Monograph:

Peng, Z.K., Lang, ZQ, Billings, S.A. et al. (2005) Analysis of Crack Induced Structure Response Using Nonlinear Output Frequency Response Functions. Research Report. ACSE Research Report 901 . Department of Automatic Control and Systems Engineering

Analysis of Crack Induced Structure Response Using Nonlinear Output Frequency Response Functions

Z K Peng, Z Q Lang, S A Billings, and Y. Lu

Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street, Sheffield
S1 3JD, UK

Research Report No. 901
July 2005

Analysis of Crack Induced Structure Responses Using Nonlinear Output Frequency Response Functions

Z.K. Peng, Z.Q. Lang, S. A. Billings, and Y. Lu

Department of Automatic Control and Systems Engineering, University of Sheffield
Mappin Street, Sheffield, S1 3JD, UK
Email: z.peng@sheffield.ac.uk, z.lang@sheffield.ac.uk

Abstract: Nonlinear Output Frequency Response Functions (NOFRFs) are a new concept recently developed for the analysis of nonlinear systems in the frequency domain. Based on this concept, the output frequency behaviors of nonlinear systems can be expressed using a number of one-dimensional functions similar to the approach used in the traditional frequency response analysis of linear systems. In this paper, the NOFRFs are employed to analyze a cracked beam described by a SODF bilinear model. The results show that the NOFRFs are highly dependent upon the ratio of stiffness of the cracked and the crack free situations, and can thus be used as crack damage indicators. In addition, the higher order NOFRF terms are significantly sensitive to crack size, and hence can be used to distinguish different situations of crack. This research study confirms the conclusion that the increasing crack size will make the structure behavior more nonlinear, and establish an important basis for the use of the NOFRF concept to detect and estimate cracks in mechanical structures.

1 Introduction

Fatigue cracks are a potential source of catastrophic failure in structures. To avoid the failure caused by cracks, many researchers have performed extensive investigations to develop structural integrity monitoring techniques. Most of the techniques are based on vibration measurement and analysis because, in most cases, vibration based methods can offer an effective and convenient way to detect fatigue cracks in structures. But to employ the vibration based methods, it needs a good understanding of the dynamical behaviors and characteristics of cracked structure. It is well known that fatigue cracks will exhibit non-linear behavior due to the opening and closing of a crack, and that this non-linearity can be modeled as a piecewise-linear model, known as a bilinear oscillator model. Many studies about crack structures have adopted the bilinear oscillator model. Zastrau [1] demonstrated the bilinear behavior by using the finite element method to determine the dynamic response of a simply supported beam. Ibrahim, Ismail, and Martin [2] employed a bondgraph model to simulate the dynamic behavior of a cantilever beam with a crack with a bilinear stiffness property. These results showed that a crack closure resulted in a smaller change in natural frequencies as that associated with a constantly open crack. Friswell and Penny [3] studied the non-linear behavior of a beam with a closing crack and then analyzed the forced response to a harmonic excitation at a frequency near the first natural frequency of the beam using a numerical integration method. The results highlighted the presence of super harmonic components in the response spectrum, a common property for non-linear systems. Later, in reference [4], the authors compared the different approaches to crack modeling and demonstrated that using a low frequency vibration, simple models of crack flexibility based on beam elements are adequate for structural health monitoring. Using the bilinear crack model, Chu and Shen [5] also found similar results of the presence of super harmonic components. Furthermore, they [6] proposed a closed-form solution by using two square wave functions to model the stiffness change of bilinear oscillators under a low-frequency excitation. With numerical simulations the authors validated that the solution can provide the spectral pattern and the magnitude of each harmonic component for a damaged rectangular beam according to the size and location of the crack. Sundermeyer and Weaver [7] exploited the weakly non-linear character of a cracked vibrating beam. Their studies supported the possibility that the bilinear behavior of a fatigue crack can be exploited for the purposes of non-destructive evaluation. Based on a bilinear crack model, Chati, Rand and Mukherjee [8] used perturbation methods to obtain the non-linear normal modes of vibration and the associated period of the motion, and the results justified the definition of the bilinear frequency as the effective natural frequency. Rivola and White [9] employed the bilinear oscillator model to simulate the nonlinear behavior of a beam with a closing crack and used the bispectrum to analyze the system response. They found that the normalized bispectrum shows high sensitivity to the bilinear nature of the crack.

All above researches have shown that when a closing crack is present in a beam, higher harmonic components will be generated in the frequency spectrum of the response to a sinusoidal force excitation. The presence of higher harmonic components can be regarded as an indicator of the nonlinear behavior of the cracked beam and should allow the identification of crack types, i.e. whether closing or always opening. The research reported in this paper is dedicated to the analysis of the nonlinear behavior of higher harmonic components caused by cracks using a new concept developed recently known as Nonlinear Output Frequency Response Functions (NOFRFs) [10]-[13]. The concept of NOFRFs in general allows the analysis of nonlinear systems to be implemented in a manner, which is similar to the analysis of the linear system frequency response. This provides a great insight into how nonlinear phenomena, for example how new frequency generation occurs but can also reveal the real mechanisms which dominate the occurrence of many nonlinear behaviors. As the NOFRFs are, like the conventional frequency response function of linear systems, one dimensional function of frequency, the analysis based on this new concept for structure cracks can provide a straightforward interpretation of the phenomena of harmonic generation associated with the existence of cracks. This can considerably facilitate the detection of cracks of different sizes. In the present study, a SDOF model of a cracked beam is considered to investigate this application of the NOFRF concept. The results establish an important basis for the use of NOFRFs to detect and estimate cracks in mechanical structures.

