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CHAPTER NUMBER

PRACTICAL OBSTACLES IN THE SENSITIVITY ANALYSIS OF NETWORK EQUILIBRIA

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ABSTRACT

Static network equilibrium continues to be the favoured paradigm used in network modelling and policy appraisal. Underlying much of this work, in particular for network optimisation problems, is the technique of sensitivity analysis, with the inherent assumption that the equilibrium flows are differentiable. Recent research has called into question the validity of conducting such analysis of the user equilibrium (UE) flows, for which the total derivatives do not always exist.

For the case of stochastic user equilibrium (SUE) it is clear that the analytical obstacles faced in the case of UE do not arise; differentiability of both logit and probit SUE model has been established, but this is not the whole story. Difficulties arise in calculating derivatives of the link-based probit model whenever rank deficiencies occur in the path covariance matrix, due to the network topology. We investigate the nature of this problem for some simple examples and show that it can sometimes be resolved.

INTRODUCTION

Sensitivity analysis has a long-established presence in the transport network research literature, with numerous applications covering problems such as trip matrix estimation, bi-level optimisation, reliability analysis and error estimation. Essentially, this technique aims to find derivatives of the implicit relationship between the input data (including policy variables) and the equilibrium flows, which may then be exploited to provide either gradient-like information, such as in a bi-level optimisation context, or a linear approximation, such as in the case of error estimation. It is perhaps surprising that a technical problem such as this should generate controversy, yet the work of Patriksson & Rockafellar (2002) achieved this by bringing into question the validity of the seminal transportation paper on the subject of network equilibrium sensitivity analysis by Tobin & Friesz (1988). The fact that virtually all the transportation applications reported in the literature were derived from Tobin & Friesz's analysis makes this a highly pertinent issue to address. However, the sophisticated mathematical tools utilised in Patriksson & Rockafellar's analysis deter many in the transport research field, even the more mathematically-minded, and appreciating the significance of the subtle arguments presented is not a straightforward task.

By coincidence, in parallel to the work of Patriksson & Rockafellar, the present authors were themselves presenting a series of new results on the sensitivity analysis of network equilibria (Clark and Watling, 2000, 2002, Connors et al., 2004a, 2004b, 2007). A key distinction between these two fields of enquiry was, however, the network equilibrium model used for the analysis: the later work was based on the Probit Stochastic User Equilibrium (probit SUE) model, whereas Patriksson's findings related (as did the original Tobin & Friesz analysis) exclusively to Wardrop's Deterministic User Equilibrium (DUE) model. Indeed, in terms of sensitivity analysis this issue turns out to be a critical feature: in theoretical terms, the PSUE model behaves in a 'smooth' way that circumvents many of the difficulties inherent in DUE sensitivity analysis. On the other hand, practical computational problems are then introduced into the analysis of the PSUE sensitivities, most notably the problem of calculating the Jacobian of the probit choice probability fractions, expressions for these fractions not being available in closed form, leading to problems of degeneracy that have apparent (but misleading) parallels with the DUE case.

The purpose of the present paper is to attempt to clarify the main difficulties in performing sensitivity analysis of the DUE and SUE models, reviewing the theoretical issues for both models and then considering concomitant practical/computational problems. Although a number of recent, as yet unpublished, manuscripts have also sought to make these clarifications for the DUE theoretical case (Josefsson and Patriksson, 2006), our objective is to add to this debate with a somewhat different approach. Namely, we explore DUE as a limiting case of SUE, and illustrate with analysis and examples how features of the different models and/or of the network under consideration give rise to distinctive characteristics of the sensitivity analysis for the two models.

DEFINITIONS AND NOTATION

The network is represented by a directed graph consisting of N links labelled $a = 1,2,...,N$; a demand matrix **q,** with entries q^r representing the travel demand on the r^{th} origin-destination (OD) movement; and a set of paths connecting the r^{th} OD movement denoted K^r with the set of all paths $K = K^1 \cup ...$ having cardinality $|K|$. The link-path incidence matrix Δ has elements that are Kronecker delta functions $\delta^r_{a,k}$, denoting the links a that are part of path k serving OD movement r. An assignment of flows to all paths is denoted by the vector $\mathbf{f}$. The assignment $\mathbf{f}$ is *feasible* for demand vector $\mathbf{q}$ if and only if

$$\sum_{k \in K^r} f^r_k = q^r \ \forall r \text{ and } f^r_k \geq 0 \ \forall k, r. \tag{1}$$

The (closed, convex) set of feasible path flows thus defined is denoted F. The vector of link flows is denoted $\mathbf{x}$. The link cost-flow relationships are assumed to be single-valued and differentiable, with $\mathbf{t}(\mathbf{x},\mathbf{s})$ the vector of link costs when the link flow is $\mathbf{x}$ and the design parameters $\mathbf{s}$. The mapping between link flows and path costs $\mathbf{c}(\mathbf{x},\mathbf{s})$ is derived from the link cost-flow relationships according to the standard link-additive model:

$$\mathbf{c}(\mathbf{x},\mathbf{s}) = \Delta^T \mathbf{t}(\mathbf{x},\mathbf{s}). \tag{2}$$

