Article:
Hazelton, ML and Watling, DP (2004) Computation of Equilibrium Distributions of Markov Traffic-Assignment Models. Transportation Science, 38 (3). pp. 331-342. ISSN: 0041-1655

https://doi.org/10.1287/trsc.1030.0052

Computation of Equilibrium Distributions of Markov Traffic Assignment Models

Martin L. Hazelton
University of Western Australia, Crawley WA 6009, Australia

David P. Watling
University of Leeds, Leeds LS2 9JT, United Kingdom

November 28, 2002

Abstract: Markov traffic assignment models explicitly represent the day-to-day evolving interaction between traffic congestion and drivers' information acquisition and choice processes. Such models can, in principle, be used to investigate traffic flows in stochastic equilibrium, yielding estimates of the equilibrium mean and covariance matrix of link or route traffic flows. However, in general these equilibrium moments cannot be written down in closed form. While Monte Carlo simulations of the assignment process may be used to produce 'empirical' estimates, this approach can be extremely computationally expensive if reliable results (relatively free of Monte Carlo error) are to be obtained. In this paper an alternative method of computing the equilibrium distribution is proposed, applicable to the class of Markov models with linear exponential learning filters. Based on asymptotic results, this equilibrium distribution may be approximated by a Gaussian process, meaning that the problem reduces to determining the first two multivariate moments in equilibrium. The first of these moments, the mean flow vector, may be estimated by a conventional traffic assignment model. The second, the flow covariance matrix, is estimated through various linear approximations, yielding an explicit expression. The proposed approximations are seen to operate well in a number of illustrative examples. The robustness of the approximations (in terms of network input data) is discussed, and shown to be connected with the 'volatility' of the traffic assignment process.

Keywords: Markov process, robustness, route choice, Stochastic User Equilibrium, transportation network.

Introduction

Traffic assignment is the method by which travel demands are translated into traffic volumes over links of a transport network, through modelling the interaction between traffic congestion and drivers' route choice decisions. For many decades, the dominant interpretation of this problem has been one of determining a 'self-consistent' point prediction of traffic volumes, based on an analogy with equilibria of economic markets. The two most popular such approaches are the Wardrop user equilibrium model (Wardrop, 1952), derived from an assumption that drivers have identical perceptions of generalised travel costs, and the stochastic user equilibrium (SUE) model (Daganzo and Sheffi, 1977), whereby heterogeneity in perceived travel costs is incorporated through additive random elements. While much recent research effort has been directed at generalising these methods to incorporate within-day dynamic effects (Ran and Boyce, 1996), two common features of the underlying techniques remain:

(a) in the space of traffic volumes they are deterministic, with a single point state presumed to persist; and

(b) the persistent state reached in (a) is independent of the adjustment processes in behaviour that preceded.

These two assumptions mean that the resulting models are particularly convenient to apply in practice, yet it is not difficult to challenge their plausibility. With regard to (a), from observations of real traffic networks, there is evidence of significant day-to-day variation in travel demands, traffic volumes and travel times (May and Montgomery, 1987; Hanson and Huff, 1988; Mohammadi, 1997). For many intents and purposes in traffic planning and management this variability must be taken into account. For instance, a proposed alteration to a traffic system, which is predicted to provide congestion relief when link volumes are at their mean level, is of little use if daily variation renders it grid-locked on one day in twenty. Such observations have motivated the emerging research area of network reliability (Bell and Cassir, 2000). Furthermore, if link cost functions are convex then (by Jensen's inequality) evaluation of these costs at a fixed 'average' flow (such as SUE) will underestimate the long-run expected cost. This has been noted previously by a number of authors, including Cascetta (1989) and Watling (2002). Assumption (b), on the other hand, is especially problematic in the context of driver information systems, which impact on drivers' learning and information acquisition processes, namely the aspects that these models specifically neglect (see, for example, Mahmassani and

Jayakrishnan, 1991; Emmerink and Nijkamp, 1999).

A number of works may be found in the literature to address these shortcomings. In particular, regarding (a) a number of authors have proposed extensions to the basic SUE model so as to allow a covariance matrix for the link traffic volumes to be estimated. Watling (2002) investigated a generalisation of SUE in which both the mean vector and covariance matrix of link volumes can be simultaneously calculated as the solution to a fixed point problem, both flow means and variances impacting on mean travel costs. Hazelton (1998) took a rather different approach, suggesting a way of defining an equilibrium probability distribution on the space of all possible patterns of link traffic volume, based on defining each traveller's route choice probability distribution conditionally on the routes selected by all other travellers. Traffic volume means and variances can then be computed by Gibbs sampling, as described in Hazelton *et al.* (1996).

A more direct way of estimating day-to-day variation, and one that simultaneously addresses both (a) and (b) above, is to explicitly model traffic assignment as a stochastic process in time. An important class of such models was introduced by Cascetta (1989), who proposed representing the evolution of a transport system as a Markov chain. In this type of model travellers independently make route choices every 'day' based upon past experience of travel costs (that is, experiences on previous days). In addition to providing a representation of the traffic system dynamics, a Markov chain assignment model satisfying certain rather general conditions defines a unique equilibrium probability distribution on the link traffic volumes. A Markov chain assignment will converge (in distribution) to its equilibrium distribution under mild conditions, so the properties of this probability distribution, such as its mean vector and covariance matrix, are of considerable interest. Unfortunately these properties are typically only definable in implicit form, as the moments of a fixed point probability distribution, and so have to be estimated in some fashion. A seemingly natural approach is to simulate the evolution of the assignment process, since the Markov structure at least renders this straightforward in principle. However, while means and variances of link volumes can be approximated to an arbitrary degree of accuracy by taking a sufficiently long simulation of the traffic system, in practice such simulation is highly time-consuming even using fast modern computers. In particular, when defining the random element of the route travel costs as normal random variables at the link level (i.e. a probit model), a shortest path algorithm must be employed for every traveller at every time point.