2 Bilinear Oscillator Model of Cracked Beam

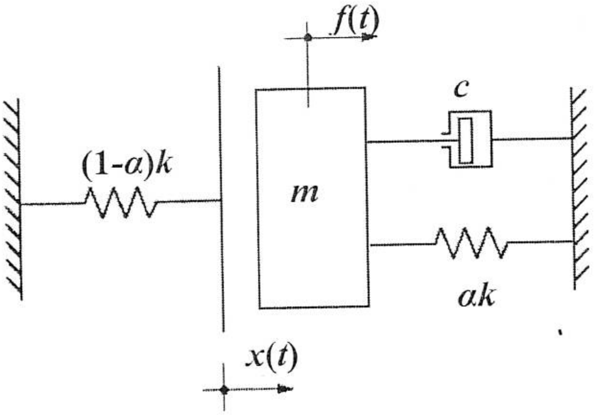


Figure 1. The bilinear oscillator model of a cracked beam

The bilinear oscillator is a simple and effective model that can replicate the nonlinear behavior of a cracked beam very well. For a crack in one edge of a beam, when excited by an external force, the crack will alternate between the open and close status. When a crack is in compression (crack closing) the equivalent stiffness can be considered to be the same as the stiffness of a beam which has no crack. On the other hand, when the crack is stretching (crack opening), the equivalent stiffness near the crack will be reduced. Therefore, the stiffness of a cracked beam will take two different values depending on the status of the crack, open or closed. The present study was based on a SDOF model of a cracked beam from reference [9]. When using the SDOF model, some fundamental assumptions are that the beam vibrates mainly in its first mode and a crack does not affect the mass of the beam. For simplification, the equivalent damping of the beam is assumed to be constant although the friction forces at the point of the crack closure will increase the system damping. The bilinear SDOF model of a cracked beam can be described as a bilinear oscillator shown in Figure 1. The corresponding motion can be expressed as Equation (1).

$$\begin{cases} m\ddot{x} + c\dot{x} + \alpha kx = f(t) & x \geq 0, \\ m\ddot{x} + c\dot{x} + kx = f(t) & x < 0, \end{cases} \tag{1}$$

where m and c are the object mass and damping coefficient respectively; x(t) is the displacement; k is the stiffness at the compression status (crack closing); α is known as the stiffness ratio ($0 \leq \alpha \leq 1$); αk is the stiffness at the stretching status (crack opening); $f(t)$ is the external force exciting the bilinear model. Obviously, if the stiffness ratio α is equal to one, then the model becomes linear. When excited by a sinusoidal force, the response will also be a sinusoidal function. Otherwise, if α is smaller than one, the response is expected to contain several harmonics of the excitation frequency. Define $S(x)$ as the restoring force of the bilinear oscillator as follows

$$S(x) = \begin{cases} \alpha kx & x \geq 0, \\ kx & x < 0, \end{cases} \tag{2}$$

Obviously $S(x)$ is a piecewise linear continuous function of the displacement x as illustrated in Figure 2.

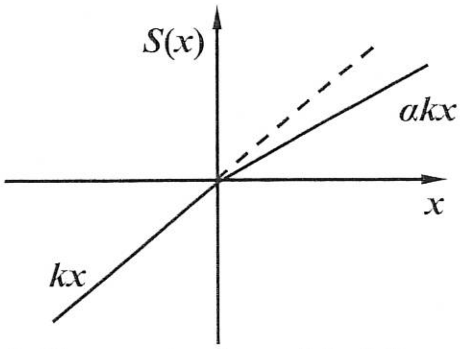


Figure 2. The restoring force of the bilinear oscillator

In mathematics, the Weierstrass Approximation Theorem [14] guarantees that any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy. This theorem is expressed as

If $f(x)$ is a continuous real-valued function on $[\bar{a}, \bar{b}]$ and if any $\varepsilon > 0$ is given, then there exists a polynomial $P(x)$ on $[\bar{a}, \bar{b}]$ such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [\bar{a}, \bar{b}]$.