The link flow vector $\mathbf{x}^* = \Delta \cdot \mathbf{f}^*$ is a solution to the DUE if $\mathbf{f}^*$ satisfies (1), and for each OD movement r:

$$f^r_k > 0 \Rightarrow c^r_k = \min\{c^r_j \ \forall j \in K^r\} \tag{3}$$

The reverse implication does not follow; there can be minimum cost paths that have zero flow. This is non-strict complementarity,

$$c^r_k = \min\{c^r_j \ \forall j \in K^r\} \Rightarrow f^r_k \geq 0. \tag{4}$$

For the case of SUE, we first define a random utility model for each OD movement r, representing the proportion of the OD flow on movement r that chooses path k when the mean (deterministic) path costs are $\mathbf{c}$:

$$P^r_k(\mathbf{c}) = \Pr\left(c^r_k + \varepsilon^r_k \leq c^r_j + \varepsilon^r_j \ \forall j \in K^r\right). \tag{5}$$

For each movement r, the stochastic terms $\{\varepsilon^r_k : k \in K^r\}$ are assumed to have a non-degenerate joint probability density function that is continuous, strictly positive, and independent of the deterministic path costs $\mathbf{c}$. The stochastic terms are assumed to be independent between OD movements. The vector of path choice proportions is $\mathbf{P}$.

The basic principle underlying the SUE model is then:

At SUE, no driver can improve their perceived travel cost by unilaterally changing route.

Formally, an SUE is defined to be a feasible path flow vector $\tilde{\mathbf{f}}$ (in the sense of (1)) that satisfies the fixed-point condition (Sheffi, 1985):

$$\tilde{\mathbf{f}} = \mathbf{Q} \cdot \mathbf{P}\left(\mathbf{c}\left[\Delta \cdot \tilde{\mathbf{f}}\right]\right), \tag{6}$$

where the demand matrix $\mathbf{Q}$ is constructed from the vector $\mathbf{q}$, such that each OD demand q^r is repeated $\left|K^r\right|$ times along its diagonal. The corresponding SUE link flow solution is $\tilde{\mathbf{x}} = \Delta \cdot \tilde{\mathbf{f}}$.

DUE AND SUE SENSITIVITY ANALYSIS: THEORETICAL ISSUES

In the search for (directional) derivatives of the DUE flows with respect to perturbations of the design parameters, there are several cases that present difficulties:

a) The set of 'active' paths (the paths with non-zero flow) changes.
   This occurs when, due to the perturbation, a new path is assigned some flow that was previously unused.
b) The occurrence of non-strictly complementary solutions.
   When the equilibrium solution includes a minimum cost path with zero flow, under any perturbation the flow on this path can only increase or remain zero, the flow cannot decrease and become negative.
c) Non-uniqueness of the path flows.
   For some network topologies (*e.g.* Figure-8 network), the DUE path flows are not uniquely defined at any value of the design parameters. The gradient of the path flow with respect to perturbations of the design parameters is therefore not well-defined.

If we consider the DUE link flows as a function of the design parameters, $\mathbf{x}^*(\mathbf{s})$, the surface of equilibrium flows is not differentiable everywhere. In particular, where non-strict complementarity occurs, not all of the directional derivatives exist. By seeking only those directional derivatives that exist, the method of Patriksson & Rockafellar (2002) calculates the available sub-gradients and naturally provides the total derivative when this exists. In this way sensitivity analysis of DUE can be conducted despite these problematic features of the equilibrium surface.

For the case of SUE, if the probability density function for the stochastic terms assigns strictly positive probabilities to all path costs, from (5) it is clear that every path is assigned some flow. This is the case, for example, for the probit and logit models. Therefore, all paths are always active. No path has zero flow. Moreover, since the SUE is defined by the fixed point condition (6), there is nothing corresponding to the issue of non-strict complementarity. Issues (a) and (b) above do not occur for the case of SUE. Regarding (c), if the link travel time functions are strictly increasing functions of the link flow (see Sheffi, 1985) then the SUE link flows are unique. It follows (see Rosa, 2003) that, at equilibrium, the path flows are also uniquely determined by the fixed point condition (6).

Assuming that the link cost functions and the probability density function of the stochastic terms are (single valued and) differentiable, then the SUE link flows and path flows are differentiable. Davis (1994) stated that this was the case for the logit and probit models. For the case of logit SUE, Davis (1994) provided the gradients of the equilibrium flows. Furthermore, Patriksson (2004) shows that where the DUE gradient exists, it is the limiting case of the logit SUE gradient. For the case of probit SUE, Clark and Watling (2002) derive sensitivity expressions for the equilibrium link flows. In the light of the SUE fixed-point condition (6), they consider the link flow gap function

$$\mathbf{d}(\mathbf{x};\mathbf{s}) = \mathbf{x} - \Delta \cdot \mathbf{Q} \cdot \mathbf{P}[\mathbf{c}(\mathbf{x};\mathbf{s})].$$

For design parameters, **s**, the link flows $\tilde{\mathbf{x}}$ are a solution to the SUE if and only if $\mathbf{d}(\tilde{\mathbf{x}};\mathbf{s}) = 0$. As stated above, the probit link flows are differentiable and clearly, so is the gap function. We can therefore write down the Taylor series expansion of the gap function about the equilibrium flows at some initial setting, $\mathbf{s}_0$, of the design parameters:

$$\mathbf{d}(\mathbf{x};\mathbf{s}) \approx \mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}_0);\mathbf{s}_0) + \left[\nabla_{\mathbf{x}}\mathbf{d}\big|_{\mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}_0);\mathbf{s}_0)}(\mathbf{x} - \tilde{\mathbf{x}}(\mathbf{s}_0))\right] + \left[\nabla_{\mathbf{s}}\mathbf{d}\big|_{\mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}_0);\mathbf{s}_0)}(\mathbf{s} - \mathbf{s}_0)\right] \quad (7)$$

