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ABSTRCT

A novel signal processing approach is developed in the present study for accurate evaluation of the output frequency responses of nonlinear systems to sinusoidal inputs. A simulation study is described, which verifies the effectiveness of the proposed approach. The approach is also applied to analyse experimental data from dynamic testing of rheological properties of a polymer material. The results show that the approach has considerable potential in a wide range of engineering applications.

Key words: Output frequency response, Sinusoidal signals, Nonlinear systems, Discrete Fourier Transform

1 INTRODUCTION

Dynamic testing is widely used in mechanical and civil engineering to investigate the behaviour and properties of structures and materials. Sinusoidal signals are often used by researchers and engineers as a standard testing signal. The basic procedure consists of exciting the system (structure or material to be tested) using a multi-tone signal which is a pure sine signal or a sum of several sine signals with different frequencies, measuring the time domain response of the tested system to the input, and evaluating the Discrete Fourier Transform (DFT) of the system response to determine the output frequency response of the system to the testing input. The obtained output frequency response reflects the frequency domain behaviour of the structure or material under study and is often used to compute the frequency response function (FRF) to establish the frequency domain model of the underlying system.

Over the whole testing and analysis process, the evaluation of the DFT of the system time domain response is a critical step. An inaccurate DFT result will result in an inaccurate description of the system output frequency response and, consequently, an inaccurate system frequency domain model. The correct implementation of this step in system dynamic analysis is to apply a Fast Fourier Transform (FFT) procedure to the sampled system output response over a time interval which is a common multiple of the periods of all frequency components in the response. This principle has to be followed otherwise accurate magnitude and phase estimates of the frequency components in the system output response cannot be obtained.

When the structure or material under investigation behave linearly, the response to a multi-tone input contains exactly the same frequency components as the frequency components in the input. In such circumstances, it is straightforward to follow the principle of choosing a proper time interval when determining the system output frequency response using a FFT procedure. This time interval can directly be determined from the frequency components of the applied multi-tone input because of the simple relationship between the input and output frequencies of linear systems. However, the behaviour of many engineering structures and materials are inherently nonlinear, and a linear approximation is valid only when the system is subject to relatively small loads. When the structure or material behave nonlinearly, the relationship between the input and output frequencies becomes much more complicated. Consequently the choice of a time interval for the FFT computation as used in the linear system case no longer works, and as far as we are aware, no systematic methods are currently available to address this issue. The standard FFT procedure is still being used to process the output responses of nonlinear systems [1][2][3]. It seems not to have been realised that the computation results may not be accurate because the time interval over which the DFT of the output response is evaluated may not be appropriate.

In this paper, a new signal processing approach for the accurate computation of the output frequency responses of nonlinear systems to multi-tone inputs is proposed. The approach is based on theoretical results developed recently by the authors on the analysis of nonlinear systems in the frequency domain [4][5]. The new approach can readily be used to determine an appropriate time interval over which the accurate computation of the system output frequency response can be achieved via a standard FFT procedure. Simulation studies are conducted, and the results demonstrate the effectiveness and practical significance of the new approach. The new method is also applied to the analysis of experimental data collected from the dynamic testing of rheological properties of a polymer material. The results show that the new procedure has considerable potential for use in a wide range of engineering applications associated with dynamic testing and analysis of nonlinear structures and materials.

2 EVALUATION OF THE SPECTRA OF MULTI-TONE SIGNALS USING FFT

The evaluation of the spectrum of a multi-tone signal is a common procedure in the analysis of data from dynamic testing of many engineering structures and materials. The objective is to identify the different frequency components in the signal and to determine the magnitude and phase of the signal at each frequency. The results are normally used to study the dynamic behaviour of the structure or material under investigation or to determine the frequency domain model of the underlying system.

In the practical implementation of this procedure, a Fast Fourier Transform (FFT) operation is normally performed on a sampled time sequence of the experimentally measured signal to evaluate the Discrete Fourier Transform (DFT) of the signal. If the magnitude and phase of the signal at different frequencies are to be accurately obtained, a basic principle has to be followed when performing the FFT operation. In the following, a simple analysis is performed and two numerical examples are provided to introduce the basic principle and to demonstrate its effect on the signal processing results.

Consider a simple case where the measured signal is a pure sinusoid

$$x(t) = A\cos(\omega_0 t + \phi) \tag{1}$$

where A, ω_0, and ϕ are the magnitude, angular frequency, and phase. From a sampled sequence of $x(t)$,

$$x(nT_s), n = 0, \ldots, N-1$$

where T_s is the sampling interval and N is the length of the sampled data, the DFT of the signal can be obtained as

$$X(k) = \sum_{n=0}^{N-1} x(nT_s) e^{-j\left(\frac{2\pi}{N}k\right)n} \qquad k = 0, \ldots, N-1 \tag{2}$$

and it is expected that the DFT result would show that $x(t)$ has a magnitude A and phase ϕ at frequency ω_0 and zero magnitude at other frequencies.

