes (typically, small trigraphs, complete trigraphs, independent trigraphs and a few others). But this is not a problem; if a trigraph belongs to several basic classes, our algorithm chooses one such class arbitrarily, and the output is correct. We choose not to make this too formal and heavy, so this is not mentioned explicitly in the descriptions of the algorithms. For doubled graphs, there is one more ambiguity. Let T be a doubled graph, and (X, Y) a good partition of T. A switchable pair uv of T is a *matching pair* if $u, v \in X$ and an *antimatching pair* if $u, v \in Y$. In some small degenerate cases, a switchable pair of a doubled graph may be a matching and an antimatching pair according to the good partition under consideration, but once a good partition is fixed, there is no ambiguity. Again, this is not a problem: when a pair is ambiguous, the algorithm chooses arbitrarily one particular good partition.

Let S be a switchable component of a trigraph from $\mathcal{F}$. A *label* for S is a pair $L' = (L, N)$ such that L is a prelabel and N is one of the following: "bipartite", "complement of bipartite", "line", "complement of line", "doubled-matching", "doubled-antimatching". We say that L' *extends* L. The tag added to extend a prelabel of a switchable component S should be thought of as "the basic class in which S ends up when the trigraph is fully decomposed". When T is a trigraph and $\mathcal{S}$ is a set of switchable components of T, a *labeling* for $(T, \mathcal{S})$ is a function that associates to each $S \in \mathcal{S}$ a label. Under these circumstances we say that T is *labeled*. As with prelabels, switchable components not in $\mathcal{S}$ receive no label.

Let T be a labeled trigraph, $\mathcal{S}$ a set of switchable components of T and $\mathcal{L}$ a labeling for $(T, \mathcal{S})$. The *expansion* of $(T, \mathcal{S}, \mathcal{L})$ is the trigraph obtained from T after performing for each $S \in \mathcal{S}$ with label L the following operation:

1. If $L = ((\text{“Complement odd 2-join”}, \alpha_A, \alpha_B, \alpha_X), N)$ for some N (so S is a switchable pair ab): transform ab into a strong edge, give weight α_A to a and weight α_B to b.

2. If $L = ((\text{“Odd 2-join”}, \alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X), N)$ for some N (so S is a switchable pair ab): transform ab into a strong edge, and:

    - If N is equal to one of "bipartite", "complement of line", or "doubled-matching", then add a vertex a', a vertex b', make a' strongly complete to $N(a) \setminus \{b\}$, make b' strongly complete to $N(b) \setminus \{a\}$, and give weights $\alpha_{AC} + \alpha_{BC} - \alpha_C - \alpha_X$, $\alpha_X - \alpha_{BC}$, $\alpha_{AC} + \alpha_{BC} - \alpha_C - \alpha_X$ and $\alpha_X - \alpha_{AC}$ to a, a', b and b' respectively.
    - If N is equal to one of "complement of bipartite", "line" or "doubled-antimatching", then add a vertex a', make a' strongly complete to $\{a\} \cup N(a) \setminus \{b\}$, and give weights $\alpha_{AC} - \alpha_C$, $\alpha_X - \alpha_{BC}$ and $\alpha_{BC} - \alpha_C$ to a, a' and b respectively.

3. If $L = ((\text{“Complement even 2-join”}, \alpha_A, \alpha_B, \alpha_X), N)$ for some N (so S is made of two switchable pairs ac and cb and c is heavy): delete the vertex c, and give weight α_A to a and weight α_B to b.

4. If $L = ((\text{“Even 2-join”}, \alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X), N)$ for some N (so S is made of two switchable pairs ac and cb, and c is light): transform ac and cb into strong edges, and give weights $\alpha_X - \alpha_{BC}$, $\alpha_X - \alpha_{AC}$, and $\alpha_X + \alpha_C - \alpha_{AC} - \alpha_{BC}$ to a, b and c respectively.

The expansion should be thought of as “what is obtained if one uses a gadget as defined in Section 6 instead of a block of decomposition as defined in Section 4”.

7.1 *Suppose that T is trigraph that is in a basic class with name N, $\mathcal{S}$ is a set of switchable components of T and $\mathcal{L}$ is a labeling for T such that for all $S \in \mathcal{S}$ with label L, one of the following holds:*

- $L = (\dots, N)$ *where N is “bipartite”, “complement of bipartite”, “line” or ”complement of line”; or*
- $N =$ *“doubled”, S is a matching pair of T and $L = (\dots,$ “doubled-matching”$)$; or*
- $N =$ *“doubled”, S is an antimatching pair of T and $L = (\dots,$ “doubled-antimatching”$)$.*

Then the expansion of $(T, \mathcal{S}, \mathcal{L})$ is a basic trigraph.

Proof. From our assumptions, T is basic. So, it is enough to prove that expanding one switchable component S preserves being basic, and the result then follows by induction on $|\mathcal{S}|$. Let T' be the expansion. For several cases from the definition of expansions (namely items 1, 3 and 4), expanding just means possibly transforming some switchable pairs into strong edges, and possibly deleting a vertex. From 2.3, this preserves being basic. Hence, in the argument below, we just study item 2 from the definition of expansions, and thus we may assume that S is a switchable pair ab.

It is easy to check that expansion as defined in item 2 preserves being bipartite and being complement bipartite; so if $N \in \{$“bipartite”, “complement of bipartite”$\}$, then we are done.

Suppose that $N =$“line”, and so T is a line trigraph. Let G be the full realization of T, and R a bipartite graph such that $G = L(R)$. So a is an edge $x_a y_a$ in R, and b is an edge $y_a x_b$. Since T is a line trigraph, it follows that every clique of size at least 3 in T is a strong clique, and so a and b have no common neighbors in T. Therefore all the neighbors of a except b are edges incident with x_a, and not with y_a. Let R' be the graph obtained from R by adding a pendant edge e at x_a. We observe that $L(R')$ is isomorphic to the full realization of T' (the edge e yields the new vertex a'), and therefore T' is a line trigraph.

Next suppose that $N =$"complement of line", so T is the complement of a line trigraph. Since every clique of size at least 3 in $\overline{T}$ is a strong clique, it follows that $V(T) = N(a) \cup N(b)$. Assume that there exist $u, v \in N(a) \setminus N(b)$ such that u is adjacent to v. Since $\overline{T}$ is a line trigraph, and if uv is a semiedge then $\{u, v, b\}$ is a clique of size 3 in $\overline{T}$, it follows that u is strongly adjacent to v in T. Let R be a bipartite graph such that the full realization of $\overline{T}$ is $L(R)$. Then in R no two of the edges u, v, a share an end, and yet b shares an end with all three of them, a contradiction. This proves that $N(a) \setminus N(b)$ (and symmetrically $N(b) \setminus N(a)$) is a strongly stable set in T. As $N(a) \cap N(b) = \emptyset$, then T is bipartite, and so a previous argument shows that T' is basic.

So we may assume that T is a doubled trigraph with a good partition (X, Y). If $S = ab$ is a matching pair of T, then adding the vertices a', b' to X, produces a good partition of T'. If $S = ab$ is an antimatching pair of T, then adding the vertex a' to Y produces a good partition of T'. Thus in all cases T' is basic and the theorem holds. ∎

Let T be a trigraph, $\mathcal{S}$ a set of switchable components of T, $\mathcal{L}$ a labeling of $(T, \mathcal{S})$ and T' the expansion of $(T, \mathcal{S}, \mathcal{L})$. Let $X \subseteq V(T)$. We define the *expansion X' of X* as follows. Start with $X' = X$ and perform the following for every $S \in \mathcal{S}$.