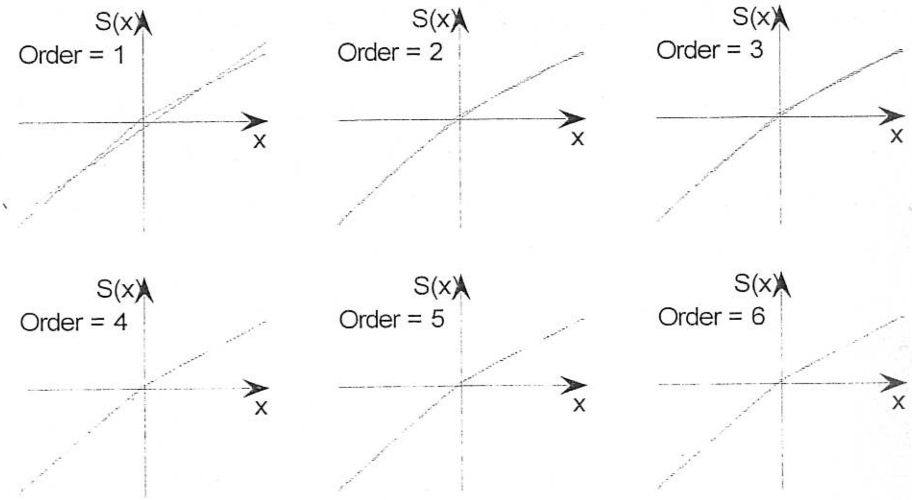


Figure 3. Approximations of $S(x)$ ($\alpha = 0.6$) with polynomials

Since the restoring force $S(x)$ is a continuous function of the displacement x, it can easily be approximated by a polynomial. Figure 3 gives the results of using polynomials with different orders to approximate $S(x)$ where the stiffness ratio α is taken as 0.6. It can be seen that a fourth order polynomial can fit $S(x)$ very well. Using a polynomial $P(x)$ to replace $S(x)$ and ignoring the tiny approximation error, the SDOF model Equation (1) of the cracked beam can be rewritten as

$$m\ddot{x} + c\dot{x} + P(x) = f(t) \tag{3}$$

where

$$P(x) = \sum_{i=1}^{\overline{N}} c_i k x^i \tag{4}$$

$\overline{N}$ is the order of the approximating polynomial, and c_i, $i = 1, \cdots, \overline{N}$ are the polynomial coefficients.

Table 1 The polynomial approximation result for the bilinear oscillator

c / α	c_1	c_2	c_3	c_4
1.00	1.0000	0.0000	0.0000	0.0000
0.90	0.9500	-0.0818	0.0000	0.0407
0.80	0.9000	-0.1637	0.0000	0.0814
0.70	0.8500	-0.2455	0.0000	0.1221
0.60	0.8000	-0.3273	0.0000	0.1629

The model described by Equation (3) is an extensively studied polynomial-type nonlinear system where the term of $c_1 x$ represents the linear part and the other high order terms represent the nonlinear part. For the bilinear oscillator model, the polynomial coefficients are determined by the stiffness ratio α. Table 1 shows the results of using a fourth order polynomial to approximate the bilinear oscillators with different stiffness ratios. It is known from Table 1 that all coefficients, apart from c_1 will increase with the decrease of α. This means that the nonlinear strength of the bilinear oscillator will increase with the decrease of α. It is worthy to note that except for c_1, the values of $c_2 \ldots$ and c_N also depend on the range of x over which the polynomial approximation is defined. In the case shown in Table 1, this range of x is [-1, 1].

For the free undamped vibration of the bilinear oscillator, the effective natural frequency can be determined as ω_0 [8],

$$\omega_0 = 2\omega_1\omega_2/(\omega_1 + \omega_2) \tag{5}$$

where

$$\omega_1 = \sqrt{k/m} \text{ and } \omega_2 = \sqrt{\alpha k/m} \tag{6}$$

Therefore

$$\omega_0 = \frac{2\sqrt{\alpha}}{(1+\sqrt{\alpha})}\sqrt{\frac{k}{m}} \tag{7}$$

For the polynomial-type nonlinear system (3), the natural frequency of its linear part can be defined as

$$\omega_L = \sqrt{c_1 k/m} \tag{8}$$

Table 2 shows a comparison between ω_L and ω_0 under different stiffness ratios. It can be seen that the ω_L is a good approximation of ω_0. To a certain extent, this further demonstrates the feasibility of using a polynomial-type nonlinear model to describe the bilinear model.

Table 2. Comparison between ω_L and ω_0

α	$\omega_L\left(\sqrt{k/m}\right)$	$\omega_0\left(\sqrt{k/m}\right)$	$\lvert\omega_0 - \omega_L\rvert/\omega_0$
1.00	1.0000	1.0000	0.0000%
0.90	0.9747	0.9737	0.1027%
0.80	0.9487	0.9443	0.4660%
0.70	0.9220	0.9111	1.1964%
0.60	0.8944	0.8730	2.4513%

For polynomial-type nonlinear systems, a powerful analysis tool called the Nonlinear Output Frequency Response Function (NOFRF) has recently been developed [12]. The objective of the present study is to analyze the nonlinear behaviors of cracked beams using the new NOFRF concept.

3 Nonlinear Output Frequency Response Function

Recently, the NOFRFs have been proposed and used to investigate the behavior of structures with polynomial-type non-linearities [12]. The definition of NOFRFs is based on the Volterra series. The Volterra series extends the familiar concept of the convolution integral for linear systems to a series of multi-dimensional convolution integrals.

For a linear system, with input $u(t)$ and output $y(t)$, the input and output relationship in the time domain can be described by a convolution integral, as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \tag{9}$$

In the frequency domain, the linear system input output relationship is given by

$$Y(j\omega) = H(j\omega)U(j\omega) \tag{10}$$

when the system is subject to an input where the Fourier Transform exists. In equation (10), $Y(j\omega)$ and $U(j\omega)$ represent the system input and output spectrum which are the Fourier Transforms of the system time domain input $u(t)$ and output $y(t)$ respectively.