However, simulation-based methods are by no means the only way of approximating equilibrium properties of Markov chain assignments. One alternative approach is to investigate theoretically the behaviour in the limiting case as the size of the travelling population becomes large. Following this direction Davis and Nihan (1993) considered the mean of a Markov chain assignment of the Cascetta (1989) type. They elegantly showed that this mean converges to the SUE assignment for the traffic system under consideration, as the travel demands tends to infinity. Furthermore they demonstrated that a Markov chain assignment can be increasingly well approximated by a Gaussian multivariate autoregressive process as this demand becomes large. By considering the equilibrium distribution of this autoregressive process, Davis and Nihan were able to implicitly define the covariance matrix in equilibrium, as the fixed point of a recursive equation.

The work of Davis and Nihan (1993) was of major importance for researchers working on Markov traffic assignment models since it provides the theory for computation of approximate properties (in particular the first two multivariate moments) of the associated equilibrium distributions. Their Gaussian approximation result means that knowledge of these two moments is sufficient to characterise the whole equilibrium distribution. Nevertheless, this research has arguably not received the attention that it deserves. In particular, while several authors have recognized that SUE can provide a reasonable approximation to the mean equilibrium flow pattern (see Cantarella and Cascetta, 1995, for example), we are not aware of any published work addressing the computation of the equilibrium covariance matrix for Markov assignment models. A possible explanation for this is that calculation of the fixed point of Davis and Nihan's recursive equation is highly computationally time consuming for networks of realistic size. In this paper we devise an approximation method for estimating the equilibrium covariance matrix for a particular class of Markov assignment models, namely those based on linear learning filters with exponentially decreasing weights. The approximation can be computed in a fraction of the time required to solve Davis and Nihan's fixed point problem. Furthermore, the relative simplicity of the proposed approach allows a clear analysis of the quality of the approximations under various situations (e.g. under changes to the network data).

The present paper is structured as follows. In the next section we describe the class of Markov assignment models with exponential learning filters, and introduce necessary notation. We also define the equilibrium distribution of this type of model. Section 2 is divided into three parts. In the first we derive an approximate equilibrium flow covariance matrix for networks with a single origin-destination movement. In the second part we consider the robustness of our approximation, and draw connections with issues of traffic system stability. A small illustrative example is given in the third part, demonstrating how our methodology can be implemented under a probit type of Markov assignment model. In section 3 we generalize to networks with multiple origin-destination pairs. Some issues arising from our work and avenues for further research are discussed in the last section.

1 Markov Traffic Assignment Models

Consider a transport network with A directed links and W inter-zonal movements (origin-destination pairs). Let N_k be the number of acyclic routes corresponding to inter-zonal movement k ($k = 1, 2, \ldots, W$) and define $N = \sum_{k=1}^{W} N_k$. Let $\mathbf{\Gamma}$ be the $N \times W$ path-movement incidence matrix of 0/1 elements, with $\Gamma_{rk} = 1$ only if route r relates to inter-zonal movement k, and let $\mathbf{\Delta}$ be the $A \times N$ link-path incidence matrix of 0/1 elements, with $\Delta_{ar} = 1$ only if link a is part of route r. Let $\boldsymbol{q}$ denote the column W-vector of inter-zonal demands (over the time interval of interest). In what follows we shall have cause to look at limiting cases where network demands become large, so it is convenient to decompose the demands by $\boldsymbol{q} = \zeta \boldsymbol{q}_0$ for scalar ζ and fixed vector $\boldsymbol{q}_0$. While the elements of $\mathbf{q}$ are assumed constant over time in our models, this does not imply that our work only applies to transport systems with fixed demand from day-to-day. This is because we can introduce a single dummy route denoting the option of not travelling for each inter-zonal movement. Let $\boldsymbol{f}$ and $\boldsymbol{v}$ be the route and link flow vectors. In order to keep notation simple in cases of limiting demand (as $\zeta \to \infty$), we define the corresponding travel costs in terms of proportions of travellers using a route (or selecting a route). Specifically, if we denote by $t_{\zeta a}(\boldsymbol{v})$ the cost of travelling along link a at a given link flow vector $\boldsymbol{v}$ and demand multiplier ζ, and let $\boldsymbol{t}_\zeta(\boldsymbol{v})$ be the corresponding A-vector of link costs, then

$$\boldsymbol{t}_\zeta(\boldsymbol{v}) = \boldsymbol{t}(\zeta^{-1}\boldsymbol{v}) \tag{1}$$

where $\boldsymbol{t}(\cdot)$ is a function independent of ζ. Define $\boldsymbol{c}_\zeta(\boldsymbol{f}) = \mathbf{\Delta}^T \boldsymbol{t}_\zeta(\mathbf{\Delta} f)$ to be the N-vector of implied route cost versus route flow performance functions (where $\mathbf{\Delta}^T$ denotes the transpose of $\mathbf{\Delta}$), and let $\boldsymbol{c}(\boldsymbol{f}/\zeta) = \mathbf{\Delta}^T \boldsymbol{t}(\mathbf{\Delta} f/\zeta)$ be the corresponding ζ independent route costs functions.

