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Phragmén–Lindelöf principles for generalized analytic functions on unbounded domains

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Abstract

We prove versions of the Phragmén–Lindelöf strong maximum principle for generalized analytic functions defined on unbounded domains. A version of Hadamard's three-lines theorem is also derived.

Keywords: Phragmén–Lindelöf principle, generalized analytic function, pseudoanalytic function, three-lines theorem

MSC: 30G20, 30C80

1 Introduction

Versions of the maximum principle for complex-valued functions defined on a domain in $\mathbb{C}$ have been of interest since the development of the classical maximum modulus theorem and Phragmén–Lindelöf principle for holomorphic functions (see, e.g. [10, Chap. V]). It is important to distinguish between two types of result here. First, there is the *weak maximum principle* asserting that under certain circumstances a nonconstant function $f : \Omega \to \mathbb{C}$ cannot attain a local maximum in its domain Ω: thus if Ω is bounded and f is continuous on $\overline{\Omega}$ we have

$$\sup_{z \in \Omega} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|. \tag{1}$$

Second – and this will be our main concern in this paper – there is the *strong maximum principle* or *Phragmén–Lindelöf principle*. This generally applies to unbounded domains, and generally a supplementary hypothesis on f is required for the conclusion (1) to hold. For example, if $f : \Omega \to \mathbb{C}$ is analytic, where $\Omega = \mathbb{C}_+$, the right-hand half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, then if f is known to be bounded we may conclude that (1) holds, whereas the example $f(z) = \exp(z)$ shows that it does not hold in general.

We shall use the following standard notation:

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x + i f_y).$$

For quasi-conformal mappings f, that is, those satisfying the Beltrami equation $\overline{\partial} f = \nu \partial f$ with $|\nu| \leq \kappa < 1$, the weak maximum principle holds (see, for example [4]). This fact was used in [1, Prop. 4.3.1] to deduce a weak maximum principle for functions solving the conjugate Beltrami equation

$$\overline{\partial} f = \nu \overline{\partial f}. \tag{2}$$

Their argument is based on the fact that if f is a solution to (2), then it also satisfies a classical Beltrami equation $\overline{\partial} f = \nu_f \partial f$, where $\nu_f(z) = \nu(z)\overline{\partial f(z)}/\partial f(z)$, and hence $f = G \circ h$ where G is holomorphic and h is a quasi-conformal mapping (cf. [7, Thm. 11.1.2]).

Carl [3] considered functions w satisfying equations of the form

$$\overline{\partial} w(z) + A(z) w(z) + B(z) \overline{w(z)} = 0 \tag{3}$$

and deduced a weak maximum principle for such functions, analogous to (1), under certain hypotheses on the functions A and B. We shall take this as our starting point.

For general background on generalized analytic functions (pseudo-analytic functions) we refer to the books [2, 9, 11]. The following definitions are taken from the recent paper [1].

Definition 1.1. *Let $1 \leq p < \infty$. For $\nu \in W^{1,\infty}(\mathbb{D})$ (i.e., a Lipschitz function with bounded partial derivatives), the class H^p_ν consists of all measurable functions $f : \mathbb{D} \to \mathbb{C}$ satisfying the conjugate Beltrami equation (2) in a distributional sense, such that the norm*

$$\|f\|_{H^p_\nu} = \left(\text{ess sup}_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}$$

is finite. Clearly for $\nu = 0$ we obtain the classical Hardy space $H^p(\mathbb{D})$. If instead ν is defined on an arbitrary subdomain $\Omega \subset \mathbb{C}$, we may define the class $H^\infty_\nu(\Omega)$ as the space of all bounded measurable functions satisfying (2), equipped with the supremum norm.

We may analogously define spaces $G^p_\alpha(\mathbb{D})$, where $\alpha \in L^\infty(\mathbb{D})$, and in general $G^\infty_\alpha(\Omega)$, where now, for a function w we replace (2) by

$$\overline{\partial} w = \alpha \overline{w}. \tag{4}$$

Once again, the case $\alpha = 0$ is classical.