1. If $L = ((\text{“Complement odd 2-join”}, \alpha_A, \alpha_B, \alpha_X), N)$ for some N (so S is a switchable pair ab), do not change X'.

2. If $L = ((\text{“Odd 2-join”}, \alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X), N)$ for some N (so S is a switchable pair ab):

   - If N is equal to one of “bipartite”, “complement of line”, or “doubled-matching”, do: if $a \in X$ then add a' to X', and if $b \in X$ then add b' to X'.
   - If N is equal to one of “complement of bipartite”, “line” or “doubled-antimatching”, do: if $a \in X$ then add a' to X'.

3. If $L = ((\text{“Complement even 2-join”}, \alpha_A, \alpha_B, \alpha_X), N)$ for some N (so S is made of two switchable pairs ac and cb and c is heavy), do: if $c \in X$, then remove c from X'.

4. If $L = ((\text{“Even 2-join”}, \alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X), N)$ for some N (so S is made of two switchable pairs ac and cb, and c is light), do not change X'.

7.2 *With the notation as above, if (X_1, X_2) is a proper (complement) 2-join of T with split $(A_1, B_1, C_1, A_2, B_2, C_2)$, then (X'_1, X'_2) is a proper (complement) 2-join of T' with split $(A'_1, B'_1, C'_1, A'_2, B'_2, C'_2)$, with same parity as (X_1, X_2) (note that the notion of parity makes sense for T', since T' is Berge by 6.1, 6.6 and 6.7).*

Proof. Follows easily from the definitions. ∎

7.3 *There exists an algorithm with the following specification.*

Input: *A triple $(T, \mathcal{S}, \mathcal{L})$ such that T is a trigraph in $\mathcal{F}$ with no balanced skew-partition, $\mathcal{S}$ is a set of switchable components of T and $\mathcal{L}$ is a prelabeling for $(T, \mathcal{S})$.*

Output: *A labeling $\mathcal{L}'$ for $(T, \mathcal{S})$ that extends $\mathcal{L}$, and a maximum weighted strong stable set of the expansion of $(T, \mathcal{S}, \mathcal{L}')$.*

Running time: $O(n^5)$

Proof. We describe a recursive algorithm. The first step of the algorithm is to use 5.1 to check whether T is basic. Note that if T is a doubled trigraph, the algorithm from 5.1 also outputs which switchable pair is a matching-pair, and which switchable pair is an antimatching pair.

Suppose first that T is in a basic class with name N (this is the case in particular when $|V(T)| = 1$). We extend the prelabeling $\mathcal{L}$ into a labeling $\mathcal{L}'$ as follows: if $N \neq$“doubled”, then we append N to every label and otherwise, for each $S \in \mathcal{S}$ with label L, we add “doubled-matching” (resp. “doubled-antimatching”) to L when S is a matching (resp. antimatching) pair. It turns out that the labeling that we obtain satisfies the requirements of 7.1, so the expansion T' of $(T, \mathcal{S}, \mathcal{L}')$ is basic, and by running the algorithm from 5.1 again for T', we obtain a maximum weighted strong stable set of T' in time $O(n^4)$. So, as claimed, we may output a labeling $\mathcal{L}'$ for $(T, \mathcal{S})$ that extends $\mathcal{L}$, and a maximum weighted strong stable set of the expansion of $(T, \mathcal{S}, \mathcal{L}')$

Suppose now that T is not basic. Since T is in $\mathcal{F}$ and has no balanced skew-partition, by 3.2, we know that T has a 2-join or the complement of a 2-join. In [3], an $O(n^4)$ time algorithm for computing a 2-join in any input graph is described. In fact 2-joins as defined in [3] are sligthly different from the ones we use: they are not required to satisfy the last item in our definition of a 2-join (this item ensures that no side of the 2-join is a path of length exactly 2). But the method from Theorem 4.1 in [3] shows how to handle these kinds of requirements with no additional time. It is easy to adapt this method to the detection of a 2-join in a trigraph (also Section 10 of the present article gives a similar algorithm). So we can find the decomposition that we need in time $O(n^4)$. We then compute the blocks T_X and T_Y as defined in Section 4. Note that every member of $\mathcal{S}$ is a switchable pair of exactly one of T_X or T_Y. We call $\mathcal{S}_X$ (resp. $\mathcal{S}_Y$) the set formed by the members of $\mathcal{S}$ that are in T_X (resp. T_Y). Let S be the marker switchable component used to create the block T_Y. Observe that for every $u \in S$ there exists a vertex $v \in X$ such that $N_T(v) \cap Y = N_{T_Y}(u) \cap Y$. The same is true for T_X. So, the prelabeling $\mathcal{L}$ for $(T, \mathcal{S})$ naturally yields a prelabeling $\mathcal{L}_X$ for $(T_X, \mathcal{S}_X)$ and a prelabeling $\mathcal{L}_Y$ for $(T_Y, \mathcal{S}_Y)$ (each $S \in \mathcal{S}_X$ receives the same prelabel it has in $\mathcal{L}$, similarly for $\mathcal{S}_Y$). In what follows, *the decomposition* refers to the decomposition that was used to build T_X and T_Y, (so one of “complement odd 2-join”, “complement even 2-join”, “odd 2-join” or “even 2-join”) and we use our standard notation for a split of the decomposition.

Up to symmetry, we may assume that $|V(T_X)| \leq |V(T_Y)|$. By 4.1, T_X, T_Y are trigraphs from $\mathcal{F}$, and by 4.2, they have no balanced skew-partition.

Let S be the marker switchable component that was used to create block T_Y. We set $\mathcal{S}'_Y = \mathcal{S}_Y \cup \{S\}$. We now build a prelabeling $\mathcal{L}_Y$ for $\mathcal{S}'_Y$ as follows. All switchable components in $\mathcal{S}_Y$ keep the prelabel that they have in $\mathcal{S}$. The marker component S receives the following prelabel:

- If the decomposition is a complement odd 2-join, then recursively compute $\alpha_A = \alpha(T_X|A_1)$, $\alpha_B = \alpha(T_X|B_1)$ and $\alpha_X = \alpha(T_X|X)$, and define the prelabel of S as (“Complement odd 2-join”, $\alpha_A, \alpha_B, \alpha_X$). Observe that in this case $|S| = 2$.

- If the decomposition is an odd 2-join, then recursively compute $\alpha_{AC} = \alpha(T_X|(A_1 \cup C_1))$, $\alpha_{BC} = \alpha(T_X|(B_1 \cup C_1))$, $\alpha_C = \alpha(T_X|C_1)$ and $\alpha_X = \alpha(T_X|X)$ and define the prelabel of S as (“Odd 2-join”, $\alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X$). Observe that in this case $|S| = 2$ and no vertex of $T'_Y \setminus S$ is strongly complete to S.

- If the decomposition is a complement even 2-join, then recursively compute $\alpha_A = \alpha(T_X|A_1)$, $\alpha_B = \alpha(T_X|B_1)$ and $\alpha_X = \alpha(T_X|X)$, and define the prelabel of S as ("Complement even 2-join", $\alpha_A, \alpha_B, \alpha_X$). Observe that in this case $|S| = 3$ and S is light.