Because $X(k)$ $k = 0, \cdots, N-1$ in (2) represent the spectrum of $x(t)$ at discrete time angular frequencies

$$\frac{2\pi}{N}k \qquad k = 0, \cdots, N-1$$

which correspond to continuous time angular frequencies

$$\frac{2\pi}{NT_s}k \qquad k = 0,\cdots,N-1$$

the specific k at which $X(k)$ shows the magnitude and phase of $x(t)$ at $\omega = \omega_0$ can be obtained as

$$k = k^* = \omega_0 \frac{NT_s}{2\pi} = \frac{2\pi}{T_0}\frac{NT_s}{2\pi} = \frac{N}{T_0/T_s} = \frac{N}{N_0} \tag{3}$$

where T_0 is the period of $x(t)$ and $N_0 = T_0/T_s$.

Substituting (1) and the definition of N_0 into (2) yields

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} A\cos(\omega_0 n T_s + \phi) e^{-j\left(\frac{2\pi}{N}k\right)n} \\ &= \sum_{n=0}^{N-1} A\cos\left(\frac{2\pi}{N}\frac{N}{N_0}n + \phi\right) e^{-j\left[\left(\frac{2\pi}{N}k\right)n+\phi\right]} e^{j\phi} \end{aligned} \qquad k = 0,\ldots,N-1 \tag{4}$$

The general term in (4) can be further written as

$$\begin{aligned} &\left\{ A\cos\left[\frac{2\pi}{N}\left(\frac{N}{N_0}\right)n + \phi\right]\cos\left[\frac{2\pi}{N}kn + \phi\right] - jA\cos\left[\frac{2\pi}{N}\left(\frac{N}{N_0}\right)n + \phi\right]\sin\left[\frac{2\pi}{N}kn + \phi\right]\right\} e^{j\phi} \\ &= \frac{1}{2}\left\{ \begin{array}{l} A\cos\left[\frac{2\pi}{N}\left(\frac{N}{N_0} + k\right)n + 2\phi\right] + A\cos\left[\frac{2\pi}{N}\left(\frac{N}{N_0} - k\right)n\right] \\ - jA\sin\left[\frac{2\pi}{N}\left(\frac{N}{N_0} + k\right)n + 2\phi\right] + jA\sin\left[\frac{2\pi}{N}\left(\frac{N}{N_0} - k\right)n\right] \end{array} \right\} e^{j\phi} \end{aligned} \tag{5}$$

Therefore

$$X(k) = \frac{1}{2}\left\{ \begin{array}{l} A\sum_{n=0}^{N-1}\cos\left[\frac{2\pi}{N}\left(\frac{N}{N_0} + k\right)n + 2\phi\right] + A\sum_{n=0}^{N-1}\cos\left[\frac{2\pi}{N}\left(\frac{N}{N_0} - k\right)n\right] \\ - jA\sum_{n=0}^{N-1}\sin\left[\frac{2\pi}{N}\left(\frac{N}{N_0} + k\right)n + 2\phi\right] + jA\sum_{n=0}^{N-1}\sin\left[\frac{2\pi}{N}\left(\frac{N}{N_0} - k\right)n\right] \end{array} \right\} e^{j\phi} \qquad k = 0,\ldots,N-1 \tag{6}$$

The following conclusions follow from Equation (6).

(i) If N/N_0 is an integer, then

$$\frac{(N-1)}{N}\left(\frac{N}{N_0} + k\right) \quad \text{and} \quad \frac{(N-1)}{N}\left(\frac{N}{N_0} - k\right)$$

which are the number of cycles involved in the first and third summation, and the second and fourth summation on the right hand side of equation (6), respectively, can be well approximated by an integer for a sufficiently large N. Consequently, when $k \neq k^* = N/N_0$, $X(k) \approx 0$, and when $k = k^* = N/N_0$

$$X(k) = X(k^*) \approx \frac{1}{2} ANe^{i\phi} \tag{7}$$

which implies that

$$\begin{cases} A \approx \dfrac{2\left|X(k^*)\right|}{N} \\ \phi \approx \angle X(k^*) \end{cases} \tag{8}$$

(ii) If N/N_0 is not an integer, the number of cycles involved in the four summations on the right hand side of equation (6) is not an integer either. Now no conclusions can generally be reached regarding the relationship between $X(k), k = 0,\ldots, N-1$, and the magnitude and phase of the sinusoid (1).