Markov traffic assignment models attempt to represent the state of the transport system during a given observation period (e.g. 9.00am-11.00am) on a sequence of days. Let $\boldsymbol{F}^{(n)}$ be the vector of route traffic flows in the period on day n, and let $\boldsymbol{V}^{(n)}$ be the corresponding vector of link flows. Now, the traffic flows observed on day n are determined by the route choices made by all travellers on that day. In Markov assignment modelling each traveller's choice on any given day is a random variable whose distribution is defined conditionally on the link costs (and possibly the link traffic flows) during the past m days, where m is a finite constant. The class of Markovian models is very rich, allowing traveller behaviour and learning to be represented in a wide variety of ways. Nevertheless, this breadth allied to the natural complexity of these types of model makes mathematical analysis extremely problematic. We therefore concentrate on a sub-class of models in which traveller learning is based upon linear filters of past costs with exponentially decreasing weights (see Ben Akiva *at al.*, 1991; Iida *et al.*, 1992; and Emmerink *et al.*, 1995). To be more specific, we define our route choice model as follows. A traveller i making inter-zonal movement k on day n is assumed to assign a disutility $\breve{U}_{ir}^{(n)}$ to each feasible route r. This disutility can be decomposed as

$$\breve{U}_{ir}^{(n)} = U_r^{(n-1)} + \eta_{ir}^{(n)} \qquad r \sim k \tag{2}$$

where $U_r^{(n-1)}$ is a (route specific) measured disutility, $\eta_{ir}^{(n)}$ is a person specific random variable (independent of $\eta_{i'r}^{(n)}$ for all $i' \neq i$) and $r \sim k$ if and only if $\Gamma_{rk} = 1$ (i.e. only if route r is feasible for inter-zonal movement k). We assume that the distribution of $\eta_{ir}^{(n)}$ is not a function of n and i. Nevertheless, we recognize that there are circumstances in which it is useful to divide the travelling population into a number of classes (e.g. travellers with and without in-vehicle guidance systems), and in such cases the distribution $\eta_{ir}^{(n)}$ may be assumed to be dependent on the class membership of individual i. The N-vector of measured disutilities for day n (based upon the states of the transport system up to and including day $n-1$) is given by

$$\boldsymbol{U}^{(n-1)} = s(\lambda)^{-1} \sum_{j=1}^{m} \lambda^{j-1} \boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-j)}) \tag{3}$$

where $s(\lambda) = \sum_{j=1}^{m} \lambda^{j-1} = (1-\lambda^m)/(1-\lambda)$ and it is assumed that $0 < \lambda < 1$. Note that this implies that the 'learning weights' $\lambda^{j-1} s(\lambda)^{-1}$ are positive, decreasing and sum to unity.

On any given day each traveller selects the route with smallest personal disutility. This route choice mechanism implies a probability distribution on the space of feasible routes. For inter-zonal movement k let $p_r(\boldsymbol{U}^{(n-1)})$ be the probability of a traveller selecting route $r \sim k$ conditional on the N-vector of measured disutilities. It is well known that if the $\eta_{ir}^{(n)}$ follow independent Gumbel distributions then the elements of the N-vector of probabilities $\boldsymbol{p}(\boldsymbol{U}^{(n-1)})$ are given according to the logit route choice model (see Sheffi, 1985, for example). A common alternative is to define the error terms at a link level, when $\eta_{ir}^{(n)}$ can be decomposed into a sum of link based random variables (implying a natural correlation between the random terms for routes sharing common links). If these link based random variables are normally distributed (i.e. a probit model) then $\boldsymbol{p}(\boldsymbol{U}^{(n-1)})$ has no closed form, but can be evaluated by Monte Carlo methods, for example. Whatever the distribution of $\eta_{ir}^{(n)}$, the number of travellers taking each possible route for inter-zonal movement k on day n is distributed conditionally as

$$\boldsymbol{F}_{[k]}^{(n)}|\boldsymbol{U}^{(n-1)} \sim \text{Multinomial}\left(q_k, \boldsymbol{p}(\boldsymbol{U}^{(n-1)})\right) \qquad k = 1, \ldots, W \tag{4}$$

where $\boldsymbol{F}_{[k]}$ is the vector containing flows on routes servicing inter-zonal movement k. The process $\{\boldsymbol{F}^{(n)} : n = 1, 2, \ldots\}$ is an m-dependent Markov chain (Cascetta, 1989). Equivalently, the concatenated process $\{\bar{\boldsymbol{F}}^{(n)} = (\boldsymbol{F}^{(n)}, \ldots, \boldsymbol{F}^{(n-m+1)}) : n = 1, 2, \ldots\}$ is a Markov chain, so we may apply standard Markov theory to the assignment model. An important consequence is that under the condition that all routes have non-zero probability of being selected on any day (a condition which holds if $\eta_{ir}^{(n)}$ has infinite support) then the probability distribution of $\bar{\boldsymbol{F}}^{(n)}$ will converge to a unique equilibrium (a fact noted by Cascetta). This equilibrium is also often termed the 'stationary' distribution (since $\bar{\boldsymbol{F}}^{(n+1)}$ has this distribution if $\bar{\boldsymbol{F}}^{(n)}$ does). This paper will concentrate on the marginal equilibrium distribution of $\boldsymbol{F}^{(n)}$, and we will write $\boldsymbol{F}$ for a random vector with this equilibrium distribution.

An important step forward in understanding the properties of the distribution of $\boldsymbol{F}$ was provided by Davis and Nihan (1993). These authors considered the limiting case of this equilibrium distribution as the network demand and capacity become large. Within our notational framework this corresponds to the limit as $\zeta \to \infty$, since the dependence of $\boldsymbol{t}_\zeta$ and $\boldsymbol{c}_\zeta$ on ζ means that the network capacity can be interpreted as increasing in tandem with the demand.