- If the decomposition is an even 2-join, then recursively compute $\alpha_{AC} = \alpha(T_X|(A_1 \cup C_1))$, $\alpha_{BC} = \alpha(T_X|(B_1 \cup C_1))$, $\alpha_C = \alpha(T_X|C_1)$ and $\alpha_X = \alpha(T_X|X)$ and define the prelabel of S as ("Even 2-join", $\alpha_{AC}, \alpha_{BC}, \alpha_C, \alpha_X$). Observe that in this case $|S| = 3$ and S is heavy.

Now, $(T_Y, \mathcal{S}'_Y)$ has a prelabeling $\mathcal{L}_Y$. We recursively run our algorithm for $(T_Y, \mathcal{S}'_Y, \mathcal{L}_Y)$.

We obtain an extension $\mathcal{L}'_Y$ of $\mathcal{L}_Y$ and a maximum weighted strong stable set of the expansion T'_Y of $(T_Y, \mathcal{S}'_Y, \mathcal{L}'_Y)$.

We use $\mathcal{L}'_Y$ to finish the construction of $\mathcal{L}'$, using for each $S \in \mathcal{S}_Y$ the same extension as we have in $\mathcal{L}'_Y$ for extending $\mathcal{L}_Y$. Hence, now, we have an extension $\mathcal{L}'$ of $\mathcal{L}$. Let T' be the expansion of $(T, S, \mathcal{L}')$.

Observe now that by 7.2, T'_Y is precisely a gadget for T', as defined in Section 6. Hence, $\alpha(T')$ may be recovered from $\alpha(T'_Y)$, as explained in one of 6.1, 6.6, or 6.7.

Hence, the algorithm works correctly when it returns $\mathcal{L}'$ and the maximum weight of a strong stable set that we have just computed.

Complexity analysis: By the way we construct our blocks of decomposition, we have $|V(T_X)| - 3 + |V(T_Y)| - 3 \leq n$ and by 3.1(viii) we have $6 \leq |V(T_X)|, |V(T_Y)| \leq n - 1$. Recall that we have assumed that $|V(T_X)| \leq |V(T_Y)|$.

Let $T(n)$ be the complexity of our algorithm. For each kind of decomposition we perform at most four recursive calls on the small side, namely T_X, and one recursive call for the big side T_Y. So we have $T(n) \leq dn^4$ when the graph is basic and otherwise $T(n) \leq 4T(|V(T_X)|) + T(|V(T_Y)|) + dn^4$, where d is the constant arising from the complexity of finding a 2-join or a complement 2-join and finding α in basic trigraphs.

We now prove that there exists a constant c such that $T(n) \leq cn^5$. Our proof is by induction on n. We show that there exists a constant N such that the induction step of our induction goes through for all $n \geq N$ (this argument, and in particular N, does not depend on c). The base case of our induction is therefore graphs that are either basic or have at most N vertices. For them, c clearly exists.

We write the proof of the induction step only when the decomposition under consideration is an even 2-join (possibly in the complement). The proof for the odd 2-join is similar. We set $n_1 = |V(T_X)|$. We have $T(n) \leq 4T(n_1) + T(n + 6 - n_1) + dn^4$ for all n_1 and n satisfying $\lfloor \frac{n}{2} \rfloor + 3 \geq n_1 \geq 7$.

Let us define $f(n_1) = n^5 - 4n_1^5 - (n + 6 - n_1)^5 - dn^4$. We show that there exists a constant N such that $f(n_1) \geq 0$ for all $n \geq N$ and all n_1 such that $7 \leq n_1 \leq \lfloor \frac{n}{2} \rfloor + 3$. By the induction hypothesis, this proves our claim. A simple computation yields:

$$f'(n_1) = -20n_1^4 + 5(n + 6 - n_1)^4$$

$$f''(n_1) = -80n_1^3 - 20(n + 6 - n_1)^3$$

Since $n + 6 - n_1$ is positive, we have $f'' \leq 0$. So, f' is decreasing, and it is easy to see that if n is large enough, it is positive for $n_1 = 7$ and negative for $n_1 = \lfloor \frac{n}{2} \rfloor + 3$. Now f is minimum for $n_1 = 7$ or $n_1 = \lfloor \frac{n}{2} \rfloor + 3$. Since $f(7) = n^5 - (n-1)^5 - P(n)$ where P is a polynomial with $\deg(P) \leq 4$, if n is large enough, then $f(7)$ is positive. Also $f(\lfloor \frac{n}{2} \rfloor + 3) \leq n^5 - 5(\lceil \frac{n}{2} \rceil + 3)^5$. Again, if n is large enough, $f(\lfloor \frac{n}{2} \rfloor + 3)$ is positive. Hence, there exists a constant N such that for all $n \geq N$, $f(n_1) \geq 0$. This means that our algorithm runs in time $O(n^5)$. ∎

7.4 *A maximum weighted strong stable set of a trigraph T in $\mathcal{F}$ with no balanced skew-partition can be computed in time $O(n^5)$.*

Proof. Run the algorithm from 7.3 for $(T, \emptyset, \emptyset)$. ∎

7.5 *A maximum weighted stable set of a Berge graph with no balanced skew-partition can be computed in time $O(n^5)$.*

Proof. Follows from 7.4 and the fact that a Berge graph may be seen as a trigraph from $\mathcal{F}$. ∎

8 Coloring perfect graphs with an algorithm for stable sets

Gröstchel, Lovász and Schrijver [10] proved that the ellipsoid method yields a polynomial time algorithm that optimally colors any input perfect graph. However, so far, no purely combinatorial method is known. But, one is known (also due to Grötschel, Lovász and Schrijver), under the assumption that a subroutine for computing a maximum stable set is available. The goal of this section is to present this algorithm, because it is hard to extract it from the deeper material that surrounds it in [10] or [13].

In what follows, n denotes the number of vertices of the graph under consideration. We suppose that $\mathcal{C}$ is a subclass of perfect graphs, and there is an $O(n^k)$ algorithm $\mathcal{A}$ that computes a maximum weighted stable set and a maximum weighted clique for any input graph in $\mathcal{C}$.

8.1 (Lovász [15]) *A graph is perfect if and only if its complement is perfect.*

8.2 *There is an algorithm with the following specification:*

Input: *A graph G in $\mathcal{C}$, and a sequence $K_1, \dots, K_t$ of maximum cliques of G where $t \leq n$.*

Output: *A stable set of G that intersects each K_i, $i = 1, \dots, t$.*

Running time: $O(n^k)$

Proof. By $\omega(G)$ we mean here the maximum *cardinality* of a clique in G. Give to each vertex v the weight $y_v = |\{i; v \in K_i\}|$. Note that this weight is possibly zero. With Algorithm $\mathcal{A}$, compute a maximum weighted stable set S of G.

Let us consider the graph G' obtained from G by replicating y_v times each vertex v. So each vertex v in G becomes a stable set Y_v of size y_v in G' and between two such stable sets Y_u, Y_v there are all possible edges if $uv \in E(G)$ and no edges otherwise. Note that vertices of weight zero in G are not in $V(G')$. Note also that G' may fail to be in $\mathcal{C}$, but it is easily seen to be perfect. By replicating y_v times each vertex v of S, we obtain a stable set S' of G' of maximum cardinality.