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_i \tag{11}$$

where $h_n(\tau_1,\ldots,\tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [10] have derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \quad \text{for } \forall\omega \\ Y_n(j\omega) = \dfrac{1/\sqrt{n}}{(2\pi)^{n-1}} \displaystyle\int_{\omega_1+\cdots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega} \end{cases} \tag{12}$$

This expression reveals how the nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (12), $Y_n(j\omega)$ represents the nth order output frequency response of the system. The nth order Generalised Frequency Response Function (GFRF) is given by

$$H_n(j\omega_1,\ldots,j\omega_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)e^{-(\omega_1\tau_1+\cdots+\omega_n\tau_n)j}d\tau_1\ldots d\tau_n \tag{13}$$

and

$$\int\limits_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)$ over the n-dimensional hyper-plane, with the constraint of $\omega_1+\cdots+\omega_n=\omega$. Equation (12) is a natural extension of the well-known linear relationship (10) to the nonlinear case.

For linear systems, equation (10) shows that the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation (11), however, the relationship between the input and output frequencies is generally given by

$$f_Y = \bigcup_{n=1}^{N} f_{Y_n} \tag{14}$$

where f_Y denotes the non-negative frequency range of the system output, and f_{Y_n} represents the non-negative frequency range produced by the nth-order system nonlinearity. This is much more complicated than that in the linear system case. For the cases where system (12) is subjected to an input with a spectrum given by

$$U(j\omega)=\begin{cases} U(j\omega) & \text{when} \quad |\omega|\in(a,b) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where $b>a\geq 0$. Lang and Billings [10] derived an explicit expression for the output frequency range f_Y of the systems. The result obtained is

$$\begin{cases} f_Y = f_{Y_N}\bigcup f_{Y_{N-(2p^*-1)}} \\ f_{Y_n} = \begin{cases} \bigcup\limits_{k=0}^{i^*-1} I_k & \text{whe n } \dfrac{nb}{(a+b)} - \left\lfloor \dfrac{na}{(a+b)} \right\rfloor < 1 \\ \bigcup\limits_{k=0}^{i^*} I_k & \text{whe n } \dfrac{nb}{(a+b)} - \left\lfloor \dfrac{na}{(a+b)} \right\rfloor \geq 1 \end{cases} \\ i^* = \left\lfloor \dfrac{na}{(a+b)} \right\rfloor + 1 \\ \text{where } \lfloor . \rfloor \text{ means to take the integer part} \\ I_k = \left(na-k(a+b), nb-k(a+b)\right) \text{ for } k=0,\ldots,i^*-1, \\ I_{i^*} = \left(0, nb-i^*(a+b)\right) \end{cases} \tag{16}$$

In (16) p^* could be taken as $1,2,\cdots,\lfloor N/2 \rfloor$, the specific value of which depends on the system nonlinearities. If the system GFRFs $H_{N-(2i-1)}(.)=0$, for $i=1,\cdots,q-1$, and $H_{N-(2q-1)}(.)\neq 0$, then $p^*=q$. This is the first analytical description for the output frequencies of nonlinear systems, which extends the well-known relationship between the input and output frequencies of linear systems to the nonlinear case.

Based on the above results on output frequency responses of nonlinear systems, a new concept known as the Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [12]. The concept was defined as

$$G_n(j\omega) = \frac{\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}} \tag{17}$$

under the condition that

$$U_n(j\omega) = \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega} \neq 0 \tag{18}$$

Notice that $G_n(j\omega)$ is valid over the frequency range f_{Y_n} as defined in (16).

By introducing the NOFRFs $G_n(j\omega)$, $n=1,\cdots N$, Equation (12) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega)U_n(j\omega) \tag{19}$$

which is similar to the description of the output frequency response of linear systems. For a linear system, the relationship between $Y(j\omega)$ and $U(j\omega)$ can be illustrated as in Figure 4. Similarly, the nonlinear system input and output relationship of Equation (19) can be illustrated as in Figure 5.

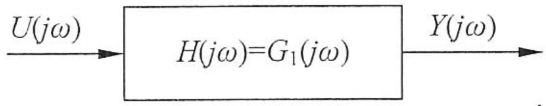


Figure 4. The output frequency response of a linear system

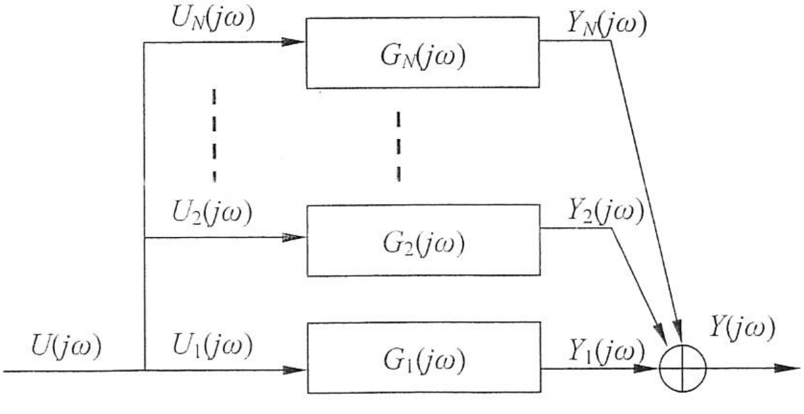


Figure 5. The output frequency response of a nonlinear system

The NOFRFs reflect a combined contribution of the system and its input to the frequency domain output behavior. It can be seen from Equation (17) that $G_n(j\omega)$ depend not only on H_n (i= 1,..., N) but also on the input $U(j\omega)$. For any structure, the dynamical properties are determined by H_n (i= 1,..., N). Because the presence of faults will cause a change in the structure dynamics, the faults in a structure should be reflected by a change in H_n (i= 1,..., N). However, it is not straightforward to measure H_n (i= 1,..., N) directly in practice. According to Equation (17), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1,...,j\omega_n)$ over $\omega_1+\cdots+\omega_n=\omega$ with the weights depending on the test input. Therefore $G_n(j\omega)$ can be used as a alternative representation of the structural dynamical properties described by H_n and can be employed to investigate structural behaviors caused by faults. Moreover,

The following section will be dedicated to the analysis of the crack induced nonlinear response of system (1) using the novel concept of NOFRF.