Theorem 1: *Let $\boldsymbol{f}^*$ be Daganzo and Sheffi's (1977) Stochastic User Equilibrium (SUE) link flow vector defined by*

$$\boldsymbol{f}^* = \zeta \text{diag}(\boldsymbol{\Gamma} \boldsymbol{q}_0)\ \boldsymbol{p}(\boldsymbol{c}_\zeta(\boldsymbol{f}^*)). \tag{5}$$

If the solution to Eqn. 5 is unique then

$$\zeta^{-1/2}(\boldsymbol{F} - \boldsymbol{f}^*) \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{\Sigma}_\infty) \tag{6}$$

as $\zeta \to \infty$, where $\xrightarrow{d}$ indicates convergence in distribution and $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Proof: This result is a straightforward corollary of the work of Davis and Nihan (1993). Proposition 3 from their paper demonstrates that the limiting equilibrium distribution of $\boldsymbol{U}$ is multivariate normal. Furthermore, the conditional distribution of $\boldsymbol{F}^{(n+1)}$ given $\boldsymbol{U}^{(n)}$ is also multivariate normal when the process is stationary. Hence the equilibrium distribution of $\boldsymbol{F}^{(n)}$ must also be multivariate normal by standard properties of the multivariate normal distribution, completing the proof.

In order to invoke Theorem 1 we shall assume henceforth that a unique SUE exists. This is guaranteed if the vector mapping $\boldsymbol{t}(\boldsymbol{v})$ between link flows and link travel costs is continuous and strictly monotone, and the choice probabilities $\boldsymbol{p}(.)$ follow a regular random utility model (see Cantarella and Cascetta, 1995). Given a unique $\boldsymbol{f}^*$, Theorem 1 indicates that a natural approximation to the distribution of $\boldsymbol{F}$ is provided by

$$\boldsymbol{F} \dot{\sim} N(\boldsymbol{f}^*, \widehat{\boldsymbol{\Sigma}}) \tag{7}$$

where $\dot{\sim}$ denotes 'approximately distributed' and $\widehat{\boldsymbol{\Sigma}}$ is an estimate of $\text{var}(\boldsymbol{F})$. Use of the approximation in Eqn. 7 requires calculation of estimates of the first two moments of $\boldsymbol{F}$. The mean vector is approximated by $\boldsymbol{f}^*$, and a number of algorithms for computing this quantity exist (such as the method of successive averages; see Sheffi, 1985, for example). Development of an estimate of the equilibrium covariance matrix is more problematic. Davis and Nihan (1993) define the link flow covariance matrix corresponding to $\boldsymbol{\Sigma}_\infty$ implicitly as the fixed point in a recursive equation. However, computation of this link covariance matrix requires route enumeration, and (more importantly) the solution of a system of linear equations with $m^2n(n+1)/2$ unknowns. For even a modestly sized network with $n = 100$ links and a memory length of $m = 10$, finding this link covariance matrix will be computationally taxing. For significantly larger networks and/or memory lengths this computation will infeasible. In the next two sections we derive an estimates $\widehat{\boldsymbol{\Sigma}}$ which are far less computationally taxing. In order to quantify the approximation we will continue to consider the limiting case as $\zeta \to \infty$. Assuming that $\boldsymbol{c}$ is continuously differentiable,

$$\mathbf{c}_\zeta(\boldsymbol{F}) = \mathbf{c}_\zeta(\boldsymbol{f}^*) + \boldsymbol{B}(\boldsymbol{F} - \boldsymbol{f}^*)\zeta^{-1} + O_p(\zeta^{-1/2}) \tag{8}$$

where $\boldsymbol{B}$ denotes the Jacobian matrix of $\boldsymbol{c}$ evaluated at $\boldsymbol{f}^*\zeta^{-1}$; and the order notation is defined such that a random variable $G_\zeta = O_p(\zeta^\alpha)$ if these exists ω for which $\lim_{\zeta\to\infty} P(|G_\zeta/\zeta^\alpha| > \omega) = 0$. Likewise, assuming that $\boldsymbol{p}$ is continuously differentiable

$$\boldsymbol{p}(\boldsymbol{U}) = \boldsymbol{p}(\boldsymbol{u}^*) + \boldsymbol{D}(\boldsymbol{U} - \boldsymbol{u}^*) + O_p(\zeta^{-1/2}) \tag{9}$$

where $\boldsymbol{D}$ denotes the Jacobian matrix of $\boldsymbol{p}(\boldsymbol{u})$ evaluated at $\boldsymbol{u} = \boldsymbol{u}^* = \boldsymbol{c}_\zeta(\boldsymbol{f}^*)$. Both results (8) and (9) are simple corollaries of Theorem 1 (using the kind of techniques presented by van der Vaart, 1998, for example).

2 Networks with a Single Inter-Zonal Movement

2.1 Derivation of an Approximate Covariance Matrix

In this section we look at the special case of a network with a single inter-zonal movement, with demand denoted by a scalar, ζ (so that $q_0 = 1$). We consider the m-dependent Markov assignment process $\{\boldsymbol{F}^{(n)}; n = 1, 2, \ldots\}$ defined by Eqns. (3) and (4) (for some given positive integer m). We assume that the distribution of the η_{ir}'s (from Eqn. (2)) has infinite (or at least semi-infinite) support, so that the process is ergodic and hence has a unique equilibrium distribution (see Cascetta, 1989, and the references therein). We write $\boldsymbol{\mu}$ for the equilibrium mean vector and $\boldsymbol{\Sigma}$ for the equilibrium covariance matrix of the route flows $\boldsymbol{F}$. A simple corollary of Theorem 1 (and of Davis and Nihan, 1993) shows that $\boldsymbol{f}^*$ is the natural approximation for $\boldsymbol{\mu}$, and quantifies the relative approximation error.