When ν is real (the most commonly-encountered situation), there is a link between the two notions: suppose that $\|\nu\|_{L^\infty(\Omega)}$ with $\|\nu\|_\infty \leq \kappa < 1$, and set $\sigma = \dfrac{1-\nu}{1+\nu}$ and $\alpha = \frac{\overline{\partial}\sigma}{2\sigma}$, so that $\sigma \in W^{1,\infty}_{\mathbb{R}}(\Omega)$. Then $f \in L^p(\mathbb{D})$ satisfies (2) if and only if $w := \dfrac{f - \nu \overline{f}}{\sqrt{1-\nu^2}}$ satisfies (4).

We shall mainly be considering the class G^∞_α, for which it is possible to prove a strong maximum principle and a generalization of the Hadamard three-lines theorem under mild hypotheses on α, which are satisfied in standard examples. The referee has suggested that there may be a link between these assumptions and the strict ellipticity of σ, although we have not been able to show this.

2 Functions defined on unbounded domains

The following result is an immediate consequence of [3, Thm. 1], taking $A = 0$ and $B(z) = -\alpha(z)$ in (3) in order to obtain (4).

Proposition 2.1. *Suppose that Ω is a bounded domain in $\mathbb{C}$ and that w is a continuous function on $\overline{\Omega}$ such that (4) holds in Ω, where α satisfies $2|\alpha|^2 \geq |\partial\alpha|$. Then $|w(z)| \leq \sup_{\zeta\in\partial\Omega} |w(\zeta)|$ for all $z \in \Omega$.*

Proof. Taking $k = 2$ in [3, Thm. 1], we require that the matrix $M = (m_{ij})_{i,j=1}^2$ be negative semi-definite, where, with $a = -2|\alpha|^2$ and $b = -\partial\alpha$, we have

$$M = \begin{pmatrix} a + \operatorname{Re} b & \operatorname{Im} b \\ \operatorname{Im} b & a - \operatorname{Re} b \end{pmatrix}.$$

On calculating m_{11}, m_{22} (which must be non-positive) and $\det M$ (which must be non-negative) we obtain the sufficient conditions $-2|\alpha|^2 \pm \operatorname{Re} \partial\alpha \leq 0$ and $2|\alpha|^2 \geq |\partial\alpha|$: clearly the second condition implies the first. □

Example 2.1. In the example $\sigma = 1/x$, occurring in the study of the tokamak reactor [5, 6], we have $\alpha(x) = -\frac{1}{4x}$ and $\partial\alpha = \frac{1}{8x^2}$; thus the inequality $2|\alpha|^2 \geq |\partial\alpha|$ is always an equality.

Note that by rescaling z we may transform the equation (4) to one with $\alpha = -\frac{1}{\lambda x}$ for any $\lambda > 0$ (with the domain also changing); then the inequality requires that $2/\lambda^2 \geq 1/2\lambda$, so that if we take $0 < \lambda < 4$ the inequality is strict.

Now for $\varepsilon > 0$ we write $h_\varepsilon(z) = 1/(1 + \varepsilon z)$, and note that whenever $\Omega \subset \mathbb{C}_+$ is a domain, we have that the functions h_ε satisfy

(i) For all $\varepsilon > 0$, $h_\varepsilon \in \operatorname{Hol}(\Omega) \cap C(\overline{\Omega})$.

(ii) For all $\varepsilon > 0$, $\lim_{|z|\to\infty, z\in\overline{\Omega}} h_\varepsilon(z) = 0$.

(iii) For all $z \in \Omega$, $\lim_{\varepsilon\to 0} |h_\varepsilon(z)| = 1$.

(iv) For all $\varepsilon > 0$, for all $z \in \partial\Omega$, $|h_\varepsilon(z)| \leq 1$.