By construction, $V(G')$ can be partitioned into t cliques of size $\omega(G)$ that form an optimal coloring of $\overline{G'}$ because $\omega(G') = \omega(G)$. Since by Theorem 8.1 $\overline{G'}$ is perfect, $|S'| = t$. So, in G, S intersects every K_i, $i \in \{1, \dots, t\}$. ∎

8.3 *There exists an algorithm of complexity $O(n^{k+2})$ whose input is a graph from $\mathcal{C}$ and whose output is an optimal coloring of G.*

Proof. We only need to show how to find a stable set S intersecting all maximum cliques of G, since we can apply recursion to $G \setminus S$ (by giving weight 0 to vertices of S). Start with $t = 0$. At each iteration, we have a list of t maximum cliques $K_1, \dots, K_t$ and we compute by the algorithm in Lemma 8.2 a stable set S that intersects every K_i, $i \in \{1, \dots, t\}$. If $\omega(G \setminus S) < \omega(G)$ then S intersects every maximum clique, otherwise we can compute a maximum clique K_{t+1} of $G \setminus S$ (by giving weight 0 to vertices of S). This will eventually find the desired stable set, the only problem being the number of iterations. We show that this number is bounded by n.

Let M_t be the incidence matrix of the cliques $K_1, \dots, K_t$. So the columns of M_t correspond to the vertices of G and each row is a clique (we see K_i as row vector). We prove by induction that the rows of M_t are independent. So, we assume that the rows of M_t are independent and prove that this holds again for M_{t+1}.

The incidence vector x of S is a solution to $M_t x = \mathbf{1}$ but not to $M_{t+1} x = \mathbf{1}$. If the rows of M_{t+1} are not independent, we have $K_{t+1} = \lambda_1 K_1 + \cdots + \lambda_t K_t$. Multiplying by x, we obtain $K_{t+1} x = \lambda_1 + \cdots + \lambda_t \neq 1$. Multiplying by $\mathbf{1}$, we obtain $\omega = K_{t+1}\mathbf{1} = \lambda_1 \omega + \cdots + \lambda_t \omega$, so $\lambda_1 + \cdots + \lambda_t = 1$, a contradiction.

So the matrices $M_1, M_2, \dots$ cannot have more than n rows. Hence, there are at most $|V(G)|$ iterations. ∎

Proof of Theorem 1.1

Proof. An $O(n^5)$ time algorithm exists for the maximum weighted stable set by 7.5, so an $O(n^7)$ time coloring algorithm for the same class exists by 8.3. ∎

9 Extreme decomposition

In this section, we prove that non-basic trigraphs in our class actually have extreme decompositions. They are decompositions where one block of decomposition is basic. Note that this is non-trivial in general, since in [20] an example is given, showing that Berge graphs in general do not necessarily have extreme 2-joins. Extreme decompositions are sometimes very useful for proofs by induction.

In fact, we are not able to prove that any trigraph in our class has an extreme 2-join or complement 2-join; to prove such a statement, we have to include a new decomposition, the homogeneous pairs, in our set of decompositions. Interestingly this decomposition is not new, it has been used in several variants of Theorem 2.5.

A *proper homogeneous pair* of a trigraph T is a pair of disjoint nonempty subsets (A, B) of $V(T)$, such that if A_1, A_2 denote respectively the sets of all strongly A-complete and strongly A-anticomplete vertices and B_1, B_2 are defined similarly, then:

- $|A| > 1$ and $|B| > 1$;
- $A_1 \cup A_2 = B_1 \cup B_2 = V(T) \setminus (A \cup B)$ (and in particular every vertex in A has a neighbor and an antineighbor in B and vice versa); and
- the four sets $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$, $A_2 \cap B_2$ are all nonempty.

In these circumstances, we say that $(A, B, A_1 \cap B_2, A_2 \cap B_1, A_1 \cap B_1, A_2 \cap B_2)$ is a *split* of the homogeneous pair.

A way to prove the existence of an extreme decomposition is to consider a "side" of a decomposition and to minimize it, to obtain what we call an *end*. But for homogeneous pairs, the two sides (which are $A \cup B$ and $V(T) \setminus (A \cup B)$ with our usual notation) are not as symmetric as the two sides of a 2-join, so we have to decide which side is to be minimized. We decide to minimize the side $A \cup B$. To make all this formal, we therefore have to distinguish between a *fragment*, which is any side of any decomposition, and a *proper fragment* which is a side to be minimized, and therefore cannot be the side $V(T) \setminus (A \cup B)$ of a homogeneous pair. All definitions are formally given below.

First we modify our definition of a fragment to include homogeneous pairs. From here on, A set $X \subseteq V(T)$ is a *fragment* of a trigraph T if one of the following holds:

1. $(X, V(T) \setminus X)$ is a proper 2-join of T;
2. $(X, V(T) \setminus X)$ is a proper complement 2-join of T;
3. there exists a proper homogeneous pair (A, B) of T such that $X = A \cup B$ or $X = V(T) \setminus (A \cup B)$.

A set $X \subseteq V(T)$ is a *proper fragment* of a trigraph T if one of the following holds:

1. $(X, V(T) \setminus X)$ is a proper 2-join of T;
2. $(X, V(T) \setminus X)$ is a proper complement 2-join of T;
3. there exists a proper homogeneous pair (A, B) of T such that $X = A \cup B$.

An *end* of T is a proper fragment X of T such that no proper induced subtrigraph of X is a proper fragment of T.

Note that a proper fragment of T is a proper fragment of $\overline{T}$, and an end of T is an end of $\overline{T}$. Moreover a fragment in T is still a fragment in $\overline{T}$. We have already defined the blocks of decomposition of a 2-join or complement-2-join. We now define the blocks of decomposition of a homogeneous pair.

If $X = A \cup B$ where (A, B, C, D, E, F) is a split of a proper homogeneous pair (A, B) of T, then we build the block of decomposition as follows. We start with $T|(A \cup B)$. We then add two new *marker vertices* c and d such that c is strongly complete to A, d is strongly complete to B, cd is a switchable pair, and there are no other edges between $\{c, d\}$ and $A \cup B$. Again, $\{c, d\}$ is called the *marker component of* T_X.

If $X = C \cup D \cup E \cup F$ where (A, B, C, D, E, F) is a split of a proper homogeneous pair (A, B) of T, then we build the block of decomposition T_X with respect to X as follows. We start with $T|X$. We then add two new *marker vertices* a and b such that a is strongly complete to $C \cup E$, b is strongly complete to $D \cup E$, ab is a switchable pair, and there are no other edges between $\{a, b\}$ and $C \cup D \cup E \cup F$. Again, $\{a, b\}$ is called the *marker component of* T_X.

9.1 *If X is a fragment of a trigraph T from $\mathcal{F}$ with no balanced skew-partition, then T_X is a trigraph from $\mathcal{F}$.*

Proof. From the definition of T_X, it is clear that every vertex of T_X is in at most one switchable pair, or is heavy, or is light. So, to prove that $T_X \in \mathcal{F}$, it remains only to prove that T_X is Berge.

If the fragment come from a 2-join or the complement of a 2-join, we have the result by 4.1.

If $X = A \cup B$ and (A, B) is a proper homogeneous pair of T, then let H be a hole or an antihole in T_X. Passing to the complement if necessary, we may assume that H is a hole. If it contains the two markers c, d, it must be a cycle on four vertices, or it must contain two strong neighbors of c in A, and two strong neighbors of d in B, so H has length 6. Hence, we may assume that H contains at most one of c, d, so a hole of the same length in T is obtained by possibly replacing c or d by some vertex of C or D. Hence, H has even length.