This result is very important for correctly using a FFT procedure for example the function fft(.) in MATLAB to evaluate the spectrum of a sinusoidal signal. It is known from MATLAB Signal Processing Toolbox [6] that the first equation in (8) should be used to obtain the accurate magnitude when using function fft() to analyse a sinusoidal signal. But the conditions for the application of equation (8), that is, N/N_0 must be an integer, is not mentioned in [6]. This can easily cause MATLAB users who are not aware of the condition to achieve incorrect analysis results. The condition is an important principle to follow when evaluating the spectrum of a sinusoidal signal using FFT. Consider

$$\frac{N}{N_0} = \frac{NT_s}{N_0 T_s} = \frac{T}{T_0} \tag{9}$$

where T is the time interval over which a sampled sequence of the original signal is obtained and used for the FFT analysis. The principle can be described in a more specific way as: the FFT analysis result for a sinusoidal signal should be obtained from a sampled sequence over a time interval which is a multiple of the signal period.

In order to illustrate the importance of this principle, consider an example where the sinusoidal signal y(t)=sin(2×π×1.5t) is sampled at the sampling period T_s=0.001s and the FFT procedure in MATLAB is applied to the sampled result. Ideally the FFT result should indicate the signal has only one frequency component at f=1.5 with magnitude 1. Figures 1 and 2 show the FFT results obtained over 25000 sampling intervals which is 25000×0.001=25s and over 10000 sampling intervals which is 10000×0.001=10s, respectively. Obviously Figure 2 shows the ideal result but Figure 1 does not. The problem with the result in Figure 1 arises because the choice of a proper time interval for the FFT computation was not followed since 25/(1/1.5)=22.4888 is not an integer. In the case of Figure 2, however, the time interval over which the sampled signal is used for the FFT computation is 10s which is exactly 10/(1/1.5)=15 multiples of the signal period.

An extension of the above analysis result to the Fourier analysis of a multi-tone signal $x(t)$ composed of K (≥ 1) sinusoids with period T_i, magnitudes A_i, and phase ϕ_i, $i = 1,\ldots,K$ indicates that if N is a common multiple of

$$N_i = T_i / T_s,\ i = 1,\ldots,K, \tag{10}$$

then the DFT

$$X(k)=\sum_{n=0}^{N-1}x(nT_s)e^{-j\left(\frac{2\pi}{N}k\right)n}\approx\begin{cases}\frac{1}{2}A_iNe^{j\phi_i} & \text{when } k=k_i^*=\frac{N}{N_i},\quad i=1,\ldots,K\\ 0 & \text{otherwise}\end{cases} \tag{11}$$

Consequently

$$\begin{cases}A_i\approx\frac{2\left|X(k_i^*)\right|}{N}\\ \phi_i\approx\angle X(k_i^*)\end{cases}\qquad k_i^*=\frac{N}{N_i},\quad i=1,\ldots,K \tag{12}$$

But, if N is not a common multiple of $N_i=T_i/T_s$, $i=1,\ldots,K$, a correct evaluation of the spectrum of $x(t)$ can, generally, not be achieved.

Considering

$$\frac{N}{N_i}=\frac{NT_s}{N_iT_s}=\frac{T}{T_i}\qquad i=1,\ldots,K \tag{13}$$

the principle to follow when evaluating the spectrum of a multi-tone signal using FFT can be stated as: the FFT analysis result for a multi-tone signal should be obtained from a sampled sequence over a time interval which is a common multiple of the periods of all the frequency components in the signal.

3 ACCURATE COMPUTATION OF OUTPUT FREQUENCY RESPONSES OF NONLINEAR SYSTEMS TO MULTI-TONE INPUTS

In practice, the evaluation of the spectrum of multi-tone signals is often needed when analysing the output response of an engineering system or structure to a multi-tone input excitation. If the system or structure under study behaves linearly, the response to a multi-tone input will be a multi-tone signal with exactly the same frequencies as in the input signal but with a different magnitude and phase. In these cases, the procedure introduced in the last section can be applied directly to conduct the analysis for the system output frequency responses. The procedure simply involves determining a time interval which is a common multiple of the periods of all frequency components in the multi-tone input and performing a FFT operation on a sampled sequence of the output signal over this time interval to produce a DFT analysis result. Because the frequency components in the input and output signals are exactly the same in this case, according to the analysis in Section 2, the DFT analysis result shows accurate magnitude and phase estimates of all the frequency components in the output signal.

However, if the system or structure under study are nonlinear, the relationship between a multi-tone input excitation and its output frequency response is much more complicated.

It was shown by Lang and Billings [4] that when an analytical nonlinear system is subject to a multi-tone input

$$u(t)=\sum_{i=1}^{K}\left|A_i\right|\cos(\omega_i t+\angle A_i) \tag{14}$$

where $\omega_i=2\pi/T_i$, $\angle A_i=\phi_i$, i=1,…,k, the output spectrum can be expressed as

$$Y(j\omega)=\sum_{n=1}^{N_Y} Y_n(j\omega)=\sum_{n=1}^{N_Y} \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n\left(j\omega_{k_1},\cdots,j\omega_{k_n}\right)A(\omega_{k_1})\cdots A(\omega_{k_n}) \tag{15}$$