The link flow Jacobian, $\nabla_{\mathbf{x}}\mathbf{d}$, and design parameter Jacobian, $\nabla_{\mathbf{s}}\mathbf{d}$, are evaluated at the initial equilibrium flows. Evaluating **d**(.) with the network flows at (the new) equilibrium, $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{s})$, by definition of the gap function, gives $\mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}),\mathbf{s}) = \mathbf{0}$. We can therefore write

$$0 \approx \nabla_{\mathbf{x}}\mathbf{d} \cdot (\tilde{\mathbf{x}}(\mathbf{s}) - \tilde{\mathbf{x}}(\mathbf{s}_0)) + \nabla_{\mathbf{s}}\mathbf{d} \cdot (\mathbf{s} - \mathbf{s}_0) \quad (8)$$

For those points $\mathbf{s}_0$ where the link flow Jacobian is non-singular, $|\nabla_{\mathbf{x}}\mathbf{d}| \neq 0$, the equilibrium flows at **s** can be expressed in terms of those at $\mathbf{s}_0$ (Clark and Watling, 2002):

$$\tilde{\mathbf{x}}(\mathbf{s}) \approx \tilde{\mathbf{x}}(\mathbf{s}_0) - \nabla_{\mathbf{x}}\mathbf{d}^{-1}\nabla_{\mathbf{s}}\mathbf{d} \cdot (\mathbf{s} - \mathbf{s}_0). \quad (9)$$

Note that this requires inversion of the link flow Jacobian. Sufficient conditions for differentiability of the probit SUE flows are presented in Connors *et al.* (2007). In calculating the two Jacobian matrices, they are naturally decomposed as follows

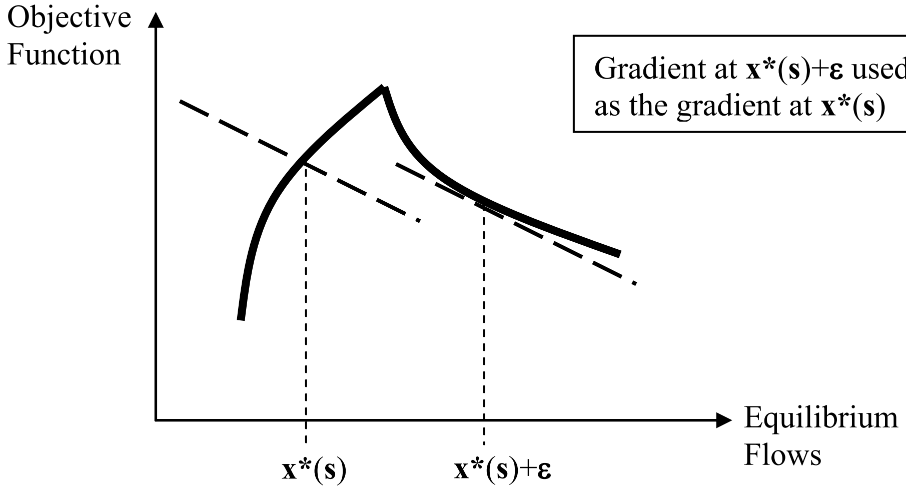
$$\nabla_{\mathbf{x}}\mathbf{d} = \mathbf{I} - \mathbf{Q} \cdot \nabla_{\mathbf{c}}\mathbf{P} \cdot \Delta^T \cdot \nabla_{\mathbf{x}}\mathbf{t} \quad \text{and} \quad \nabla_{\mathbf{s}}\mathbf{d} = -\mathbf{Q} \cdot \nabla_{\mathbf{c}}\mathbf{P} \cdot \Delta^T \cdot \nabla_{\mathbf{s}}\mathbf{t}, \quad (10)$$

for the case where the design parameters do not represent changes to demand. Clark & Watling (2002) showed that the path choice probability Jacobian, $\nabla_{\mathbf{c}}\mathbf{P}$, can itself be calculated using a probit equilibrium assignment of reduced dimension (c.f. the original equilibrium problem).

DUE SENSITIVITY ANALYSIS: PRACTICAL ISSUES

While Patriksson & Rockafellar (2002) provide a method to derive the available (sub-) gradient information for DUE, some practical issues remain. The set of paths to be included in the analysis, including those paths that might be non-strictly complementary, must be determined; thus far only heuristics methods have been proposed to accomplish this task (Josefsson and Patriksson, 2006).

For the case of DUE, the set of non-differentiable points constitutes a set of measure zero: it is on a 'knife edge' that only some directional derivatives are available, rather than the total derivative. This might appear to diminish the amount of effort worth expending in dealing with the non-differentiable points, but there are two reasons why this would be short sighted. Firstly, while the set of non-differentiable points is of zero measure, this does not mean that such features are rare. The zero measure is due to non-strict complementarity occurring at specific settings of the design parameters rather than for whole ranges: it infers nothing about the profusion of non-differentiable points. Secondly, in practice the impact of non-differentiable points extends into the region surrounding them due to the imperfect convergence of any equilibrium assignment algorithm. Consider, for example, a bi-level optimisation (maximisation) program that computes the gradients of the flows and hence of the objective function, with respect to the design parameters being optimised.