If there exists a proper homogeneous pair (A, B) of T such that $X = V(T) \setminus (A \cup B)$, then since every vertex of A has a neighbor and an antineighbor in B, we see that every realization of T_X is an induced subgraph of some realization of T. It follows that T_X is Berge. ∎

9.2 *If X is a fragment of a trigraph T from $\mathcal{F}$ with no balanced skew-partition, then the block of decomposition T_X has no balanced skew-partition.*

Proof. To prove this, we suppose that T_X has a balanced skew-partition (A', B') with a split (A'_1, A'_2, B'_1, B'_2). From this, we find a skew-partition in T. Then we use 2.8 to prove the existence of a *balanced* skew-partition in T. This gives a contradiction that proves the theorem.

If the fragment come from a 2-join or the complement of a 2-join, we have the result by 4.2.

If $X = A \cup B$ and (A, B) is a homogeneous pair of T, then let (A, B, C, D, E, F) be a split of (A, B). Because cd is a switchable pair, the markers c and d have no common neighbor and cd dominates T_X, there is up to symmetry only one case: $c \in A'_1$ and $d \in B'_1$. Since B'_2 is complete to d, and A'_2 is anticomplete to c, it follows that $A'_2, B'_2 \subseteq B$.

Now $(A'_1 \setminus \{c\} \cup C \cup F, A'_2, B'_1 \setminus \{d\} \cup D \cup E, B'_2)$ is a split of a skew-partition in T. The pair (A'_2, B'_2) is balanced in T because it is balanced in T_X. Hence, by 2.8, T admits a balanced skew-partition, a contradiction.

If $X = V(T) \setminus (A \cup B)$ and (A, B) is a proper homogeneous pair of T, then let (A, B, C, D, E, F) be a split of (A, B). Because ab is a switchable pair we may assume, using symmetry and complementation that $a \in A'_1$ and $b \in A'_1 \cup B'_1$. If $b \in A'_1$, then $(A \cup B \cup A'_1 \setminus \{a, b\}, A'_2, B'_1, B'_2)$ is a split of a skew-partition in T, and if $b \in B'_1$, then $(A \cup A'_1 \setminus \{a\}, A'_2, B \cup B'_1 \setminus \{b\}, B'_2)$ is a split of a skew-partition in T. In both cases, the pair (A'_2, B'_2) is balanced in T because it is balanced in T_X. Hence, by 2.8, T admits a balanced skew-partition, a contradiction. ∎

9.3 *If X is an end of a trigraph T from $\mathcal{F}$ with no balanced skew-partition, then the block of decomposition T_X is basic.*

Proof. Let T be a trigraph from $\mathcal{F}$ with no balanced skew-partition and X an end of T. By 9.1, we know that $T_X \in \mathcal{F}$ and by 9.2, we know that T_X has no balanced skew-partition. By 3.2, it is enough to show that T_X has no proper 2-join and no proper complement 2-join.

Passing to the complement if necessary, we may assume that one of the following three statements hold:

- $X = A \cup B$ and (A, B) is a proper homogeneous pair of T;
- $(X, V(T) \setminus X)$ is a proper even 2-join of T;
- $(X, V(T) \setminus X)$ is a proper odd 2-join of T.

Case 1: $X = A \cup B$ where (A, B) is a proper homogeneous pair of T. Let (A, B, C, D, E, F) be a split of (A, B).

Suppose T_X admits a proper 2-join (X_1, X_2). Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of (X_1, X_2). Because cd is a switchable pair we may assume that c, d are both in X_2. As $\{c, d\}$ strongly dominates T_X we may assume that $c \in A_2$ and $d \in B_2$, so $C_1 = \emptyset$. Since c is strongly complete to A, $A_1 \subseteq A$, and analogously $B_1 \subseteq B$. By 9.2 and 3.1, $|A_1| \geq 2$ and $|B_1| \geq 2$, and because $C_1 = \emptyset$, every vertex from A_1 has a neighbor and an antineighbor in B_1 and vice versa. Now $(A_1, B_1, C \cup A_2 \setminus \{c\}, D \cup B_2 \setminus \{d\}, E, F \cup C_2)$ is a split of a proper homogeneous pair of T. Because $|X_2| \geq 3$, $A_1 \cup B_1$ is strictly included in $A \cup B$, a contradiction.

Because $A \cup B$ is also a homogeneous pair of $\overline{T}$, by the same argument as above, T_X cannot admit a proper complement 2-join.

Case 2: $(X, V(T) \setminus X)$ is a proper even 2-join (X_1, X_2) of T, where $X = X_1$. Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of $(X, V(T) \setminus X)$.

Suppose that T_X admits a proper 2-join (X'_1, X'_2). Let $(A'_1, B'_1, C'_1, A'_2, B'_2, C'_2)$ be a split of (X'_1, X'_2). Since ac and bc are switchable pairs, we may assume that $a, b, c \in X'_2$. Now we claim that $(X'_1, V(T) \setminus X'_1)$ is a proper 2-join of T and X'_1 is strictly included in X, which gives a contradiction. Note that because of the definition of a 2-join and the fact that c has no strong neighbor, X'_2 cannot only be $\{a, b, c\}$ and hence, X'_1 is strictly included in X. Since c has no strong neighbor, we have $c \in C'_2$. Since a and b have no common strong neighbor in T_{X_1}, there are up to symmetry three cases: either $a \in A'_2$, $b \in B'_2$, or $a \in A'_2$, $b \in C'_2$, or $a, b \in C'_2$.

If $a \in A'_2$ and $b \in B'_2$, then $(A'_1, B'_1, C'_1, A_2 \cup A'_2 \setminus \{a\}, B_2 \cup B'_2 \setminus \{b\}, C_2 \cup C'_2 \setminus \{c\})$ is a split of a 2-join of T.

If $a \in A'_2$ and $b \in C'_2$, then $(A'_1, B'_1, C'_1, A_2 \cup A'_2 \setminus \{a\}, B'_2, B_2 \cup C_2 \cup C'_2 \setminus \{b, c\})$ is a split of a 2-join of T.

If $a \in C'_2$ and $b \in C'_2$, then $(A'_1, B'_1, C'_1, A'_2, B'_2, X_2 \cup C'_2 \setminus \{a, b, c\})$ is a split of a 2-join of T.

By 9.2 and 3.1 each of these 2-joins is proper, and we have a contradiction.

Suppose T_X admits a proper complement 2-join (X'_1, X'_2). Because c has no strong neighbor we get a contradiction.

Case 3: $(X, V(T) \setminus X)$ is a proper odd 2-join (X_1, X_2) of T, where $X = X_1$. Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of $(X, V(T) \setminus X)$.

Suppose T_X admits a proper 2-join (X'_1, X'_2). Let $(A'_1, B'_1, C'_1, A'_2, B'_2, C'_2)$ be a split of (X'_1, X'_2). Since ab is a switchable pair, we may assume that $a, b \in X'_2$. Now we claim that $(X'_1, V(T) \setminus X'_1)$ is a proper 2-join of T, obtaining a contradiction, because X'_2 cannot be only $\{a, b\}$ (by the definition of a 2-join), so X'_1 is strictly included in X. Because a and b have no common strong neighbor in T_{X_1} there are up to symmetry three cases: either $a \in A'_2$, $b \in B'_2$, or $a \in A'_2$, $b \in C'_2$, or $a, b \in C'_2$.