where N_Y is the maximum order of the system nonlinearity,

$$Y_n(j\omega)=\frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n\left(j\omega_{k_1},\cdots,j\omega_{k_n}\right)A(\omega_{k_1})\cdots A(\omega_{k_n})$$

is the spectrum of the system nth order output,

$$k_l \in \{-K,\cdots,-1,1,\cdots,K\},\ l=1,\cdots,n,$$

$$A(\omega)=\begin{cases} |A_k|e^{j\angle A_k} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\} \\ 0 & \text{otherwise} \end{cases},$$

$$\omega_{-k}=-\omega_k\ ,\quad |A_{-k}|e^{j\angle A_{-k}}=|A_k|e^{-j\angle A_k}$$

and

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \tag{16}$$

is called the nth order transfer function or generalised frequency response function (GFRF) which, as shown in equation (16), is defined as the multi-dimensional Fourier Transform of the system nth order impulse response function $h_n(\tau_1,\cdots,\tau_n)$.

In (15), the constraint

$$\omega=\omega_{k_1}+\cdots+\omega_{k_n}\quad k_l\in\{-K,\cdots,-1,1,\cdots,K\},\ l=1,\cdots,n \tag{17}$$

for the summation of the n-dimensional function

$$H_n\left(j\omega_{k_1},\cdots,j\omega_{k_n}\right)A(\omega_{k_1})\cdots A(\omega_{k_n})$$

defines the frequency components in the system nth order output. Denote the range of frequencies defined by (17) as Ω_n. The frequency components in the system output can be expressed as [5]

$$\Omega=\Omega_{N_Y}\bigcup\Omega_{N_Y-(2p^*-1)} \tag{18}$$

where the value to be taken by p^* could be $1,2,\ldots,\lfloor N_Y/2\rfloor$ where $\lfloor\ \rfloor$ denotes to take the integer part. The specific value of p^* depends on the system nonlinearities. If the system GFRF's $H_{N_Y-(2i-1)}(.)=0$ for $i=1,\cdots,q-1$, and $H_{N_Y-(2q-1)}(.)\neq0$, then $p^*=q$.

Given the order n of system nonlinearity and the frequency components $\omega_1,\cdots,\omega_K$ in the applied multi-tone input, Lang and Billings [4][5] showed that Ω_n can be determined as

$$\Omega_n = \{\{W_n\}\} \tag{19}$$

where $\{\{W_n\}\}$ means a set composed of all the different elements of vector W_n, and W_n is given by

$$\begin{cases} W_n = \begin{bmatrix} \left|\sum \overline{W}_n(1,:)\right| \\ \vdots \\ \left|\sum \overline{W}_n(K(2K)^{n-1},:)\right| \end{bmatrix} \\ \overline{W}_n = \begin{bmatrix} I\overline{W}_{n-1}(1,:) & \overline{W} \\ \vdots & \vdots \\ I\overline{W}_{n-1}(K(2K)^{n-2},:) & \overline{W} \end{bmatrix} \\ n \geq 2 \quad \overline{W} = [\omega_1, \cdots, \omega_K]^T \quad I = \Big[\underbrace{1, \cdots, 1}_{n}\Big]^T \end{cases} \tag{20}$$

Equations (18) (19) (20) indicate that unlike linear systems or structures where the input frequencies are exactly the same as the frequencies in the output, the output frequencies of nonlinear systems are normally much richer than the frequencies in the input. The frequency components that may be available in the output of a nonlinear system will depend not only on the input frequencies but also on the system properties. This complexity implies that when analysing the output frequency response of a nonlinear structure or system to a multi-tone input excitation, the procedure for accurately evaluating the spectrum of a multi-tone signal cannot be followed in a straightforward way as in the linear system case since the frequency components in the output signal are unlikely to be the same as the frequency components in the input.

However, equations (18) (19) (20) provide an effective algorithm for determining the output frequencies of a nonlinear system subject to the excitation of a multi-tone input (14). Based on these results the following procedure should be used for the accurate computation of the output frequency response of nonlinear systems to multi-tone inputs.

(i) Choose a possibly maximum order N_Y of the system nonlinearity.

(ii) From N_Y and the frequency components $\omega_1, \cdots, \omega_K$ of the applied multi-tone input, evaluate the output frequency components Ω of the system using equations (18) (19) (20), and denote the result as $\Omega = \{\omega_{o1}, \cdots, \omega_{oL}\}$

(iii) Determine the periods of the frequency components in the system output as

$$T_{Oi} = \frac{2\pi}{\omega_{Oi}} \qquad i = 1, \ldots, L \tag{21}$$

choose a time interval T as a common multiple of $T_{Oi}, i = 1, \ldots, L$, and sample the output signal over this time interval under an appropriate sampling period T_s.

(iv) Perform a $N = T/T_s$ point FFT operation on the sampled system output $y(0T_s), \cdots, y((N-1)T_s)$ to yield a DFT analysis result $Y(k)$, $k = 0, \cdots, N-1$, for the output signal.