Suppose that $\overline{\partial} w = \alpha\overline{w}$ and that h is holomorphic; then $\overline{\partial}(hw) = \beta\overline{hw}$, where $\beta = \alpha h/\overline{h}$. Moreover,

$$\partial\beta = \partial(\alpha h)/\overline{h} = (\partial\alpha)(h/\overline{h}) + \alpha(\partial h)/\overline{h}.$$

That is, with $h = h_\varepsilon$, we have $|\beta| = |\alpha|$ and $|\partial\beta| \leq |\partial\alpha| + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$.

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{C}_+$ (not necessarily bounded) and that w is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in Ω where α is a C^1 function satisfying $2|\alpha|^2 \geq |\partial\alpha| + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$ for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta\in\partial\Omega} |w(\zeta)|$ for all $z \in \Omega$.*

Proof. Fix $\varepsilon > 0$ and $M = \sup_{\zeta\in\partial\Omega} |w(\zeta)|$. Suppose that $M > 0$. Then by property (ii) there is an $\eta > 0$ such that for all $z \in \overline{\Omega}$ with $|z| \geq \eta$ we have $|w(z)h_\varepsilon(z)| \leq M$.

Now, by property (i) and Proposition 2.1 we have

$$\sup_{z\in\Omega\cap D(0,\eta)} |w(z)h_\varepsilon(z)| = \sup_{z\in\partial(\Omega\cap D(0,\eta))} |w(z)h_\varepsilon(z)|,$$

at least if $2|\alpha|^2 \geq |\partial\alpha| + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$.

Now $\partial(\Omega \cap D(0,\eta)) \subset (\partial\Omega \cap \overline{D(0,\eta)}) \cup (\partial D(0,\eta) \cap \overline{\Omega})$.

By hypothesis, $|w(z)| \leq M$ if $z \in \partial\Omega$, and by property (iv), $|h_\varepsilon(z)| \leq 1$ for $z \in \partial\Omega$. So $\sup_{z\in\partial\Omega\cap\overline{D(0,\eta)}} |w(z)h_\varepsilon(z)| \leq M$.

By the definition of η we also have $|w(z)h_\varepsilon(z)| \leq M$ if $|z| \geq \eta$ with $z \in \overline{\Omega}$, and in particular for $z \in \overline{\Omega} \cap \partial D(0,\eta)$.

We conclude that $\sup_{z\in\Omega\cap D(0,\eta)} |w(z)h_\varepsilon(z)| \leq M$. However, $|w(z)h_\varepsilon(z)| \leq M$ whenever $z \in \overline{\Omega}$ with $|z| \geq \eta$, and hence $\sup_{z\in\Omega} |w(z)h_\varepsilon(z)| \leq M$. Now, letting ε tend to 0, and using property (iii), we have the result in the case $M > 0$.

If $M = 0$, then by the above we have that $\sup_{z\in\partial\Omega} |w(z)| \leq \gamma$ for all $\gamma > 0$, and the same holds for $z \in \Omega$ by the above. Letting $\gamma \to 0$ we conclude that w is identically 0 on Ω.

□

Example 2.2. Consider the case $\alpha = -\frac{1}{\lambda x}$ and $\partial\alpha = \frac{1}{2\lambda x^2}$. For the hypotheses of the theorem to be valid we require

$$\frac{2}{\lambda x^2} \geq \frac{1}{2\lambda x^2} + \frac{1}{\lambda x}\frac{\varepsilon}{|1+\varepsilon z|}.$$

If $\lambda = 1$ (and by rescaling the domain we can assume this) then this always holds, since $|1 + \lambda z| \geq \lambda x$.

In the following theorem, it will be helpful to note that we shall be considering composite mappings as follow:

$$\Lambda \xrightarrow{h} \Omega \xrightarrow{w} \mathbb{C} \qquad \text{and} \qquad \Lambda \xrightarrow{h} \Omega \xrightarrow{\alpha} \mathbb{C}.$$

Theorem 2.2. *Suppose that $\Omega \subset \mathbb{C}$ is simply-connected and that the disc $D(a,r)$ is contained in $\mathbb{C} \setminus \overline{\Omega}$. Let $h : \mathbb{C} \to \mathbb{C}$ be defined by $h(z) = re^z + a$, and let Λ be a component of $h^{-1}(\Omega)$. Set $g_\varepsilon(z) = 1/(1 + \varepsilon g(z))$, where $g(z) = \log\left(\dfrac{z-a}{r}\right)$ is a single-valued inverse to h defined on Ω. Suppose that w is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in Ω with α a C^1 function satisfying*

$$2|\alpha|^2 \geq |\partial\alpha| + |\alpha||\partial g_\varepsilon|/|g_\varepsilon| \tag{5}$$

for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta \in \partial\Omega} |w(\zeta)|$ for all $z \in \Omega$.