Corollary 1: *Under the assumptions of Theorem 1,*

$$\zeta^{-1}\boldsymbol{\mu} = \zeta^{-1}\boldsymbol{f}^* + O(\zeta^{-1/2}) \tag{10}$$

where $g_\zeta = O(\zeta^\alpha)$ *if* $\lim_{\zeta\to\infty} g_\zeta/\zeta^\alpha = \omega$ *for some finite constant* ω.

We now go on to derive a novel estimate of $\boldsymbol{\Sigma}$. The overall plan in this derivation is first to approximate $\boldsymbol{\Sigma}$ in terms of $\mathrm{var}(\boldsymbol{U})$ (the covariance of the disutility), which we do in lemma 1. We then derive an implicit approximation for $\mathrm{var}(\boldsymbol{U})$ in lemma 2. Combining these lemmas allows us to find an approximation for $\boldsymbol{\Sigma}$ as an infinite sum, as we present in Theorem 2. Finally, we consider a simpler approximation based on a truncated form of this sum.

Lemma 1: *For a network with a single inter-zonal movement, and under the assumptions of Theorem 1,*

$$\zeta^{-1}\boldsymbol{\Sigma} = \zeta^{-1}\boldsymbol{\Theta}^* + \zeta\mathrm{var}(D\boldsymbol{U})\left(1 + O(\zeta^{-1/2})\right) + O(\zeta^{-1/2}) \tag{11}$$

where

$$\boldsymbol{\Theta}^* = \mathrm{diag}(\boldsymbol{f}^*) - \zeta^{-1}\boldsymbol{f}^*(\boldsymbol{f}^*)^T$$

Proof: Following on from Eqn. 4, standard results for the Multinomial distribution give

$$\mathrm{E}[\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}] = \zeta\boldsymbol{p}^{(n-1)} \tag{12}$$

and

$$\mathrm{var}(\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}) \equiv \boldsymbol{\Theta}(\boldsymbol{p}^{(n-1)})) = \zeta\left(\mathrm{diag}(\boldsymbol{p}^{(n-1)}) - \boldsymbol{p}^{(n-1)}(\boldsymbol{p}^{(n-1)})^T\right) \tag{13}$$

where $\boldsymbol{p}^{(n-1)} = \boldsymbol{p}(\boldsymbol{U}^{(n-1)})$. Note that the conditional covariance matrix is not equal to $\boldsymbol{\Sigma}$ because the Multinomial distribution is defined in terms of an invariant set of probabilities, whilst the probability vector $\boldsymbol{p}(\boldsymbol{U}^{(n-1)})$ determining route flows on day n varies according to the state of the transport system over the previous m days. Now, we may decompose the covariance matrix of $\boldsymbol{F}^{(n)}$ by

$$\begin{aligned} \text{var}\left(\boldsymbol{F}^{(n)}\right) &= \text{E}\left[\text{var}\left(\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}\right)\right] + \text{var}\left(\text{E}\left[\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}\right]\right) \\ &= \text{E}\left[\boldsymbol{\Theta}(\boldsymbol{p}^{(n-1)}))\right] + \text{var}\left(\text{E}\left[\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}\right]\right) \end{aligned} \quad (14)$$

Consider the first term on the right hand side. Employing Eqn. 9 we obtain

$$\begin{aligned} \text{E}\left[\boldsymbol{\Theta}(\boldsymbol{p}^{(n-1)}))\right] &= \text{E}\left[\text{diag}\left\{\zeta\left(\boldsymbol{p}(\boldsymbol{u}^*) + \boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)\right\}\right] - \text{E}\left[\zeta\left(\boldsymbol{p}(\boldsymbol{u}^*) + \boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)\left(\boldsymbol{p}(\boldsymbol{u}^*) + \boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)^T\right] \\ &\quad + O(\zeta^{1/2}) \\ &= \text{E}\left[\text{diag}\left(\boldsymbol{f}^* + \zeta\boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)\right] - \zeta\text{E}\left[\left(\boldsymbol{f}^*\zeta^{-1} + \boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)\left(\boldsymbol{f}^*\zeta^{-1} + \boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)^T\right] \\ &\quad + O(\zeta^{1/2}) \end{aligned}$$

where $\breve{\boldsymbol{U}}^{(n)} = \boldsymbol{U}^{(n)} - \boldsymbol{u}^*$. In equilibrium this simplifies to give

$$E\left[\boldsymbol{\Theta}(\boldsymbol{p}))\right] = \text{diag}(\boldsymbol{f}^*) - \zeta^{-1}\boldsymbol{f}^*(\boldsymbol{f}^*)^T + O(\zeta^{1/2}) = \boldsymbol{\Theta}^* + O(\zeta^{1/2}). \quad (15)$$

Note that $\boldsymbol{\Theta}^* = \boldsymbol{\Theta}(\zeta^{-1}\boldsymbol{f}^*)$ is the conditional covariance matrix evaluated at the SUE probability vector $\boldsymbol{p}(\boldsymbol{u}^*) = \zeta^{-1}\boldsymbol{f}^*$. The second term on the right hand side in Eqn. (14) can be approximated using the result

$$\begin{aligned} \text{var}\left(\text{E}\left[\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}\right]\right) &= \text{var}\left(q\left(\boldsymbol{p}(\boldsymbol{u}^*) + \boldsymbol{D}\breve{\boldsymbol{U}}^{(n-1)}\right)\right)\left(1 + O(\zeta^{-1/2})\right) \\ &= \zeta^2\boldsymbol{D}\text{var}\left(\boldsymbol{U}^{(n-1)}\right)\boldsymbol{D}^T\left(1 + O(\zeta^{-1/2})\right) \end{aligned} \quad (16)$$

since $\text{var}(\boldsymbol{U}) = \text{var}(\breve{\boldsymbol{U}})$. Combining Eqns. (15) and (16) when the process is equilibrium and dividing through by ζ completes the proof of lemma 1.