(v) Evaluate the magnitudes and phases of all frequency components in the output signal as

$$\begin{cases} A_i \approx \dfrac{2\left|Y(k_i^*)\right|}{N} \\ \phi_i \approx \angle Y(k_i^*) \end{cases} \qquad k_i^* = \frac{T}{T_{Oi}}, \quad i = 1, \ldots, L \tag{22}$$

Note that if there is no prior knowledge about the maximum order N_Y of system nonlinearity, a sufficiently large N_Y may need to be taken in step (i), and for the same reason, if the specific value of p^* in (18) is unknown, p^* can be chosen as $p^* = 1$. In addition, apart from avoiding aliasing effects, the appropriate choice of the sampling interval T_s should be selected to ensure $T_{Oi}/T_s, i = 1,...,L$ are all integers, if possible, so that the amplitudes and phases of the output signal at the output frequencies $\Omega = \{\omega_{o1},\cdots,\omega_{oL}\}$ can be obtained exactly from (22).

4 SIMULATION STUDY AND EXPERIMENTAL DATA ANALYSIS

In order to verify the effectiveness of the nonlinear signal processing approach proposed in Section 3 and to demonstrate the significance of the approach for dynamic testing and analysis of nonlinear structures and materials, in this section, a simulation study is first conducted, and then the new procedure is applied to process the stress response of a polymer material to a harmonic strain input. The practical stress data are rheological measurements on a strain controlled rheometer, which was used to test the material and to study non-linear phenomena in polymer rheology.

4.1 Simulation study

The simulation study considers the nonlinear system shown in Figure 3, which is composed of a static polynomial nonlinearity $u(t) + 0.5u^2(t)$ followed by a linear dynamic with transfer function $5/(s^2 + s + 4)$. A two-tone input signal

$$u(t) = \sin 0.8t + \sin 1.2t$$

was used to excite the system and conduct a numerical simulation to generate the response of the system to the input. The new nonlinear signal processing approach was then applied to the generated system response to perform a frequency domain analysis to study the energy distribution over all possible output frequencies of the system. The results obtained in each of the five steps of the approach are as follows:

(i) Choose $N_Y = 2$, the order of the static polynomial nonlinearity.

(ii) From $N_Y = 2$ and the input frequencies $[\omega_1,\cdots,\omega_K] = [\omega_1, \omega_2] = [0.8, 1.2]$, the output frequency components of the system Ω were determined using (18)(19)(20). The result is $\Omega = [0, 0.4, 0.8, 1.2, 1.6, 2, 2.4]$.

(iii) The periods of the frequency components in the system output are

$$[T_{O1},\cdots,T_{O7}] = \left[\infty, \frac{10}{4}2\pi, \frac{10}{8}2\pi, \frac{10}{12}2\pi, \frac{10}{16}2\pi, \frac{5}{10}2\pi, \frac{10}{24}2\pi\right]$$

The time interval, which is a common multiple of these periods, was chosen as $T = 10\pi$ seconds. Over this time interval, the simulated system output is sampled under the sampling period $T_s = 0.0001$ seconds.

(iv) Perform a $N = T/T_s = 10\pi / 0.0001 = 314159$ point FFT on the sampled system output. This yields a DFT analysis result $Y(k)$, $k = 0,\cdots,314158$ of the system output. The result $(2/N)|Y(k)|$ is shown in Figure 4 for $k = 0,\cdots,49$.

(v) From Figure 4, the magnitudes of the system output at seven different output frequencies can be obtained as $2|Y(k_i^*)|/N$, $i = 1,\cdots,7$, where $[k_1^*,\cdots,k_7^*] = \left[\frac{T}{T_{o1}},\cdots,\frac{T}{T_{O7}}\right] = [0,2,4,6,8,10,12]$. Note that the accurate results of $2|Y(k_i^*)|/N$, $i = 1,\cdots,7$, can be determined from the system model and input, the results are 1.25, 0.6475, 1.4476, 1.7685, 0.5807, 1.25, 0.42, which are almost the same as the direct FFT analysis results shown in Figure 4. The simulation study therefore verifies the effectiveness of the proposed approach.

Figure 5 shows the analysis result obtained using the sampled system output over a time interval $T = 35$ second, which is not a common multiple of the seven periods $T_{O1},\cdots,T_{O7}$ in the system output signal. Obviously, this analysis did not follow the procedure of the new approach. Consequently, correct magnitudes of different frequency components in the output were not obtained. This demonstrates the importance of using the proposed approach for the data analysis.

4.2 Experimental data analysis

The stress data of a polymer material called polydimethylsiloxane (PDMS) were collected from the rheological measurements on a strain controlled rheometer and studied using the proposed signal processing approach. The stress is the response of the material to a harmonic strain input with frequency 10 Hz.

(i) Choose $N_Y = 9$.