Proof. First we identify the equation satisfied by $v = w \circ h$, where h is holomorphic. Namely,

$$\begin{aligned} \overline{\partial} v &= \overline{\partial}(w \circ h) = \overline{\partial(\overline{w} \circ h)} = \overline{(\partial\overline{w} \circ h)(\partial h)} = (\overline{\partial} w \circ h)(\overline{\partial h}) \\ &= ((\alpha\overline{w}) \circ h)(\overline{\partial h}) = (\alpha \circ h)(\overline{w} \circ h)(\overline{\partial h}) = \beta\overline{v}, \end{aligned}$$

where $\beta = (\alpha \circ h)(\overline{\partial h})$. Note that $\partial\beta = (\partial\alpha \circ h)|\partial h|^2$, since $\partial(\overline{\partial h}) = 0$.

The condition

$$2|\beta|^2 \geq |\partial\beta| + |\beta||\partial h_\varepsilon|/|h_\varepsilon| \tag{6}$$

at a point of Λ can be rewritten

$$2|\alpha \circ h|^2|\partial h|^2 \geq |\partial\alpha \circ h|\,|\partial h|^2 + |\alpha \circ h|\,|\partial h||\partial h_\varepsilon|/|h_\varepsilon|.$$

Now $g_\varepsilon = h_\varepsilon \circ g$; thus $\partial h_\varepsilon = (\partial g_\varepsilon \circ h)(\partial h)$.

That is, (6) is equivalent to

$$2|\alpha \circ h|^2|\partial h|^2 \geq |\partial\alpha \circ h|\,|\partial h|^2 + |\alpha \circ h|\,|\partial h|^2|\partial g_\varepsilon \circ h|/|g_\varepsilon \circ h|,$$

or

$$2|\alpha \circ h|^2 \geq |\partial\alpha \circ h| + |\alpha \circ h||\partial g_\varepsilon \circ h|/|g_\varepsilon \circ h|.$$

The set Λ is open, and thus $\partial\Lambda \cap \Lambda = \emptyset$ and also $h(\partial\Lambda) \cap \Omega = \emptyset$. Moreover, since $h(\partial\Lambda) \subset h(\overline{\Lambda}) \subset \overline{h(\Lambda)}$, we get $h(\partial\Lambda) \subset \overline{\Omega} \setminus \Omega = \partial\Omega$.

Since w is bounded on Ω, the function $v = w \circ h$ is bounded on Λ, and using the calculations above and Theorem 2.1 with condition (6), we see that

$$\sup_{z \in \Lambda} |v(z)| = \sup_{z \in \partial\Lambda} |v(z)|.$$

Since $h(\Lambda) = \Omega$, $\sup_{z\in\Lambda} |v(z)| = \sup_{z\in\Omega} |w(z)|$. Moreover, since $h(\partial\Lambda) \subset \partial\Omega$, we have also

$$\sup_{z\in\partial\Lambda} |v(z)| \le \sup_{z\in\partial\Omega} |w(z)|.$$

It follows that $\sup_{z\in\Omega} |w(z)| \le \sup_{z\in\partial\Omega} |w(z)|$ and we obtain equality.

□

We now provide a generalization of the three-lines theorem of Hadamard (see, for example [8, Thm. 9.4.8] for the classical formulation with $\alpha = 0$).