At a given setting of the design parameters, **s,** the equilibrium flows, **x***(**s**) are sought, but the equilibrium assignment algorithm terminates when the flows are calculated to be **x***(**s**)+**ε**. The gradient is then calculated at **x***(**s**)+**ε**, (dashed tangent line in Figure) but this gradient is assumed to be that of the flows at **s**, when it is not. The error induced would be small if the surface of equilibrium flows were smooth, but whenever such calculations occur near to non-differentiable points, misleading gradient information may be passed to the optimisation algorithm. Common use of the Frank Wolfe algorithm with its well documented lack of uniform convergence (e.g. Sheffi, 1985) compound this problem in a bilevel setting (Shepherd and Sumalee, 2004).

SUE SENSITIVITY ANALYSIS: PRACTICAL ISSUES

While in principle all paths are used at SUE, in practice (especially for large networks) this is not the case, and which paths are actually assigned flow at the termination of a numerical solution algorithm will depend on the properties of that algorithm. There are two reasons why a given path may be unused at the estimate of equilibrium obtained by such an algorithm:

1. The algorithm assigned zero flow to this path, instead of the correct, positive equilibrium flow.
2. The algorithm did not consider this path.

The probit path choice probabilities are defined by an integral that cannot be evaluated exactly (see (12) below) necessitating use of an estimation method, most commonly Monte Carlo simulation or analytic approximation (e.g. Clark, 1961, or Mendell and Elston, 1974). Case 1 may result from inaccuracies inherent in such methods coupled with the fact that machine precision may round to zero small choice probabilities multiplied by finite OD demand.

One of the standard methods for calculating probit SUE is the link-based Method of Successive Averages (MSA) algorithm proposed by Sheffi (1985), in which all paths are implicitly available. The active paths are generated incrementally during the course of a Monte Carlo-based solution algorithm, using auxiliary solutions generated by a stochastic shortest path method. While in an infinite number of iterations this algorithm would assign flow to all conceivable paths, in practice (at the end of a finite number of iterations) many paths will have never been generated during the procedure, and will therefore not be assigned any flow. In such a case, provided a large number of iterations had been used, the correct equilibrium flow to such an unused path will be extremely small, but nevertheless positive. This is an example of the second case stated above. One problem with this approach is that the active path set may change between equilibrium assignments calculated at 'adjacent' settings of the design parameters, calling into question the precise meaning of the gradient calculated at any point.

An alternative method for calculating probit SUE is to define the active path set upfront. For small networks this may include every conceivable path, for large networks it will almost certainly not. At each iteration of the equilibrium assignment algorithm (MSA for example), the choice probabilities are calculated for all paths in the active path set and each of these paths is assigned some flow. One benefit of this approach is that the surface of equilibrium flows and its gradient are consistent for the (fixed) active path set. Re-running the model with additional paths included in the active path set will give new equilibrium flows, and hence will alter the equilibrium solution surface and its gradient. The path set may be generated heuristically in an attempt to include all paths that carry "significant" flow at equilibrium, or may be generated according to other criteria. Neglected paths are not assigned any flow (under case 2 listed above).

Degeneracy Issues with Probit SUE

For the most common implementation of probit SUE, where the path covariance matrix is constructed from variances of the constituent links, a practical obstacle remains: the possibility of degeneracies arising from the network topology. To understand the significance of this we present the details of the probit model and then investigate the nature of these degeneracies using some simple examples.

The probit model is a particular instance of the SUE formulation described above; it is constructed by introducing stochastic terms to the link costs. The *perceived* cost on the *a*-th link is $T_a = t_a + \xi_a$; the stochastic terms ξ_a are independent and normally distributed about zero with (non-zero but finite) flow-independent variances σ_a. The resulting perceived path costs are

$$c_k^r(\mathbf{f}) + \varepsilon_k^r = \sum_{a \in A} \left[t_a(\Delta \cdot \mathbf{f}) + \xi_a \right] \delta_{a,k}^r \ . \tag{11}$$

The random error terms from the constituent links have been collected in the stochastic terms $\{\varepsilon_k^r\}$, whose joint probability density function is multivariate normal (MVN) with zero mean, and variance-covariance matrix Σ^r for the *r*-th OD movement. By construction, the network topology is reflected in the correlation structure (the variance-covariance matrix) of the perceived path costs. Since the perceived costs are MVN distributed, the path choice probabilities cannot be written in closed form:

$$P_k^r = \frac{1}{\sqrt{|\Sigma^r|(2\pi)^{K^r}}} \int \cdots \int \exp\left[-\frac{1}{2}\mathbf{c}^T(\Sigma^r)^{-1}\mathbf{c}\right] d\mathbf{c}. \tag{12}$$

The region of integration is where the k-th path is the cheapest: $c_k^r \leq \min\{c_j^r : \forall j \neq k\}$. The probit path choice probabilities therefore rely on the existence of the (so called) precision matrix, $(\Sigma^r)^{-1}$. For network topologies where the link-path incidence matrix is rank deficient, even for only one constituent OD movement, construction of the path-costs from the link-costs will result in a singular covariance matrix for this OD. For such cases, the path choice probability, (11), and hence the probit model itself, are not well defined. Clark & Watling (2002) suggest several mechanisms for working around this issue, but do not find a way to include it in their analysis.