(ii) From $N_Y = 9$ and the input frequency $[\omega_1,\cdots,\omega_K] = [\omega_1] = [10\times 2\pi]$, the output frequency components of the system Ω were determined using (18)(19)(20). The result is
$\Omega = [10\times 2\pi, 20\times 2\pi, 30\times 2\pi, 40\times 2\pi, 50\times 2\pi, 60\times 2\pi, 70\times 2\pi, 80\times 2\pi, 90\times 2\pi]$

(iii) The periods of the frequency components in the system output are
$$[T_{O1},\cdots,T_{O9}] = \left[\frac{1}{10},\frac{1}{20},\frac{1}{30},\frac{1}{40},\frac{1}{50},\frac{1}{60},\frac{1}{70},\frac{1}{80},\frac{1}{90}\right]$$
The time interval, which is a common multiple of these periods, was chosen as $T = 10$. Over this time interval, the stress data were sampled under the sampling period $T_s = 1/1280$ second.

(iv) Perform a $N = T/T_s = 10/(1/1280) = 12800$ point FFT on the sampled stress response. This yields a DFT analysis result $Y(k)$, $k = 0,\cdots,12799$ of the stress output. The result $(2/N)|Y(k)|$ is shown in Figure 6 for $k = 0,\cdots,1000$.

(v) From Figure 6, the magnitudes of the stress output at nine different output frequencies can be obtained as $2|Y(k_i^*)|/N$, $i = 1,\cdots,9$, where $[k_1^*,\cdots,k_9^*] = \left[\frac{T}{T_{O1}},\cdots,\frac{T}{T_{O9}}\right] = [100,200,300,400,500,600,700,800,900]$.

Figure 7 shows the analysis result obtained using the sampled stress output over a time interval $T = 10.55$ second, which is not a common multiple of the nine periods $T_{O1},\cdots,T_{O9}$ in the stress output signal. This analysis did not follow the procedure of the new approach. Consequently, an obviously different analysis result was obtained.

This result demonstrates the significance of the proposed approach in processing and analysing practical data from dynamic materials testing. Because similar testing experiments are widely used in mechanical and civil engineering to examine the behaviours and properties of structures and materials, the new approach has great potential to be applied in a wide range of associated engineering applications.

5 CONCLUSIONS

In this paper, a new signal processing approach for the accurate computation of the output frequency response of nonlinear systems to multi-tone inputs has been proposed. The approach is based on a theoretical analysis of the direct FFT result for a multi-tone signal and the theoretical results recently developed by the authors on the analysis of nonlinear systems in the frequency domain. Simulation studies verify the effectiveness of the new approach and the associated theoretical analysis. The approach has also been applied to process experimental data from dynamic testing of rheological properties of a polymer material. The application shows that the new approach has great potential for processing experimental data from testing materials and structures in different engineering areas.

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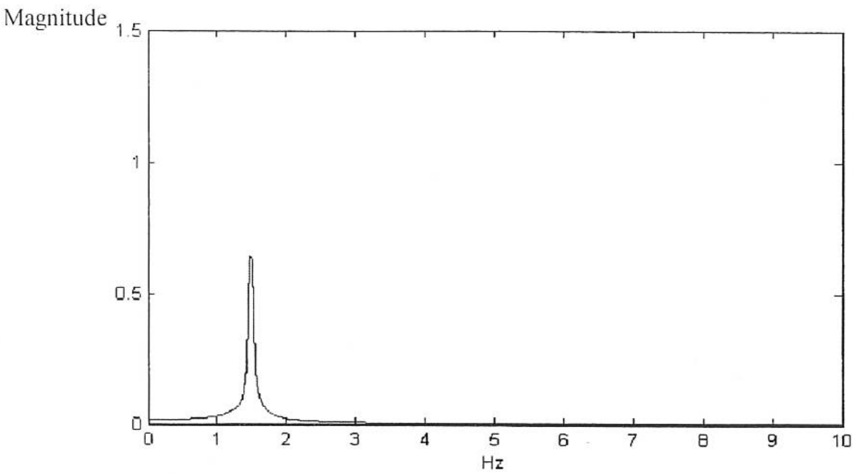

Figure 1 The FFT result of the sinusoidal signal $y(t)=\sin(2\times\pi\times1.5t)$ evaluated over a 25 second time interval