If $a \in A'_2$ and $b \in B'_2$, then $(A'_1, B'_1, C'_1, A_2 \cup A'_2 \setminus \{a\}, B_2 \cup B'_2 \setminus \{b\}, C_2 \cup C'_2)$ is a split of a 2-join of T.

If $a \in A'_2$ and $b \in C'_2$, then $(A'_1, B'_1, C'_1, A_2 \cup A'_2 \setminus \{a\}, B'_2, B_2 \cup C_2 \cup C'_2 \setminus \{b\})$ is a split of a 2-join of T.

If $a \in C'_2$ and $b \in C'_2$, then $(A'_1, B'_1, C'_1, A'_2, B'_2, X_2 \cup C'_2 \setminus \{a, b\})$ is a split of a 2-join of T.

By 9.2 and 3.1 each of these 2-joins is proper, and we have a contradiction.

Suppose T_X admits a proper complement 2-join (X'_1, X'_2). Let $(A'_1, B'_1, C'_1, A'_2, B'_2, C'_2)$ be a split of (X'_1, X'_2). Because ab is a switchable pair, we may assume that $a, b \in X'_2$. Because a and b have no common strong neighbor we may assume that $a \in A'_2$, $b \in B'_2$ and $C'_1 = \emptyset$. If C_2 and C'_2 are not empty, then $(A'_1, B'_1, B_2 \cup B'_2 \setminus \{b\}, A_2 \cup A'_2 \setminus \{a\}, C'_2, C_2)$ is a split of a proper homogeneous pair of T and $A'_1 \cup B'_1$ is strictly included in X, a contradiction (note that by 9.2 and 3.1, $|A'_1| \geq 2$, $|B'_1| \geq 2$, and each vertex from A'_1 has a neighbor and an antineighbor in B'_1 and vice versa). If C_2 is not empty and C'_2 is empty, then $(A'_1, B'_1, \emptyset, B_2 \cup B'_2 \setminus \{b\}, A_2 \cup A'_2 \setminus \{a\}, C_2)$ is a split of a proper 2-join of T (the 2-join is proper by 9.2 and 3.1). If C_2 is empty, then $(A'_1, B'_1, \emptyset, A_2 \cup A'_2 \setminus \{a\}, B_2 \cup B'_2 \setminus \{b\}, C'_2)$ is a split of a proper complement 2-join of T (again, it is proper by 9.2 and 3.1). ∎

10 Means to an end

The goal of this section is to describe a polynomial time algorithm that outputs an end (defined in Section 9) of an input trigraph (if any). To do so, one may rely on existing algorithms for detecting 2-joins and homogeneous pairs. The fastest one is in [3] for 2-joins and [11] for homogeneous pairs. But there are several problems with this approach. First, all the classical algorithms work for graphs, not for trigraphs. They are easy to convert into algorithms for trigraphs, but it is hard to get convinced by that without going through all the algorithms. Worse, most of the algorithms output a fragment, not an end. In fact, for the 2-join, an algorithm from [3] does output a minimal set X such that $(X, V(G) \setminus X)$ is a 2-join, but there still could be a homogeneous pair inside X. So, we prefer to write our own algorithm, even if most ideas are from existing work.

Our algorithm looks for a proper fragment X. In order to unify the definitions of a fragment used for 2-joins, complements of 2-joins and homogeneous pairs, we introduce a new notion. A *weak fragment* of a trigraph T is a set $X \subseteq V(T)$ such that there exist disjoint sets A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 satisfying:

- $X = A_1 \cup B_1 \cup C_1 \cup D_1$;
- $V(T) \setminus X = A_2 \cup B_2 \cup C_2 \cup D_2$;
- A_1 is strongly complete to $A_2 \cup D_2$ and strongly anticomplete to $B_2 \cup C_2$;
- B_1 is strongly complete to $B_2 \cup D_2$ and strongly anticomplete to $A_2 \cup C_2$;
- C_1 is strongly anticomplete to $A_2 \cup B_2 \cup C_2$;
- D_1 is strongly complete to $A_2 \cup B_2 \cup D_2$;
- $|X| \geq 4$ and $|V(T) \setminus X| \geq 4$;
- $|A_i| \geq 1$ and $|B_i| \geq 1$, $i = 1, 2$;
- and at least one of the following statement:
  - $C_1 = D_1 = \emptyset$, $C_2 \neq \emptyset$, and $D_2 \neq \emptyset$, or
  - $D_1 = D_2 = \emptyset$, or
  - $C_1 = C_2 = \emptyset$.

In these circumstances, we say that $(A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2)$ is a *split* for X. Given a weak fragment we say it is of type *homogeneous pair* if $C_1 = D_1 = \emptyset$, $C_2 \neq \emptyset$, and $D_2 \neq \emptyset$, of type 2-*join* if $D_1 = D_2 = \emptyset$, and of type complement 2-join if $C_1 = C_2 = \emptyset$. Note that a weak fragment may be simultaneously a 2-join fragment and a complement 2-join fragment (when $C_1 = D_1 = C_2 = D_2 = \emptyset$).

10.1 *If T is a trigraph from $\mathcal{F}$ with no balanced skew-partition, then X is a weak fragment of T if and only if X is a proper fragment of T.*

Proof. If X is a proper fragment, then it is clearly a weak fragment (the conditions $|X| \geq 4$ and $|V(T) \setminus X| \geq 4$ are satisfied when X is a side of a 2-join by 3.1). Let us prove the converse. Let X be a weak fragment, and let $(A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2)$ be a split for X. If X is of type 2-join or complement 2-join, then it is proper by 3.1. Thus we may assume that X is of type homogeneous pair, and so $C_1 = D_1 = \emptyset$, $C_2 \neq \emptyset$, and $D_2 \neq \emptyset$. Since all 4 sets A_1, A_2, B_1, B_2 are non-empty, it remains to check the following:

(i) Every vertex of $A_1(B_1)$ has a neighbor and antineighbor in $B_1(A_1)$.

(ii) $|A_1| > 1$ and $|B_1| > 1$.

Suppose (i) does not hold. By passing to $\overline{T}$ if necessary, we may assume that some $v \in A_1$ is strongly complete to B_1. Since $\{v\} \cup B_1 \cup A_2 \cup D_2$ is not a star cutset in T by 2.6, it follows that $A_1 = \{v\}$. Now every vertex of B_1 is strongly complete to A_1, and so, by the same argument, $|B_1| = 1$, contradicting the assumption that $|X| \geq 4$. Therefore (i) holds.

To prove (ii) assume that $|A_1| = 1$. Since $|X| \geq 4$, it follows that $|B_1| \geq 3$. By (i) every vertex of B_1 is semi-adjacent to the unique vertex of A_1, which is impossible since $|B_1| \geq 3$ and $T \in \mathcal{F}$. Therefore (ii) holds. ∎

A 4-tuple (a_1, b_1, a_2, b_2) of vertices from a trigraph T is *proper* if:

- a_1, b_1, a_2, b_2 are pairwise distinct;
- $a_1a_2, b_1b_2 \in \eta(T)$;
- $a_1b_2, b_1a_2 \in \nu(T)$.

A proper 4-tuple (a_1, b_1, a_2, b_2) is *compatible* with a weak fragment X if there is a split $(A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2)$ for X such that $a_1 \in A_1$, $b_1 \in B_1$, $a_2 \in A_2$ and $b_2 \in B_2$.