Lemma 2: *For a network with a single inter-zonal movement, and under the assumptions of Theorem 1,*

$$\text{var}(\boldsymbol{U}) = (\zeta s(\lambda))^{-2}\boldsymbol{B}\boldsymbol{\Theta}^*\boldsymbol{B}^T + \boldsymbol{M}\text{var}(\boldsymbol{U})\boldsymbol{M}^T + O(\zeta^{-3/2} + \zeta^{-1}\lambda^{m-1}) \quad (17)$$

where $\boldsymbol{M} = s(\lambda)^{-1}\boldsymbol{B}\boldsymbol{D} + \lambda\boldsymbol{I}$ *and* $\boldsymbol{I}$ *is the* $N \times N$ *identity matrix.*

Proof: It is convenient to amend our notation temporarily from $\boldsymbol{U}^{(n-1)}$ and $s(\lambda)$ to (respectively) $\boldsymbol{U}_m^{(n-1)}$ and $s(m, \lambda)$, to reflect the dependence of both on the fixed input parameter m.

We now derive two expressions for $\boldsymbol{U}_m^{(n-1)}$ which will be needed later. First, b based on the exponential learning filter for Eqn. (3), we have

$$\begin{aligned}\boldsymbol{U}_m^{(n-1)} &= \frac{1}{s(m,\lambda)}\sum_{i=1}^{m}\lambda^{i-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-i)}) \\ &= \frac{1}{s(m,\lambda)}\left(\sum_{i=1}^{m-1}\lambda^{i-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-i)})+\lambda^{m-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-m)})\right) \\ &= \frac{1}{s(m,\lambda)}\left(s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-1)}+\lambda^{m-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-m)})\right)\end{aligned}$$

which can be approximated via the result

$$\begin{aligned}\boldsymbol{U}_m^{(n-1)} &= \frac{1-\lambda^{m-1}}{1-\lambda^m}\boldsymbol{U}_{m-1}^{(n-1)}+\frac{\lambda^{m-1}(1-\lambda^{m-1})}{1-\lambda^m}\boldsymbol{c}_\zeta(\mathbf{F}^{(n-m)}) \\ &= \boldsymbol{U}_{m-1}^{(n-1)}+O(\lambda^{m-1}). \end{aligned} \tag{18}$$

Intuitively speaking, expression (18) expresses the fact that, provided the learning parameter λ is not too large and the number of remembered experiences m is not too small, then little information is lost if, in forming drivers' mean predicted route costs, the oldest experience is neglected, since in any case it will be given a relatively low weight in comparison with other remembered costs. Our second expression for $\boldsymbol{U}_m^{(n-1)}$ is provided by the disaggregation:

$$\begin{aligned}\boldsymbol{U}_m^{(n-1)} &= \frac{1}{s(m,\lambda)}\sum_{i=1}^{m}\lambda^{i-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-i)}) \\ &= \frac{1}{s(m,\lambda)}\left(\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-1)})+\sum_{i=2}^{m}\lambda^{i-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-i)})\right) \\ &= \frac{1}{s(m,\lambda)}\left(\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-1)})+\sum_{j=1}^{m-1}\lambda^{j}\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-j-1)})\right) \\ &= \frac{1}{s(m,\lambda)}\left(\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-1)})+\lambda\sum_{j=1}^{m-1}\lambda^{j-1}\boldsymbol{c}_\zeta(\boldsymbol{F}^{((n-1)-j)})\right) \\ &= \frac{1}{s(m,\lambda)}\left(\boldsymbol{c}_\zeta(\boldsymbol{F}^{(n-1)})+\lambda s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-2)}\right)\ . \end{aligned} \tag{19}$$

Now, a fundamental decomposition of the variance of $U_m^{(n-1)}$ is

$$\begin{aligned}\operatorname{var}\left(\boldsymbol{U}_m^{(n-1)}\right) &= \mathrm{E}\left[\operatorname{var}\left(\boldsymbol{U}_m^{(n-1)}\middle|\boldsymbol{U}_{m-1}^{(n-2)}\right)\right] \\ &\qquad + \operatorname{var}\left(\mathrm{E}\left[\boldsymbol{U}_m^{(n-1)}\middle|\boldsymbol{U}_{m-1}^{(n-2)}\right]\right)\ . \end{aligned} \tag{20}$$

Consider the first term of Eqn. (20). Based on Eqn. (19), conditionally on $\boldsymbol{U}_{m-1}^{(n-2)}$ (the weighted mean cost over the $m-1$ days $n-2, n-3, ..., n-m$), the only random variation in $\boldsymbol{U}_m^{(n-1)}$ (the weighted mean cost over the m days $n-1, n-2, ..., n-m$) is due to $\mathbf{c}(\boldsymbol{F}^{(n-1)})$. Hence,

$$\operatorname{var}\left(\boldsymbol{U}_m^{(n-1)}\middle|\boldsymbol{U}_{m-1}^{(n-2)}\right)=\frac{1}{(s(m,\lambda))^2}\operatorname{var}\left(\boldsymbol{c}_\zeta\left(\boldsymbol{F}^{(n-1)}\right)\middle|\boldsymbol{U}_{m-1}^{(n-2)}\right)$$

and using Eqn. (8), this becomes:

$$\begin{aligned}\mathrm{var}\left(\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right) &= (s(m,\lambda)\zeta)^{-2}\boldsymbol{B}\ \mathrm{var}\left(\boldsymbol{F}^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right)\boldsymbol{B}^T\left(1+O(\zeta^{-1/2})\right)\\ &= (s(m,\lambda)\zeta)^{-2}\boldsymbol{B}\ \mathrm{var}\left(\boldsymbol{F}^{(n-1)}\left|\boldsymbol{U}_{m}^{(n-2)}\right.\right)\boldsymbol{B}^T\left(1+O(\zeta^{-1/2})+O(\lambda^{m-1})\right)\\ &= (s(m,\lambda)\zeta)^{-2}\boldsymbol{B\Theta}\left(\boldsymbol{p}(\boldsymbol{U}_m^{(n-2)})\right)\boldsymbol{B}^T\left(1+O(\zeta^{-1/2})+O(\lambda^{m-1})\right). \quad (21)\end{aligned}$$