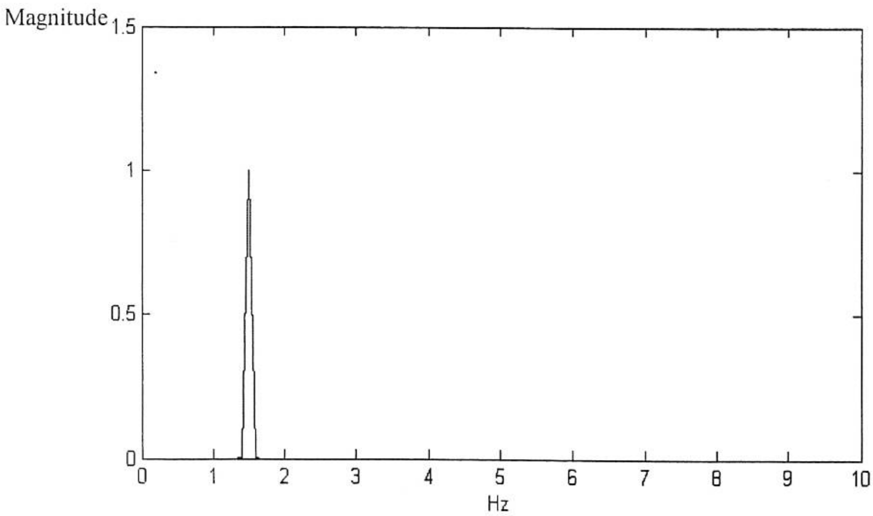

Figure 2 The FFT result of the sinusoidal signal $y(t)=\sin(2\times\pi\times1.5t)$ evaluated over a 10 second time interval

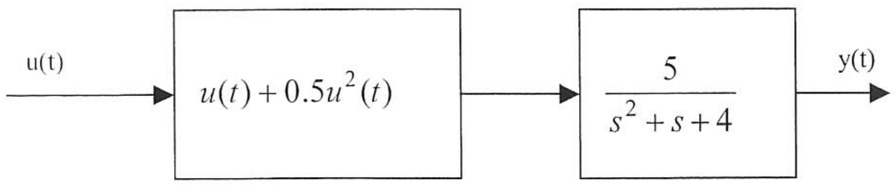

Figure 3 The nonlinear system considered in the simulation study in Section 4.1

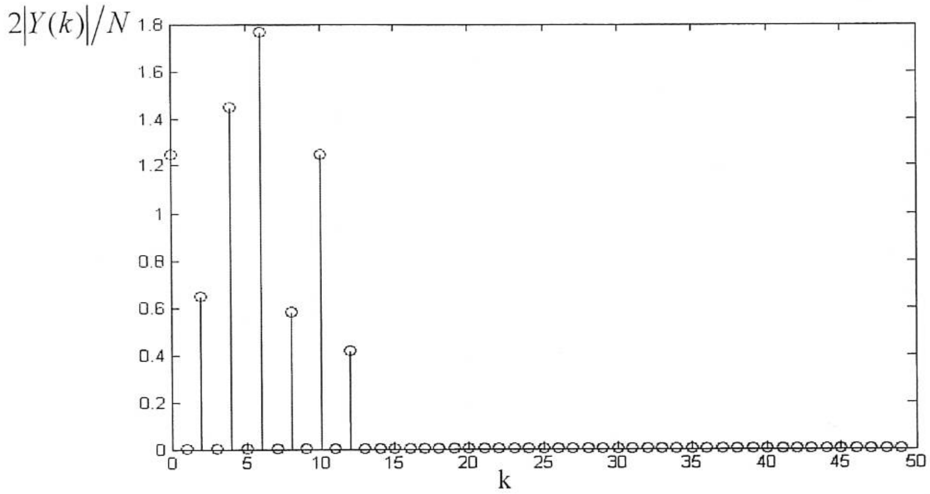

Figure 4 The spectrum of the output signal obtained using the proposed approach in the simulation study in Section 4.1

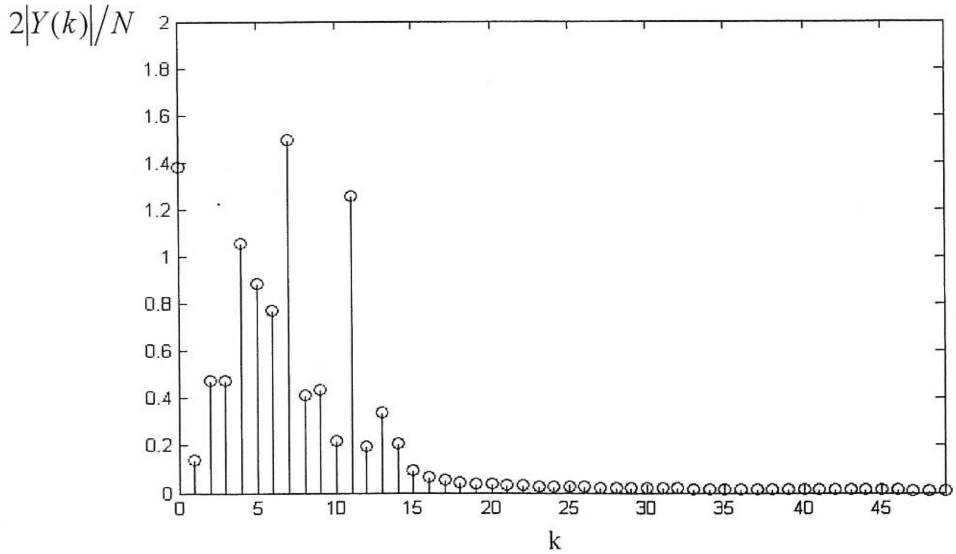

Figure 5 The spectrum of the output signal obtained without using the proposed approach in the simulation study in Section 4.1