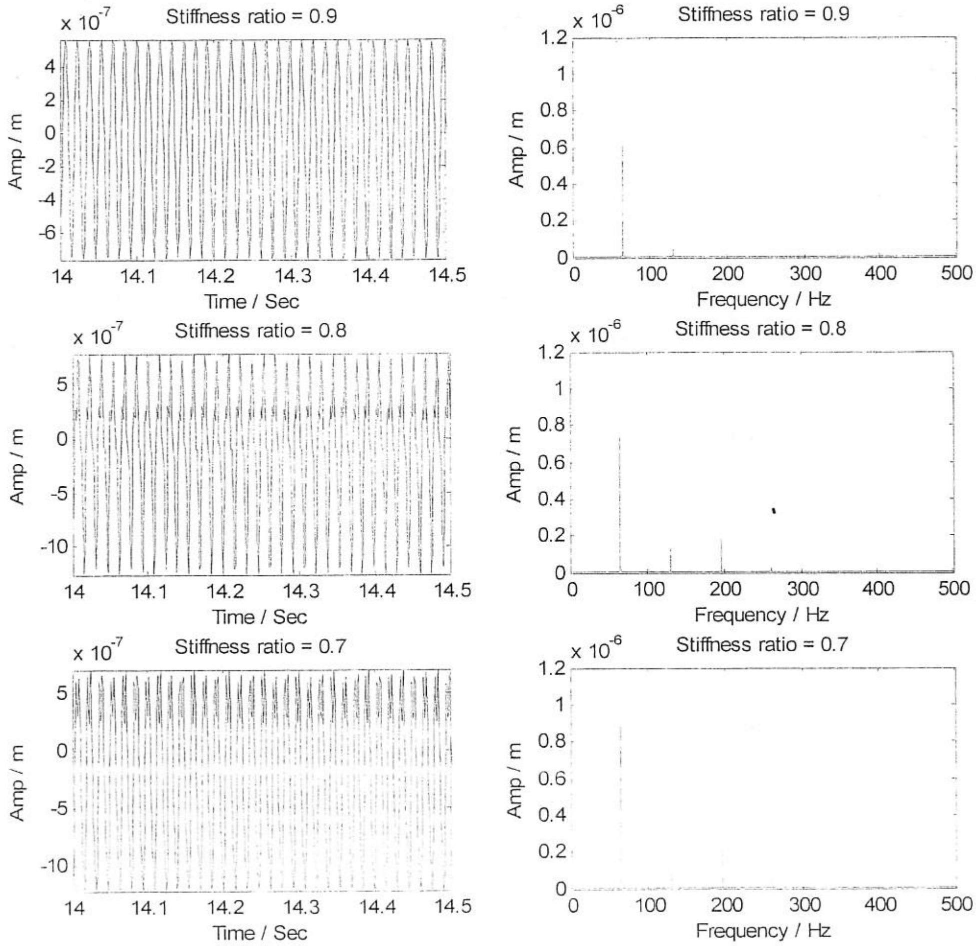
4 NOFRF Analysis of Crack Induced Nonlinear Responses

In Section 2, it was pointed out that the bilinear oscillator used extensively to model cracked structures can be approximated with a polynomial-type nonlinear system, which can easily be analyzed by the NOFRF derived from the Volterra series. In this section, the NOFRFs are applied to investigate the nonlinear behavior of a cracked beam in the frequency domain. The Volterra series based method has been used to analyze the vibration of a cracked beam in reference [15] where the higher order transform function (HOTF) of a cracked cantilevered beam was estimated. The HOTF was defined as the ratio between the output spectrum $Y(j\omega)$ and $U_n(j\omega)$ for a particular n of interest and is also based on the Volterra series of nonlinear systems. Compared to NOFRFs, HOTF is not a theoretically well established concept although under certain conditions, as discussed later, NOFRFs and HOTF can be related to each other.

The objective of the following analysis is to study the effect of the stiffness ratio α on the behavior of system (1). For this purpose, five different situations, where the stiffness ratio α changes between 1.0 and 0.6, are considered. The constant system parameters are summarized in Table 3. The external excitation $f(t)$ is a sinusoidal type force with unit amplitude and frequency of 65Hz. The fourth-order Runge–Kutta method is used to integrate Equation (1) to obtain the simulation data for the crack induced response.

Table 3 System parameters of the bilinear SDOF model	
Mass	m = 1kg
Damping coefficient	c = 6.936 Ns/m
Stiffness	$k = 1.9247{*}10^6$ N/m
Fundamental radian frequency of $f(t)$	$\omega = 130\,\pi$ rad/s

The method of Lang and Bilings [12] was served to determine the NOFRFs directly from input output test data. The algorithm generally requires experimental or simulation results for the system under investigation under N different input signal excitations, which have the same waveform but different intensities. In this study, the algorithm is applied to estimate the NOFRFs of system (1) under a sinusoidal input to reveal how the NOFRFs can be used as crack damage indicators for structure fault detection.