Note that this problem can be avoided by restricting the probit path covariance matrix to be diagonal; after all, the logit SUE path covariance matrix is a scalar multiple of the identity matrix. Alternatively, by adding a small path-specific component to the perceived path costs the degeneracy would be removed. However, construction of the path covariance from the constituent link variances is an intuitively appealing aspect of the probit model and, as we will discuss below, may not necessarily pose a problem.

The MVN distribution can be defined for the case of singular covariance matrix as follows: If the eigenvalues of Σ^r are $d_1 > ... > d_m > d_{m+1} = 0,...,d_{K^r} = 0$. Defining $D^r = \text{diag}(d_1,...,d_m)$ gives the spectral decomposition $\Sigma^r = ED^rE^T$ where E is size [k x m], of rank m, and comprises columns that are eigenvectors of Σ^r corresponding to the non-zero eigenvalues. The probability density function of the singular MVN distribution is then

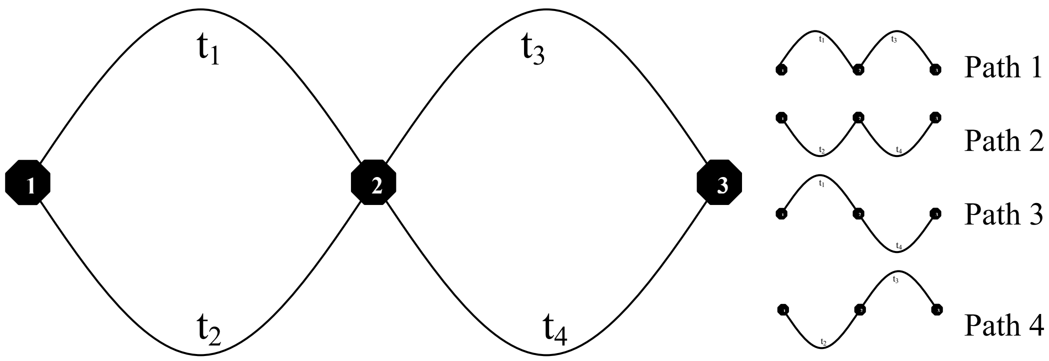
$$g(\mathbf{c}) = \frac{1}{\sqrt{|D^r|(2\pi)^{K^r}}} \exp\left[-\frac{1}{2}\mathbf{c}^T E(D^r)^{-1}E^T\mathbf{c}\right].$$

This transformation performs a rotation of the original coordinate axes (the link costs) such that the new coordinates are aligned with the eigen-vectors of the covariance matrix; the singular dimension of the probability density function can then easily be neglected as it corresponds to one of the coordinate axes. The path choice probabilities are well defined with reference to the new coordinates, although the limits of integration that define P_k^r in (12) must be transformed. Whereas there are several well known techniques (for example the method of Mendell-Elston) to efficiently calculate the MVN probability $\Pr(C_k^r \leq C_j^r \ \forall j \neq k)$, the transformed integral is not of this type and does not allow convenient estimation methods to be used (though of course Monte Carlo simulation can be applied, as it can for the initial singular integral).

The simplest network having singular a path-covariance matrix is the Figure-8. But although the degeneracy problems stated above appear in the Figure-8 network, the equilibrium flows can be shown to exist and be differentiable. This illustrates that some of the practical obstacles are introduced by our method of decomposing the Jacobians for calculation, and are not inherent to the sensitivity analysis of the network.

The Figure-8 Network

The Figure-8 network and its four paths are drawn below.



The perceived path costs are derived from their constituent link costs as follows:

$$C_1 = T_1 + T_3 = t_1 + \varepsilon_1 + t_3 + \varepsilon_3$$
$$C_2 = T_2 + T_4 = t_2 + \varepsilon_2 + t_4 + \varepsilon_4$$
$$C_3 = T_1 + T_4 = t_1 + \varepsilon_1 + t_4 + \varepsilon_4$$
$$C_4 = T_2 + T_3 = t_2 + \varepsilon_2 + t_3 + \varepsilon_3$$

The link path incidence matrix is of rank three, and the covariance matrix of the probit model for this single OD pair network is therefore singular:

$$\Sigma = \begin{bmatrix} \sigma_1^2 + \sigma_3^2 & 0 & \sigma_1^2 & \sigma_3^2 \\ 0 & \sigma_2^2 + \sigma_4^2 & \sigma_4^2 & \sigma_2^2 \\ \sigma_1^2 & \sigma_4^2 & \sigma_1^2 + \sigma_4^2 & 0 \\ \sigma_3^2 & \sigma_2^2 & 0 & \sigma_2^2 + \sigma_3^2 \end{bmatrix}$$

Despite analytical concerns regarding whether or not the probit choice probabilities are well defined, it seems reasonable that equilibrium flows exist, and that under smooth perturbations to the link cost function parameters, the equilibrium flows will smoothly vary.

By taking a link based approach, this network is surprisingly amenable to analysis. The path choice probability for path 1 is $P_1 = \mathbf{P}(C_1 \le C_2, C_1 \le C_3, C_1 \le C_4)$:

$$P_1 = \mathbf{P}(T_1 + T_3 \le T_2 + T_4, T_1 + T_3 \le T_1 + T_4, T_1 + T_3 \le T_2 + T_3)$$
$$P_1 = \mathbf{P}(T_1 + T_3 \le T_2 + T_4, T_3 \le T_4, T_1 \le T_2)$$
$$P_1 = \mathbf{P}(T_1 \le T_2, T_3 \le T_4)$$

Note that one of the conditions, comparison with path 2, is redundant. The path that can be ignored shares no links with the path in question (path 1 in this case).