The first term on the right hand side in Eqn. (20) is obtained by taking the expectation of Eqn. (21). In the equilibrium case we obtain, using the result of Eqn. (15),

$$\mathrm{E}\left[\mathrm{var}\left(\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_m^{(n-2)}\right.\right)\right] = (s(m,\lambda)\zeta)^{-2}\boldsymbol{B\Theta}^*\boldsymbol{B}^T\left(1+O(\zeta^{-1/2}+\lambda^{m-1})\right). \quad (22)$$

Moving on to the second term of Eqn. (20), note that

$$\mathrm{E}\left[\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right] = \frac{1}{s(m,\lambda)}\left\{\mathrm{E}\left[\boldsymbol{c}_\zeta\left(\boldsymbol{F}^{(n-1)}\right)\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right]+\lambda s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-2)}\right\}$$

from Eqn. (19). Using Eqn. (8) this becomes:

$$\begin{aligned}\mathrm{E}\left[\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right] &= \frac{1}{s(m,\lambda)}\left\{\boldsymbol{c}_\zeta(\boldsymbol{f}^*)+\zeta^{-1}\boldsymbol{B}\mathrm{E}\left[\breve{\boldsymbol{F}}^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right]+\lambda s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-2)}\right\}\left(1+O(\zeta^{-1/2})\right)\\ &= \frac{1}{s(m,\lambda)}\left\{\boldsymbol{u}^*+\boldsymbol{Bp}(\boldsymbol{U}_{m-1}^{(n-2)})+\lambda s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-2)}\right\}\left(1+O(\zeta^{-1/2})\right)\\ &= \frac{1}{s(m,\lambda)}\left\{\boldsymbol{u}^*+\boldsymbol{B}\left(\boldsymbol{p}(\boldsymbol{u}^*)+\boldsymbol{D}\breve{\boldsymbol{U}}_{m-1}^{(n-2)})\right)+\lambda s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-2)}\right\}\left(1+O(\zeta^{-1/2})\right).\end{aligned}$$

Using the result

$$s(m-1,\lambda)\boldsymbol{U}_{m-1}^{(n-2)} = s(m,\lambda)\boldsymbol{U}_m^{(n-2)}+O(\lambda^{m-1})$$

it follows that, under stationarity,

$$\mathrm{E}\left[\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right] = \frac{1}{s(m,\lambda)}\left[\boldsymbol{u}^*+\boldsymbol{B}\left(\boldsymbol{f}^*\zeta^{-1}-\boldsymbol{Du}^*\right)\right]+\boldsymbol{MU}_{m-1}^{(n-2)}\left(1+O(\zeta^{-1/2}+\lambda^{m-1})\right).$$

We can now approximate the second term in Eqn. (20) via

$$\mathrm{var}\left(\mathrm{E}\left[\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right]\right) = \boldsymbol{M}\mathrm{var}(\boldsymbol{U}_m^{(n-1)})\boldsymbol{M}^T\left(1+O(\zeta^{-1/2}+\lambda^{m-1})\right).$$

From Theorem 1 it can be deduced that $\mathrm{var}(\boldsymbol{U}) = O(\zeta^{-1})$ and so

$$\mathrm{var}\left(\mathrm{E}\left[\boldsymbol{U}_m^{(n-1)}\left|\boldsymbol{U}_{m-1}^{(n-2)}\right.\right]\right) = \boldsymbol{M}\mathrm{var}(\boldsymbol{U}_m^{(n-1)})\boldsymbol{M}^T+O(\zeta^{-3/2}+\zeta^{-1}\lambda^{m-1}).$$

Combining this equation with Eqn. (22) completes the proof of lemma 2.

Repeated substitution of Eqn. (17) in (11) proves the following theorem.

Theorem 2: *For a network with a single inter-zonal movement, and under the assumptions of Theorem 1,*

$$\zeta^{-1}\boldsymbol{\Sigma} = \zeta^{-1}\boldsymbol{\Theta}^*+s(\lambda)^{-2}\zeta^{-1}\boldsymbol{D}\left\{\sum_{i=0}^{\infty}\boldsymbol{M}^i(\boldsymbol{B\Theta}^*\boldsymbol{B}^T)(\boldsymbol{M}^i)^T\right\}\boldsymbol{D}^T+O(\zeta^{-1/2}+\lambda^{m-1}). \quad (23)$$

Note that Lemma 1 and Theorem 2 indicate that we can get accurate approximations to the equilibrium mean and covariance matrix relative to the demand. It is clearly unreasonable to expect to be able to obtain approximations with small absolute error since $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ both increase in proportion to ζ.