We use the following notation. When x is a vertex of a trigraph T, $N(x)$ denotes the set of the neighbors of x, $\overline{N}(x)$ denotes the set of the antineighbors of x, $\eta(x)$ the set of the strong neighbors of x, and $\sigma(x)$ the set of vertices v such that $xv \in \sigma(T)$.

10.2 *Let T be a trigraph and $Z = (a_1, b_1, a_2, b_2)$ a proper 4-tuple of T. There is an $O(n^2)$ time algorithm that given a set $R_0 \subseteq V(T)$ of size at least 4 such that $Z \cap R_0 = \{a_1, b_1\}$, outputs a weak fragment X compatible with Z and such that $R_0 \subseteq X$, or outputs the true statement "There exists no weak fragment X compatible with Z and such that $R_0 \subseteq X$".*

Moreover, when X is outputted, it is minimal with respect to these properties, meaning that $X \subset X'$ for every weak fragment X' satisfying the properties.

Proof. We use the procedure described in Table 1. It tries to build a weak fragment R, starting with $R = R_0$ and $S = V(T) \setminus R_0$. Then, several forcing rules are implemented, stating that some sets of vertices must be moved from S to R. The variable "State" contains the type of the weak fragment

Input: R_0 a set of vertices of a trigraph T and a proper 4-tuple $Z = (a_1, b_1, a_2, b_2)$ such that $a_1, b_1 \in R_0$ and $a_2, b_2 \notin R_0$.

Initialization:

$R \leftarrow R_0$; $S \leftarrow V(T) \setminus R_0$; $A \leftarrow \eta(a_1) \cap S$; $B \leftarrow \eta(b_1) \cap S$;
State $\leftarrow$ Unknown;

Vertices a_1, b_1, a_2, b_2 are left unmarked. For the other vertices of T:

Mark(x) $\leftarrow \alpha\beta$ for every vertex $x \in \eta(a_2) \cap \eta(b_2)$;

Mark(x) $\leftarrow \alpha$ for every vertex $x \in \eta(a_2) \setminus \eta(b_2)$;

Mark(x) $\leftarrow \beta$ for every vertex $x \in \eta(b_2) \setminus \eta(a_2)$;

Every other vertex of T is marked by ε;

Move($\sigma(a_1) \cap S$); Move($\sigma(b_1) \cap S$);

Main loop:

While there exists a vertex $x \in R$ marked

Do Explore(x); Unmark(x);

Output $(R \cap (\eta(a_2) \setminus \eta(b_2)), R \cap (\eta(b_2) \setminus \eta(a_2)), R \setminus (\eta(a_2) \cup \eta(b_2)), R \cap (\eta(a_2) \cap \eta(b_2)), A \setminus B, B \setminus A, S \setminus (A \cup B), A \cap B)$ as a split of the weak fragment R.

Function Explore(x):

If Mark(x) $= \alpha\beta$ **and** State = Unknown **then**
State $\leftarrow \overline{\text{2-join}}$, Move($S \setminus (A \cup B)$);

If Mark(x) $= \alpha\beta$ **and** State = $\overline{\text{2-join}}$ **then** Move($\overline{N}(x) \cap S$);

If Mark(x) $= \alpha\beta$ **and** State = 2-join **then**
Output *No weak fragment is found*, **Stop**;

If Mark(x) $= \alpha$ **then** Move($A \Delta (\eta(x) \cap S)$), Move($\sigma(x) \cap S$);

If Mark(x) $= \beta$ **then** Move($B \Delta (\eta(x) \cap S)$), Move($\sigma(x) \cap S$);

If Mark(x) $= \varepsilon$ **and** State = Unknown **then**
State $\leftarrow$ 2-join, Move($A \cap B$);

If Mark(x) $= \varepsilon$ **and** State = 2-join **then** Move($N(x) \cap S$);

If Mark(x) $= \varepsilon$ **and** State = $\overline{\text{2-join}}$ **then**
Output *No weak fragment is found*, **Stop**;

Function Move(Y):

This function just moves a subset $Y \subset S$ from S to R.

If $Y \cap \{a_2, b_2\} \neq \emptyset$ **then**
Output *No weak fragment is found*, **Stop**;

$R \leftarrow R \cup Y$; $A \leftarrow A \setminus Y$; $B \leftarrow B \setminus Y$; $S \leftarrow S \setminus Y$;

Table 1: Procedure used in Theorem 10.2

that is being considered. At the beginning, it is “Unknown”. The following properties are easily checked to be invariant during all the execution of the procedure (meaning that they are satisfied after each call to Explore):

- R and S form a partition of $V(T)$, $R_0 \subseteq R$ and $a_2, b_2 \in S$.
- For all unmarked $v \in R$, and all $u \in S$, uv is not a switchable pair.
- All unmarked vertices belonging to $R \cap (\eta(a_2) \setminus \eta(b_2))$ have the same neighborhood in S, namely A (and A is a strong neighborhood).
- All unmarked vertices belonging to $R \cap (\eta(b_2) \setminus \eta(a_2))$ have the same neighborhood in S, namely B (and B is a strong neighborhood).
- All unmarked vertices belonging to $R \cap (\eta(b_2) \cap \eta(a_2))$ have the same neighborhood in S, namely $A \cup B$.
- All unmarked vertices belonging to R not adjacent to a_2 nor b_2 are strongly anticomplete to S.
- For every weak fragment X such that $R_0 \subseteq X$ and $a_2, b_2 \in V(T) \setminus X$, we have that $R \subseteq X$ and $V(T) \setminus X \subseteq S$.

By the last item all moves from S to R are necessary. Hence, if some vertex in R is strongly adjacent to a_2 and b_2, any weak fragment compatible with Z that contains R must be a complement 2-join fragment. This is why the variable State is assigned value $\overline{\text{2-join}}$ and all vertices of $S \setminus (A \cup B)$ are moved to R. Similarly, if some vertex in R is strongly antiadjacent to a_2 and b_2, any weak fragment compatible with Z that contains R must be a 2-join fragment. This is why the variable State is assigned value 2-join and all vertices of $A \cap B$ are moved to R.

When State $=$ $\overline{\text{2-join}}$ and a vertex in R is discovered to be strongly antiadjacent to a_2 and b_2, there is a contradiction with the definition of the complement 2-join, so the algorithm must stop. When State $=$ 2-join and a vertex in R is discovered to be strongly adjacent to a_2 and b_2, there is a contradiction with the definition of the 2-join, so the algorithm must stop. When the function Move tries to move a_2 or b_2 in R (this may happen if some vertex in R is semiadjacent to a_2 or b_2), then R cannot be contained in any fragment compatible with Z.

If the process does not stop for all the reasons above, then all vertices of R have been explored and therefore are unmarked. So, if $|S| \geq 4$, at the end, R, is a weak fragment compatible with Z. More specifically, $(R \cap (\eta(a_2) \setminus$

$\eta(b_2)), R \cap (\eta(b_2) \setminus \eta(a_2)), R \setminus (\eta(a_2) \cup \eta(b_2)), R \cap (\eta(a_2) \cap \eta(b_2)), A \setminus B, B \setminus A, S \setminus (A \cup B), A \cap B)$ is a split for the weak fragment R.

Since all moves from S to R are necessary, the fragment is minimal as claimed. This also implies that if $|S| \leq 3$, then no desired fragment exists, in which case, the algorithm outputs that no weak fragment exists.