We consider the simplest case, and assume that the link error terms are independent. Following from above, for this case we have for path 1

$$P_1 = \mathbf{P}(T_1 \le T_2)\cdot \mathbf{P}(T_3 \le T_4) = \mathbf{P}(T_1 - T_2 \le 0)\cdot \mathbf{P}(T_3 - T_4 \le 0)$$

and similarly for the other paths. The path choice probabilities for the Figure-8 split into independent terms around the central node: the chosen path must be 'piecewise cheapest' on each of the independent parts of the network. This view extends to further examples presented below.

We can calculate these probabilities directly by defining $Y_{ij} = T_i - T_j$ so that $\bar{Y}_{ij} = t_i - t_j$ and $\sigma_Y^2 = \sigma_i^2 + \sigma_j^2$. With $Z_{ij} = Y_{ij} - \bar{Y}_{ij} / \sigma_Y$, $Z_{ij} \sim N(0,1)$ and so

$$P_1 = \mathbf{P}\left(Z_{12} \le \frac{-(t_1 - t_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot \mathbf{P}\left(Z_{34} \le \frac{-(t_3 - t_4)}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right) = F\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right)$$

Where $F(\cdot)$ is the cumulative (standard) normal distribution function. The path choice probabilities are thus

$$P_1 = F\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right) \quad P_2 = F\left(\frac{t_1 - t_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_3 - t_4}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right)$$

$$P_3 = F\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_3 - t_4}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right) \quad P_4 = F\left(\frac{t_1 - t_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right)$$

Recalling (9) and (10), in order to write down sensitivity expressions we need to calculate derivatives of the choice probabilities with respect to the link flows. With the standard normal density function $f(\cdot)$ these immediately follow

$$\frac{\partial P_1}{\partial t_1} = \frac{-1}{\sqrt{\sigma_1^2 + \sigma_2^2}} f\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right) \quad \frac{\partial P_1}{\partial t_2} = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} f\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot F\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right)$$

$$\frac{\partial P_1}{\partial t_3} = \frac{-1}{\sqrt{\sigma_3^2 + \sigma_4^2}} F\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot f\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right) \quad \frac{\partial P_1}{\partial t_4} = \frac{1}{\sqrt{\sigma_3^2 + \sigma_4^2}} F\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)\cdot f\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}} \right)$$

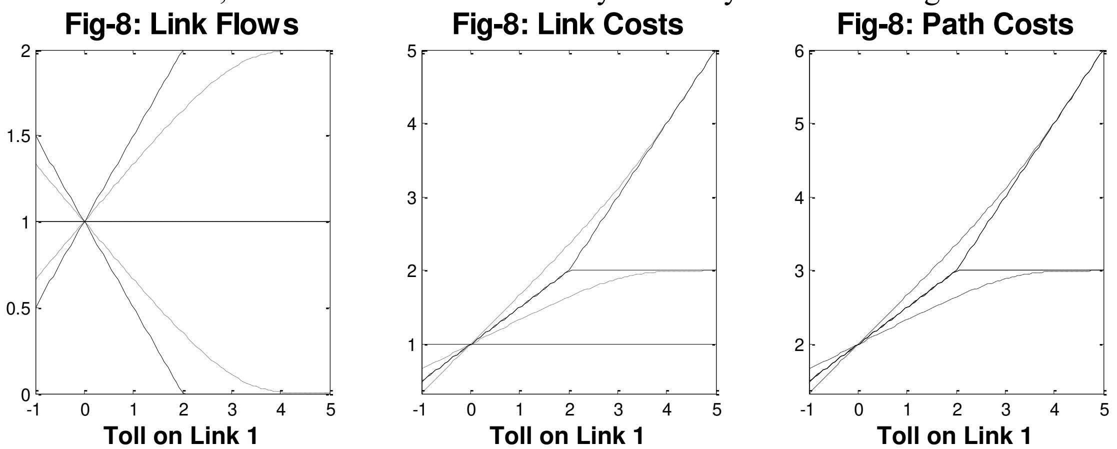
With $\sigma_{ij} = \sqrt{\sigma_i^2 + \sigma_j^2}$, $f_{ij} = f\left(\left| t_i - t_j \right| / \sigma_{ij} \right)$, and F_{ij} similarly, this gives

$$\begin{bmatrix} \frac{\partial P_1}{\partial t_1} \cdots & \frac{\partial P_1}{\partial t_4} \\ \vdots \quad \ddots & \\ \frac{\partial P_4}{\partial t_1} & \frac{\partial P_4}{\partial t_{41}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sigma_{12}} f_{21}F_{43} & \frac{1}{\sigma_{12}} f_{21}F_{43} & \frac{-1}{\sigma_{34}} F_{21}f_{43} & \frac{1}{\sigma_{34}} F_{21}f_{43} \\ \frac{1}{\sigma_{12}} f_{12}F_{34} & \frac{-1}{\sigma_{12}} f_{12}F_{34} & \frac{1}{\sigma_{34}} F_{12}f_{34} & \frac{-1}{\sigma_{34}} F_{12}f_{34} \\ \frac{-1}{\sigma_{12}} f_{21}F_{34} & \frac{1}{\sigma_{12}} f_{21}F_{34} & \frac{1}{\sigma_{34}} F_{21}f_{34} & \frac{-1}{\sigma_{34}} F_{21}f_{34} \\ \frac{1}{\sigma_{12}} f_{12}F_{43} & \frac{-1}{\sigma_{12}} f_{12}F_{43} & \frac{-1}{\sigma_{34}} F_{12}f_{43} & \frac{1}{\sigma_{34}} F_{12}f_{43} \end{bmatrix}$$