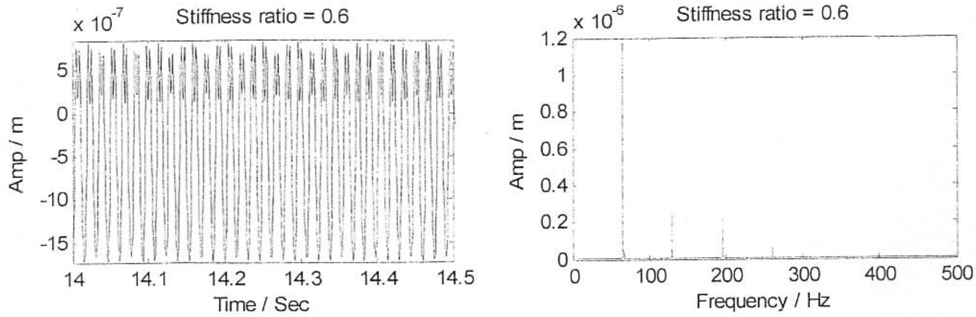


Figure 6. Simulation results for $x(t)$ obtained with stiffness ratios $\alpha = 0.9$, 0.8, 0.7 and 0.6, and the corresponding frequency spectra

Figure 6 shows the simulation results for $x(t)$ and the corresponding FFT spectra obtained by integrating Equation (1) at the different stiffness ratios - $\alpha = 0.9$, 0.8, 0.7 and 0.6. As the system is subjected to only a mono-component sinusoidal force whose spectrum contains only a fundamental harmonic, the spectra of the output shown in Figure 6 indicates that some of the input energy is transferred from the fundamental harmonic frequency to the higher harmonic frequencies. Also it can be seen from Figure 6 that only the first four harmonics are significant, which means a fourth order nonlinear model can describe the bilinear SDOF crack model very well. This result coincides with the result that the bilinear oscillator can be approximated by a four order polynomial-type nonlinear model.

Because the input is a sinusoidal force of frequency 65Hz, it is known from Equation (16) that the non-negative output frequency components contributed by the first, second, third and fourth order system nonlinearity are f_{Y_1} ={65Hz} f_{Y_2} = {0, 130Hz} f_{Y_3} ={65Hz, 195Hz} are f_{Y_4} ={0, 130Hz, 260Hz} respectively.

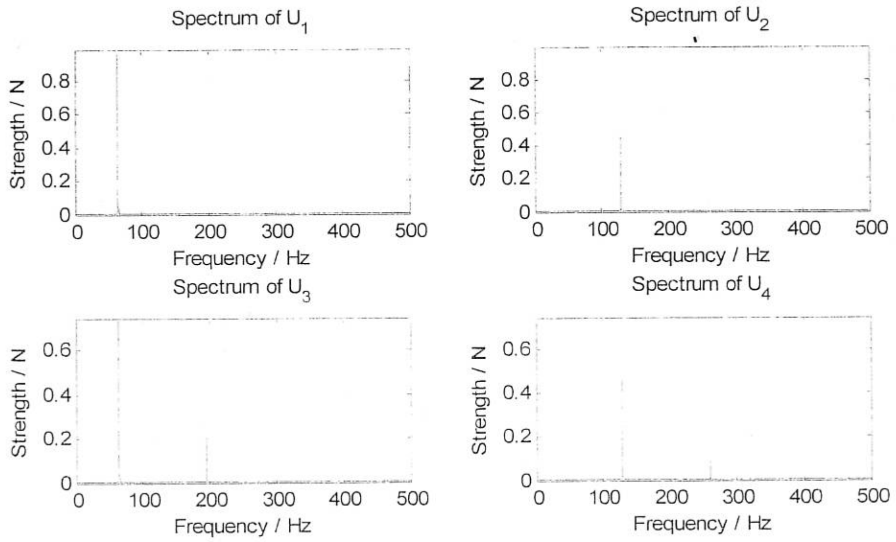


Figure 7. The spectra of $U_1(j\omega)$, $U_2(j\omega)$, $U_3(j\omega)$ and $U_4(j\omega)$

Figure 7 shows the spectra of $U_1(j\omega)$, $U_2(j\omega)$, $U_3(j\omega)$ and $U_4(j\omega)$. Obviously, the frequency components of $Y_1(j\omega)$, $Y_2(j\omega)$, $Y_3(j\omega)$ and $Y_4(j\omega)$ are the same as the frequency components of $U_1(j\omega)$, $U_2(j\omega)$, $U_3(j\omega)$ and $U_4(j\omega)$ respectively.