Complexity Issues: The neighborhood and antineighborhood of a vertex in R is considered at most once. So, globally, the process requires $O(n^2)$ time. ∎

10.3 *There exists an $O(n^5)$ time algorithm whose input is a trigraph T from $\mathcal{F}$ with no balanced skew-partition, and whose output is an end X of T (if any such end exists) and the block T_X.*

Proof. Recall that by 10.1, the weak fragments of T are its proper fragments. We first describe an $O(n^8)$ time algorithm, and then we explain how to speed it up. We assume that $|V(T)| \geq 8$ for otherwise no proper fragment exists. For all proper 4-tuple $Z = (a_1, a_2, b_1, b_2)$ and for all pairs of vertices u, v of $V(T) \setminus \{a_1, a_2, b_1, b_2\}$, we apply 10.2 to $R_0 = \{a_1, b_1, u, v\}$. This method detects for each Z and each u, v a proper fragment compatible with Z, containing u, v, and minimal with respect to these properties (if any). Among all these fragments, we choose one with minimum cardinality, this is an end. Once the end is given, it is easy to know the type of decomposition that is used and to build the corresponding block (in particular, by 2.4, one may test by just checking one path whether a 2-join is odd or even). Let us now explain how to speed this up.

We look for 2-joins and homogeneous pairs separately. We describe an $O(n^5)$ time procedure that outputs a 2-join weak fragment, an $O(n^5)$ time procedure that outputs a complement 2-join weak fragment, and an $O(n^5)$ time procedure that outputs a homogeneous pair weak fragment. Each of them outputs a fragment of minimum cardinality among all fragments of its respective kind. Hence, a fragment of minimum cardinality chosen among the three is an end.

Let us first deal with 2-joins. A set $\mathcal{Z}$ of proper 4-tuples is *universal* if for every proper 2-join with split $(A_1, B_1, C_1, A_2, B_2, C_2)$, there exists $(a_1, a_2, b_1, b_2) \in \mathcal{Z}$ such that $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$, $b_2 \in B_2$. Instead of testing all 4-tuples as in the $O(n^8)$ time algorithm above, it is obviously enough to restrict the search to a universal set of 4-tuples. As proved in [3], there exists an algorithm that generates in time $O(n^2)$ a universal set of size at most $O(n^2)$ for any input graph. It is easy to obtain a similar algorithm for trigraphs.

The next idea for 2-joins is to apply the method from Table 1 to $R_0 = \{a_1, b_1, u\}$ for all u's instead of $R_0 = \{a_1, b_1, u, v\}$ for all u, v's. As we explain now, this finds a 2-join compatible with $Z = (a_1, a_2, b_1, b_2)$ when there is one. For suppose (X_1, X_2) is such a 2-join. If X_1 contains a vertex v whose neighborhood (in T) is different from $\{a_1, b_1\}$, then by 3.1, v has at least one neighbor in $V(T) \setminus \{a_1, b_1\}$. Hence, when the loop considers $u = v$, the method from Table 1 moves some new vertices in R. So, at the end, $|R| \geq 4$ and the 2-join is detected. So, the method fails to detect a 2-join only when v has degree 2 and a_1-v-b_1 is a path while a 2-join compatible with Z exists, with v in the same side as a_1, b_1. In fact, since all vertices u are tried, this is a problem only if this failure occurs for every possible u, that is if the 2-join we look for has one side made of a_1, b_1, and a bunch of vertices $u_1, \dots, u_k$ of degree 2 all adjacent to a_1 and b_1. But in this case, either one of the u_i's is strongly complete to $\{a_1, b_1\}$ and it is the center of a star cutset, or all the u_i's are adjacent to at least one of a_1, b_1 by a switchable pair. In this last case, all the u_i's are moved to R when we run the method from Table 1, so the 2-join is in fact detected.

Complement 2-joins are handled by the same method in the complement.

Let us now consider homogeneous pairs. It is convenient to define *weak homogeneous pairs* exactly as proper homogeneous pairs, except that we require that "$|A| \geq 1$, $|B| \geq 1$ and $|A \cup B| \geq 3$" instead of "$|A| > 1$ and $|B| > 1$". A theorem similar to 10.2 exists, where the input of the algorithm is a graph G, a triple $(a_1, b_1, a_2) \in V(G)^3$ and a set $R_0 \subseteq V(G)$ that contains a_1, b_1 but not a_2, and the output is a weak homogeneous pair (A, B) such that $R_0 \subseteq A \cup B$, $a_1 \in A$, $b_1 \in B$ and $a_2 \notin A \cup B$, and such that a_2 is complete to A and anticomplete to B, if any such weak homogeneous pair exits. As in 10.2, the running time is $O(n^2)$ and the weak homogeneous pair is minimal among all possible weak homogeneous pairs. This is proved in [8].

As for 2-joins, we define the notion of a universal set of triples (a_1, b_1, a_2). As proved in [11], there exists an algorithm that generates in time $O(n^2)$ a universal set of size at most $O(n^2)$ of triples for any input graph. It is very easy to obtain a similar algorithm for trigraphs. As in the 2-join case, we apply the analogue of 10.2 to all vertices u instead of all pairs u, v. The only problem is when after the call to the analogue of 10.2, we have a weak and non-proper homogeneous pair (so $|A \cup B| = 3$). But then, it can be checked that the trigraph has a star cutset or a star cutset in the complement. ∎

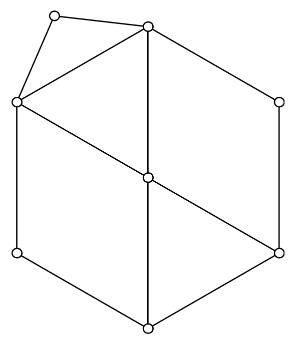

Figure 1: A graph with a balanced skew-partition

11 Enlarging the class: open questions

The class $\mathcal{C}$ of Berge graphs for which we are able to compute maximum stable sets, namely Berge graphs with no balanced skew-partitions, has a strange disease: it is not closed under taking induced subgraphs. But from an algorithmic point of view, since we are able to do the computations with weights on the vertices, we can simulate "taking an induced subgraph" by putting weight zero on the vertices that we want to delete.

This suggests that in fact, we work on the more general class $\mathcal{C}'$ of graphs that are induced subgraphs of some graph in $\mathcal{C}$. The class $\mathcal{C}'$ is closed under taking induced subgraphs so it must be defined by a list of forbidden induced subgraphs. We leave the following questions open: what are the forbidden induced subgraphs for $\mathcal{C}'$? One could think that $\mathcal{C}'$ is in fact the class of all Berge graphs, but it is not the case as shown by the graph G represented in Figure 1. The graph G is Berge and admits an obvious balanced skew-partition. Moreover, Robertson, Seymour and Thomas proved that a Berge graph that contains G as an induced subgraph also admits a balanced skew-partition, see [17], page 78. So, G is not in $\mathcal{C}'$ and G might be the smallest example of a Berge graph not in $\mathcal{C}'$.

Here are more questions on $\mathcal{C}$ and $\mathcal{C}'$. For any graph G in $\mathcal{C}'$, is there a graph H in $\mathcal{C}$ whose size is polynomial in the size of G and such that G is an induced subgraph of H? If yes, or when yes, can we compute H from G in polynomial time?

Acknowledgment

Thanks to Antoine Mamcarz for useful discussions on how to detect an end of a trigraph. Thanks to Fabien de Montgolfier for having suggesting to us that the complexity analysis in the proof of Theorem 7.3 should exist. The work on this paper began when the first and last author were visiting LIAFA under the generous support of Université Paris 7.

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