When all links have covariance $\sigma_i^2 = 0.3$ the path choice probability Jacobian with respect to the link costs is

$$\begin{bmatrix} \frac{\partial P_1}{\partial t_1} & \cdots & \frac{\partial P_1}{\partial t_4} \\ \vdots & \ddots & \\ & & \\ \frac{\partial P_4}{\partial t_1} & & \frac{\partial P_4}{\partial t_{41}} \end{bmatrix} = \begin{bmatrix} -0.01 & 0.01 & -0.51 & 0.51 \\ 0.01 & -0.01 & 0 & 0 \\ -0.01 & 0.01 & 0.51 & -0.51 \\ 0.01 & -0.01 & 0 & 0 \end{bmatrix}$$

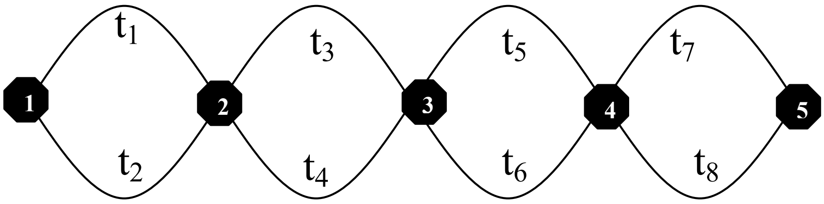
These analytical derivatives match those calculated numerically by finite differencing. The figure below illustrates the changes in flows and costs as the toll on link 1 is varied. The solid line represents DUE and the broken line SUE with link covariances 0.3 as above. When the toll reaches 2, link 1 becomes unused and the DUE solution is non-complementary; the link flow gradient is not well defined at this point. At higher tolls, the cost of link 1 increases and the DUE flow on it remains zero; all the flow is on link 2 which attains the appropriate (max) cost of 2. Meanwhile, the SUE flows and costs vary smoothly across the range of tolls.



It is worth noting that the analytic approximation method of Mendell-Elston *appears* to work for the Figure-8 network, calculating the choice probabilities despite the singular covariance matrix. Moreover this method seems to work for other MVN distributions with singular covariance, although the authors are not aware of any analysis relating to the use of this method with singular MVN distributions. This is doubly useful because the method (see Clark and Watling, 2002) used to calculate the probit path choice probability Jacobian, $\nabla_{\mathbf{c}}\mathbf{P}$ in (10), and hence the gradients, also relies on calculating choice probabilities for a singular MVN distribution.

Multiple Figure-8 Network

The analytic approach used for the Figure-8 network extends to networks of similar geometry



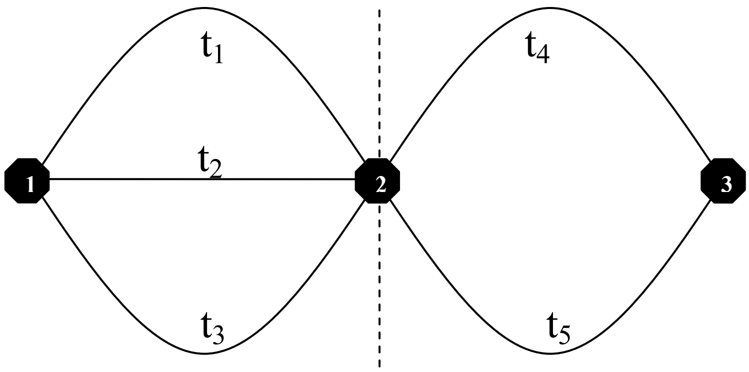
Between each pair of adjacent nodes there are only two competing links; the path taking every 'upper' link, [1, 3, 5, 7], has choice probability

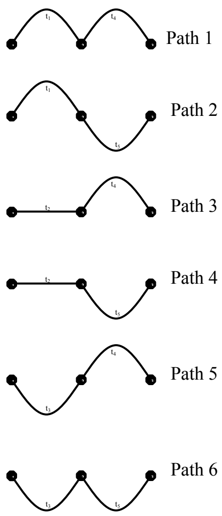
$$P_{1,3,5,7} = F\left(\frac{t_2 - t_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) \cdot F\left(\frac{t_4 - t_3}{\sqrt{\sigma_3^2 + \sigma_4^2}}\right) \cdot F\left(\frac{t_6 - t_5}{\sqrt{\sigma_5^2 + \sigma_6^2}}\right) \cdot F\left(\frac{t_8 - t_7}{\sqrt{\sigma_7^2 + \sigma_8^2}}\right)$$

and the derivatives follow easily, as above.

Figure-8 Network with Additional Link

A further extension includes more than two links between adjacent nodes.