From the spectra of $U_1(j\omega)$, $U_2(j\omega)$, $U_3(j\omega)$ and $U_4(j\omega)$ and the output frequency components of f_{Y_1}, f_{Y_2} f_{Y_3} and f_{Y_4}, it can be shown that $U_4(j\omega)$ contributes to the fourth harmonic of $Y(j\omega)$, and $U_3(j\omega)$ contributes to the third harmonic of $Y(j\omega)$. The first harmonic of $Y(j\omega)$, however, is determined by both $U_1(j\omega)$ and $U_3(j\omega)$. Similarly, the second harmonic of $Y(j\omega)$ is determined by both $U_2(j\omega)$ and $U_4(j\omega)$. According to the general expression of Equation (19), these results can be summarized as

$$Y(j130\pi) = G_1(j130\pi)U_1(j130\pi) + G_3(j130\pi)U_3(j130\pi) \quad (20)$$

$$Y(j260\pi) = G_2(j260\pi)U_2(j260\pi) + G_4(j260\pi)U_4(j260\pi) \quad (21)$$

$$Y(j390\pi) = G_3(j390\pi)U_3(j390\pi) \quad (22)$$

$$Y(j520\pi) = G_4(j520\pi)U_4(j520\pi) \quad (23)$$

From Equation (20) – (23), the NOFRFs $G_3(j390\pi)$ and $G_4(j520\pi)$ can be determined using the algorithm in [12] with only one level of input excitation. Two levels input excitations are required to determine the NOFRFs $G_1(j130\pi)$, $G_3(j130\pi)$, $G_2(j260\pi)$ and $G_4(j260\pi)$. In order to achieve this, the object structure is excited twice with the same form of force, but each time a different strength is used. More specifically, in this simulation study, for each stiffness ratio, two responses are obtained under the same sinusoidal-type excitations the strengths of which are $1N$ and $2N$ respectively.

Tables 4- 7 show the evaluated results of the NOFRFs. They were determined at $\omega = 130\pi$, $\omega = 260\pi$, $\omega = 390\pi$, and $\omega = 520\pi$ respectively. Figure 8 illustrates the modulus of these results.

Table 4. NOFRFs at $\omega = 130\pi$

Stiffness Ratio α	Re($G_1(j130\pi)$) ($\times 10^{-5}$)	Im($G_1(j130\pi)$) ($\times 10^{-5}$)	Re($G_3(j130\pi)$) ($\times 10^{-21}$)	Re($G_3(j130\pi)$) ($\times 10^{-21}$)
1.0	0.05539964933553	0.00924597482734	0.00000000000000	0.00000000000000
0.9	0.05791997200267	-0.01224837184562	-0.28455013071824	-0.17205356741103
0.8	0.06794758051883	-0.01589379386901	-0.35072457972248	-0.18528845721188
0.7	0.08226499550284	-0.01906143636801	-0.41028158382630	-0.23822801641527
0.6	0.09425435456456	-0.02023543894078	-0.52939559203394	-0.34410713482206

increase with the reduction of α. This again indicates that the system will be more nonlinear when α becomes smaller. Therefore, the NOFRF based analysis is also consistent with the analysis from this different perspective. As indicated by the effective procedure used to determine the results in Table 4-7, one advantage of the NOFRF based analysis is that NOFRFs can easily be estimated using well-established measurements. So that, NOFRFs have a great potential to be used for crack detection of practical engineering structures.

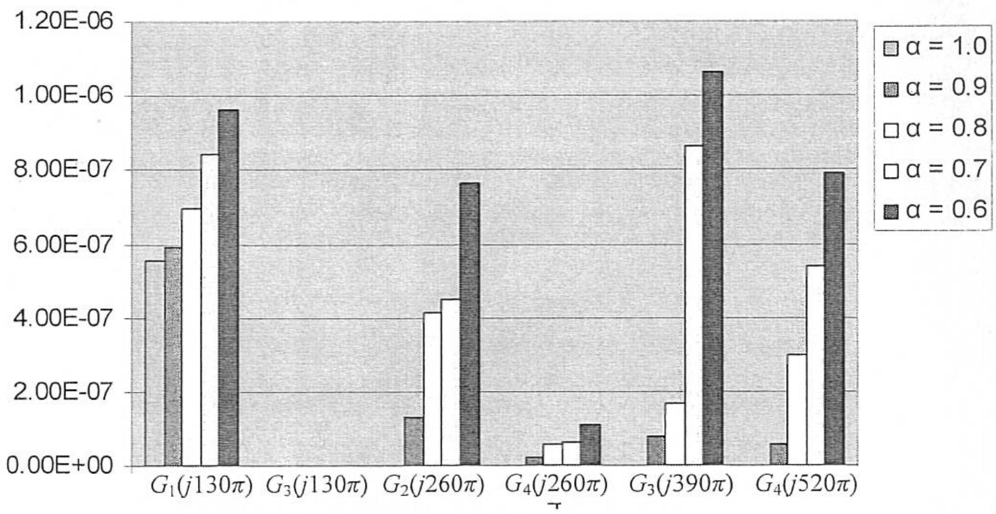


Figure 8 Modulus of NOFRFs $G_i(j\omega)$

It can clearly be observed from Figure 6 that the output spectra under $\alpha = 0.8$, 0.7 and 0.6 show no significant differences. This implies that the output spectrum is often not very sensitive to different crack sizes and could therefore not be used to distinguish different cracks. In contrast, Figure 8 shows that the NOFRFs $G_3(j390\pi)$ and $G_4(j520\pi)$ are extremely sensitive to the change of stiffness ratios, the magnitudes of these NOFRFs increasing considerately with the decrease of α. Consequently, compared to traditional output spectrum based crack detection methods, the NOFRFs based method proposed in the present study should be more effective in practical applications.