Theorem 2.3. *Suppose that a and b are real numbers with $0 < a < b$, and let $\Omega = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$. Suppose that w is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in Ω where α is a C^1 function satisfying*

$$2|\alpha|^2 \ge |\partial\alpha| + \frac{|\alpha||\log(M(a)/M(b))|}{b-a} + |\alpha||\partial h_\varepsilon|/|h_\varepsilon| \tag{7}$$

for each $\varepsilon > 0$. Then the function M defined on $[a, b]$ by

$$M(x) = \sup_{y\in\mathbb{R}} |w(x + iy)|$$

satisfies, for all $x \in (a, b)$,

$$M(x)^{b-a} \le M(a)^{b-x} M(b)^{x-a}.$$

That is, $\log M$ is convex on (a, b).

Proof. Consider the function g defined on $\overline{\Omega}$ by

$$h(z) = M(a)^{(z-b)/(b-a)} M(b)^{(a-z)/(b-a)},$$

where quantities of the form M^ω are defined for $M > 0$ and $\omega \in \mathbb{C}$ as $\exp(\omega \log M)$, taking the principle value of the logarithm.

Now $v := hw$ satisfies $|v(z)| \le 1$ for $z \in \partial\Omega$, since $|h(a + iy)| = 1/M(a)$ and $|h(b + iy)| = 1/M(b)$.

Given that $\overline{\partial} w = \alpha\overline{w}$ and that h is holomorphic, then, as we have seen, $\overline{\partial}(hw) = \beta\overline{hw}$, where $\beta = \alpha h/\overline{h}$. Moreover, $\partial\beta = \partial(\alpha h)/\overline{h} = (\partial\alpha)(h/\overline{h}) + \alpha(\partial h)/\overline{h}$.

Now $\log h = \frac{z-b}{b-a} \log M(a) + \frac{a-z}{b-a} \log M(b)$, and so

$$\left|\frac{\partial h}{h}\right| = \frac{|\log M(a)/M(b)|}{b-a}.$$

Thus the condition (7) on α implies that β satisfies $2|\beta|^2 \geq |\partial\beta| + |\beta||\partial h_\varepsilon|/|h_\varepsilon|$. Hence we can apply Theorem 2.1 to v, and the result follow.

□

Remark 2.1. As in Example 2.2, rescaling z is helpful here, since if z is reparametrized as λz, then $\partial\alpha$ is divided by λ and $b - a$ is also divided by λ: thus the inequality (7) becomes easier to satisfy.

3 Weights depending on one variable

We look at two cases here, for functions defined on a subdomain of $\mathbb{C}_+$, namely weights $\alpha = \alpha(x)$ and radial weights $\alpha = \alpha(r)$. We revisit Theorem 2.1.

Since we now have $\partial\alpha = \alpha'/2$, we obtain the following corollary.

Corollary 3.1. *Suppose that $\Omega \subset \mathbb{C}_+$ (not necessarily bounded) and that w is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in Ω where $\alpha = \alpha(x)$ is a C^1 function satisfying $2|\alpha|^2 \geq |\alpha'|/2 + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$ for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta\in\partial\Omega} |w(\zeta)|$ for all $z \in \Omega$.*

Likewise, in polar coordinates (r, θ) we have

$$\partial = \frac{1}{2}\left(e^{-i\theta}\partial_r - \frac{ie^{-i\theta}}{r}\partial_\theta\right),$$

giving the following result.

Corollary 3.2. *Suppose that $\Omega \subset \mathbb{C}_+$ (not necessarily bounded) and that w is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in Ω where $\alpha = \alpha(r)$ is a C^1 function satisfying $2|\alpha|^2 \geq |\alpha'|/2 + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$ for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta\in\partial\Omega} |w(\zeta)|$ for all $z \in \Omega$.*

Suppose now that $\alpha(x) = ax^\mu$. The condition we require is then

$$2|a|^2 x^{2\mu} \geq |a\mu|x^{\mu-1}/2 + |a|x^\mu \frac{\varepsilon}{|1 + \varepsilon z|},$$

which is only possible for $\mu = -1$. However, it is easy to write down polynomials in x that do not vanish at 0 but which satisfy the conditions of Corollary 3.2.

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