The choice probability for path 1 is

$$P_1 = \mathbf{P}(T_1 + T_4 \le \{T_2 + T_4, T_3 + T_4, T_1 + T_5, T_2 + T_5, T_3 + T_5\})$$
$$P_1 = \mathbf{P}(T_1 \le T_2, T_1 \le T_3, T_4 \le T_5)$$
$$P_1 = \mathbf{P}(T_1 \le T_2, T_1 \le T_3) \cdot \mathbf{P}(T_4 \le T_5)$$

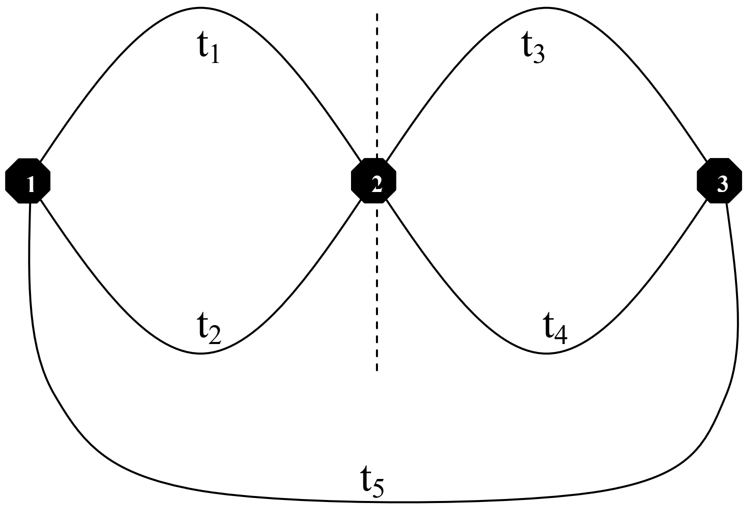
The network decomposes in the same way as for the Figure-8. For a given path, the constituent links simply have to be the most attractive between the adjacent nodes. However in this case the relevant choice probabilities for the component 3-link network cannot be written in terms of the cumulative normal distribution function, instead the probit choice probabilities (and their derivatives) for the non-degenerate 3-link network must be calculated. For example:

$$P_1 = \mathbf{P}\left(T_1 \le \min\{T_2, T_3\}\right) \cdot F\left(\frac{t_5 - t_4}{\sqrt{\sigma_4^2 + \sigma_5^2}}\right)$$

The derivatives can be calculated using the method already referred to (Clark and Watling, 2002) for calculating the probit choice probability Jacobian for a non-degenerate network. For the three link network, denote the link choice probabilities (choosing links 1,2 or 3) by PL_i

$$\begin{bmatrix} \frac{\partial P_1}{\partial t_1} \cdots & & \\ \vdots & \ddots & \\ & & \frac{\partial P_6}{\partial t_5} \end{bmatrix} = \begin{bmatrix} \frac{\partial PL_1}{\partial t_1} F_{54} & \cdots & \frac{1}{\sigma_{45}} PL_1 f_{54} \\ \vdots & \ddots & \vdots \\ \frac{\partial PL_3}{\partial t_1} F_{45} & \cdots & \frac{-1}{\sigma_{45}} PL_3 f_{45} \end{bmatrix}$$

Figure-8 Network with Alternative OD Path



Here we have

$$P_1 = \mathbf{P}\left(T_1 + T_3 \le T_2 + T_4, T_1 + T_3 \le T_1 + T_4, T_1 + T_3 \le T_2 + T, T_1 + T_3 \le T_5\right)$$
$$P_1 = \mathbf{P}\left(T_1 \le T_2, T_3 \le T_4, T_1 + T_3 \le T_5\right)$$

This network does not decompose into independent non-degenerate parts because link 5 spans the central node. In this case we cannot simply follow a link based approach and find analytic expressions for the derivatives of the path choice probabilities. However, there is surely no doubt that this network has equilibrium flows that vary smoothly with changes to the link cost function parameters, and is not fundamentally different from those presented above. Nevertheless, for this case we have to revert to less elegant techniques for calculating the path choice probabilities, using either Monte Carlo on the transformed MVN integral, or applying estimation techniques without proper justification to the singular MVN distribution.

CONCLUSION

Patriksson and Rockafellar (2002) provide a full theoretical treatment of DUE sensitivity analysis. While this provides a technically correct methodological approach, practical problems remain due to the fundamentally non-smooth variation of the equilibrium flows under perturbations. When conducting sensitivity analysis of DUE flows, the presence of inaccuracies in the calculations needs to be accounted for.

The SUE flows are unique and vary smoothly under perturbations, so the analytic concerns that have plagued DUE analysis evaporate. However, while the theoretical behaviour of the SUE model is benign, issues arise in practical applications. In particular for the probit SUE model, network topology can give rise to degeneracies in the path covariance matrix that undermine the definition of the probit path choice probabilities. For some cases these analytical obstacles can be shown to disappear, simply by following a link based analysis, but problem networks remain where this approach does not provide an analytical solution. In all cases the method of Mendell-Elston (1974) can be used to estimate the choice integrals and, following Clark & Watling (2002), the path choice probability Jacobian and hence the gradient of the equilibrium flows. However, the application of this method to singular MVN distributions lacks rigorous theoretical foundation.

Further work is required to extend the link-based analytical approach to more general network topologies, in particular to accommodate the final example presented above. In addition, justification is required for the use of the Mendell-Elston method in the case of MVN distributions with singular covariance.

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