From Equation (20), the fundamental harmonic of $Y(j\omega)$, $Y(j130\pi)$, is the sum of $U_1(j130\pi)G_1(j130\pi)$ and $U_3(j130\pi)G_3(j130\pi)$. However, the results in Table 4 show that $G_3(j130\pi)$ is far less than $G_1(j130\pi)$, moreover, $U_1(j130\pi)$ is also bigger than $U_3(j130\pi)$, therefore it is reasonable to ignore the term $U_3(j130\pi)G_3(j130\pi)$ when calculating $Y(j130\pi)$, and rewrite Equation (20) as

$$Y(j130\pi) \approx G_1(j130\pi)U_1(j130\pi) \tag{24}$$

indicating that the fundamental harmonic of $Y(j\omega)$ is mainly determined by the first order NOFRF, and the amplitude of $Y(j130\pi)$ is nearly linear to the strength of the input force. Similarly, it can deduced that $Y(j260\pi)$ is mainly determined by $G_2(j260\pi)U_2(j260\pi)$, and Equation (21) can be rewritten as

$$Y(j260\pi) \approx G_2(j260\pi)U_2(j260\pi) \tag{25}$$

Equations (24) and (25) suggest that the NOFRFs can be approximately estimated based on $G_1(j130\pi)$ and $G_2(j260\pi)$ with one level of input excitation, which is the same as the estimation of $G_3(j390\pi)$ and $G_4(j520\pi)$. In such cases, the NOFRFs $G_n(j\omega)$ degenerates to the HOTF proposed in [15] where it was validated that the HOTFs are an accurate and extremely sensitive indicator of non-linear behavior of a system and the evolution of HOTFs for the crack size is very useful for structural damage assessment. This observation and comparison with the HOTFs confirms the significance of applying the well defined and more general concept of NOFRFs to the detection of cracks in engineering structures.

5 Conclusions

This paper presents an analysis of the dynamic behavior of a beam with a closing crack in the frequency domain using the NOFRF concept recently developed by the authors. A simple SODF model was employed to obtain the forced response of the beam by the fourth-order Runge–Kutta method. It is verified that the bilinear oscillator, a well accepted model for the cracked structure, can be approximated by a four-order polynomial-type nonlinear model, which can easily be described by a Volterra series model. The new NOFRF concept, which is defined in the frequency domain for systems which can be described by the Volterra model, is then used to analyze the crack induced nonlinear response of the beam. The NOFRF based analysis shows that the NOFRFs are highly dependent upon the stiffness ratio, and can therefore be used as crack damage indicators to indicate whether there exists a crack and, if the answer is yes, the size of the crack. Meanwhile, the research also confirms a well known conclusion about structures with cracks that the increase of crack size makes the structure behave more nonlinearly. This research study establishes an important basis for the application of the NOFRF concept in fault diagnosis of mechanical structures.

Acknowledgements

The authors gratefully acknowledge the support of the Engineering and Physical Science Research Council, UK, for this work.

References

1. B. ZASTRAU 1985 *Archives of Mechanics* **37**, 731-743. Vibrations of cracked structures.
2. A. IBRAHIM, F. ISMAIL and H. MARTIN 1987 *Journal of Analytical, Experimental Modal Analysis*, **2**, 76-82. Modeling of the Dynamics of a Continuous Beam Including Nonlinear Fatigue Crack.
3. M. I. FRISWELL, J. E. T. PENNY 1992 *Proceedings 10th International Modal Analysis Conference, San Diego,* **CA 1**, 516-521. A simple nonlinear model of a cracked beam.
4. M. I. FRISWELL, J. E. T. PENNY 2002 *Structural Health Monitoring* **1**, 139-148. Crack Modeling for Structural Health Monitoring.
5. Y. C. CHU, M. H. H. SHEN 1992 *AIAA Journal* **30**, 2512-2519. Analysis of forced bilinear oscillators and the application to cracked beam dynamics.
6. M. H. H. SHEN, Y. C. CHU 1992 *Computers & Structures* **45**, 79-93. Vibrations of beams with a fatigue crack.
7. J. N. SUNDERMEYER, R. L. WEAVER 1995 *Journal of Sound and Vibration* **183**, 857-871. On crack identification and characterization in a beam by non-linear vibration analysis.
8. M. CHATI, R. RAND AND S. MUKHERJEE 1997 *Journal of Sound and Vibration* **207**, Modal analysis of a cracked beam.
9. A. RIVOLA, P. R. WHITE 1998 *Journal of Sound and Vibration* **216**, Bispectral analysis of the bilinear oscillator with application to the detection of fatigue cracks.
10. Z. Q. LANG, S. A. BILLINGS 1996 *International Journal of Control* **64**, 1049-1067. Output frequency characteristics of nonlinear system.
11. S. A. BILLINGS, Z. Q. LANG 2002 *International Journal of Control* 75, 1066-1081. Non-linear systems in the frequency domain: energy transfer filters.
12. Z. Q. LANG, S. A. BILLINGS 2005 *International Journal of Control* **78**, 354-362. Energy transfer properties of nonlinear systems in the frequency domain.
13. G. M. LEE 1997 *Mechanical Systems and Signal Processing* **11**, 219-229. Estimation of non-linear system parameters using higher-order frequency response functions.
14. H. JEFFREYS, B. S. JEFFREYS 1988 *Methods of Mathematical Physics, 3rd ed.* Cambridge, England: Cambridge University Press.
15. R. RUOTOLO, C. SURACE, P. CRESPO, D. STORER 1996 *Computers & Structure* **61**, 1057-1074. Harmonic analysis of the vibration of a cantilevered beam with closing crack.