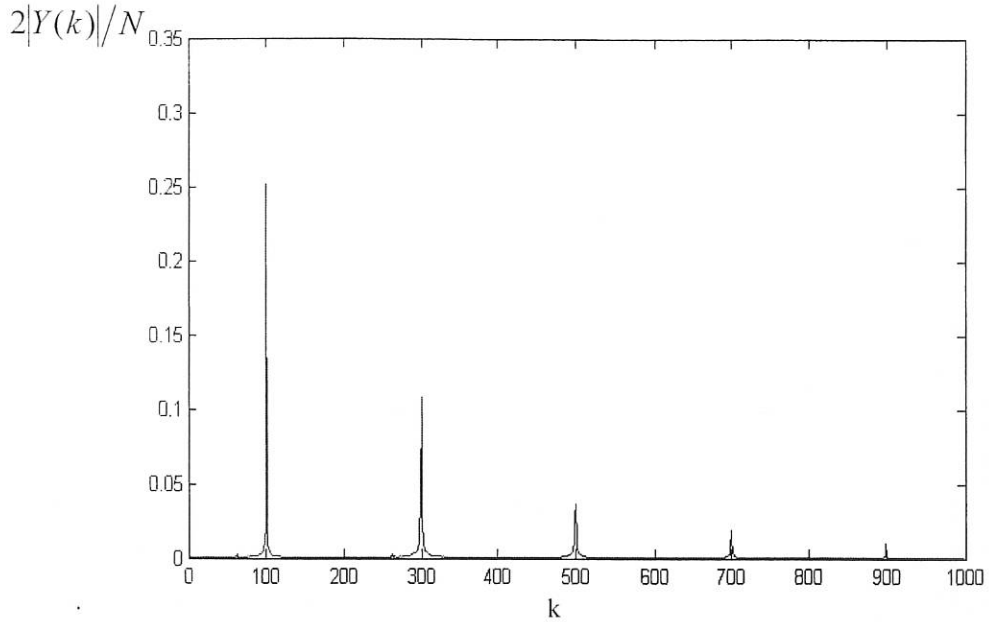

Figure 6 The spectrum of the stress signal obtained using the proposed approach in the experimental data analysis in Section 4.2

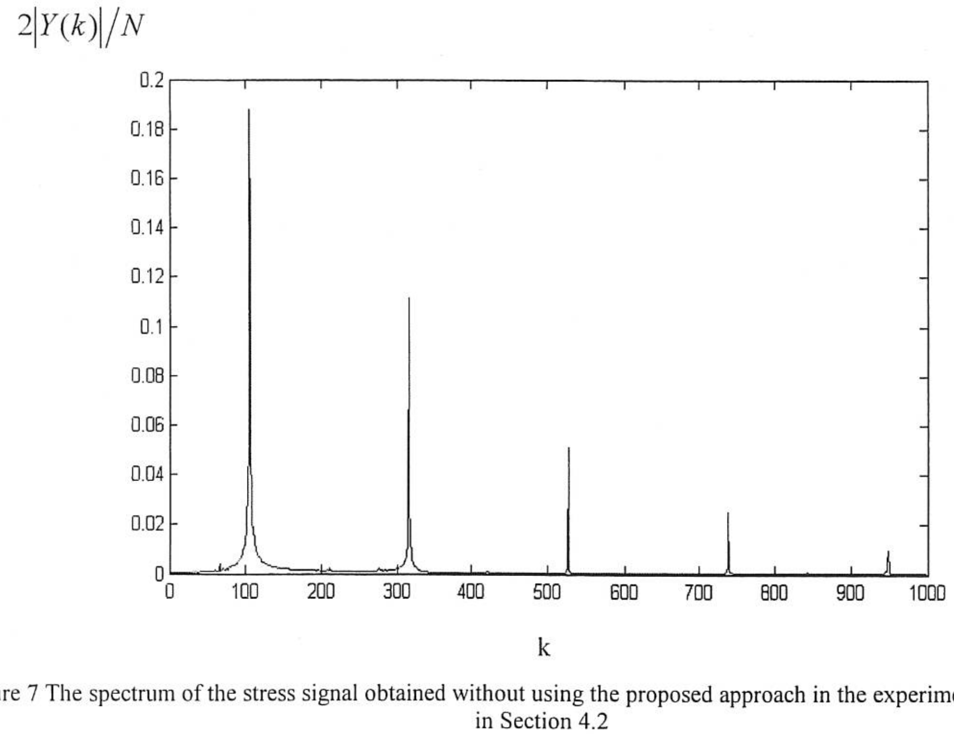

Figure 7 The spectrum of the stress signal obtained without using the proposed approach in the experimental data analysis in Section 4.2