2.2 Practical Approximation

The result of Theorem 2 does not provide a practical method for estimating $\boldsymbol{\Sigma}$ since it involves an infinite sum. Nonetheless, we can develop a practicable approximation by truncating the sum in Eqn. (17). More specifically, we propose the estimate

$$\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Theta}^* + s(\lambda)^{-2}\left\{\boldsymbol{DB}\boldsymbol{\Theta}^*(\boldsymbol{DB})^T + \boldsymbol{DMB}\boldsymbol{\Theta}^*(\boldsymbol{DMB})^T\right\} \tag{24}$$

which makes use of only the first two terms in Eqn. (17). In order to assess the quality of our approximation $\hat{\boldsymbol{\Sigma}}$, note that the terms omitted in moving from Eqn. (17) to (24) are of the form

$$\left(s(\lambda)^{-1}\boldsymbol{DB}\right)^j \boldsymbol{\Theta}^* \left\{\left(s(\lambda)^{-1}\boldsymbol{DB}\right)^k\right\}^T \lambda^{i-j-k}$$

for i, j, k non-negative integers satisfying $i \geq j+k$ and $i \geq 2$. It follows that we should expect the quality of our approximation of $\boldsymbol{\Sigma}$ to depend upon the magnitude of λ and powers of the matrix $qs(\lambda)^{-1}\boldsymbol{DB}$. Now $0 < \lambda < 1$ by assumption, so the high power terms in λ will become increasingly small. Turning to $s(\lambda)^{-1}\boldsymbol{DB}$, high powers of the matrix will contribute little to the infinite sum in (17) if all its eigenvalues have modulus less than one. In particular, we have reason to hope that the approximation in (24) will operate well if the magnitudes of all the eigenvalues of $s(\lambda)^{-1}\boldsymbol{DB}$ are substantially smaller than one. Conversely, we cannot expect our approximation to be useful if any of the eigenvalues of $s(\lambda)^{-1}\boldsymbol{DB}$ have modulus one or greater.

The matrix $s(\lambda)^{-1}\boldsymbol{DB}$ has an interesting connection with the volatility of the traffic assignment process. To see this, define

$$\boldsymbol{\psi}(\boldsymbol{f}^{(n-1)}) = \mathrm{E}[\boldsymbol{F}^{(n)}|\boldsymbol{F}^{(n-1)} = \boldsymbol{f}^{(n-1)}, \boldsymbol{U}_m^{n-2} = \boldsymbol{u}^*].$$

This function quantifies the marginal effect on the day n expected traffic flow of the realized flow pattern $\boldsymbol{f}^{(n-1)}$ on day $n-1$, given that the disutility based on the previous m days has been at its equilibrium expected value. Hazelton (2002) suggests that the assignment process will be non-volatile in an intuitive sense if

$$||\boldsymbol{\psi}(\boldsymbol{f}^{(n-1)}) - \boldsymbol{f}^*|| < ||\boldsymbol{f}^{(n-1)} - \boldsymbol{f}^*||$$

(where $||\cdot||$ is the Euclidean norm) since this implies that we expect any deviation from SUE (the equilibrium mean) on day $n-1$ to be damped down on day n. A reversal of this inequality suggests a highly volatile assignment process in which deviations from SUE at day $n-1$ are expected to be magnified on the following day. Now, it is simple to show that the Jacobian matrix of $\boldsymbol{\psi}(\boldsymbol{f}^{(n-1)})$ evaluated at $\boldsymbol{f}^*$ is $s(\lambda)^{-1}\boldsymbol{DB}$. Hence

$$\boldsymbol{\psi}(\boldsymbol{f}^{(n-1)}) - \boldsymbol{f}^* = s(\lambda)^{-1}\boldsymbol{DB}(\boldsymbol{f}^{(n-1)} - \boldsymbol{f}^*)\left(1 + O(\zeta^{-1/2})\right).$$

It follows that the volatility of the assignment process depends on the matrix $s(\lambda)^{-1}\boldsymbol{DB}$ in quite a direct fashion. This in turn implies that we should expect the quality of our approximation for $\boldsymbol{\Sigma}$ to be dependent on the assignment process volatility. We note that there is a strong analogy here with the classical techniques for examining asymptotic stability of deterministic dynamical assignment processes (cf. Watling, 1999).

Example We illustrate the preceding discussion on the robustness of our truncated series approximation with a simple example involving a simple two zone network with only one inter-zonal movement, in which the zones are connected by two parallel non-overlapping routes. Suppose that the cost functions at demand ζ for each route are quadratic, parameterized as

$$c_{\zeta r}(f_r) = \alpha_i + \left(\frac{f_r}{\beta\zeta}\right)^2 \qquad r = 1, 2$$

so that β is proportional to route capacity (assumed the same for both routes). Suppose further that a logit model is used to define the route choice probabilities; that is,

$$\boldsymbol{p}_r(\boldsymbol{u}) = \frac{e^{-\theta u_r}}{e^{-\theta u_1} + e^{-\theta u_2}}.$$

Then it is simple to show that

$$s(\lambda)^{-1}\boldsymbol{DB} = 2\theta p_1(\boldsymbol{u}^*)\left(1 - p_1(\boldsymbol{u}^*)\right)\beta^{-2}s(\lambda)^{-1}\zeta^{-1}\begin{pmatrix} -f_1^* & f_2^* \\ f_1^* & -f_2^* \end{pmatrix}.$$

The eigenvalues ϕ_1, ϕ_2 of this matrix are given by

$$\phi_1 = 0 \qquad\qquad \phi_2 = -2\theta p_1(\boldsymbol{u}^*)\left(1 - p_1(\boldsymbol{u}^*)\right)\beta^{-2}s(\lambda)^{-1}.$$

Since $p_1(\boldsymbol{u}^*)[1 - p_1(\boldsymbol{u}^*)] \leq 4^{-1}$ it follows that

$$|\phi_2| \leq \frac{1}{2}\theta\beta^{-2}s(\lambda)^{-1}.$$

We can expect our approximation for $\boldsymbol{\Sigma}$ to operate satisfactorily if $|\phi_2| << 1$, and hence if

$$\theta << 2\beta^2 s(\lambda). \tag{25}$$

The approximation is likely to fail badly if this inequality is reversed.

To illustrate this point we carried out three simulation experiments for this network with $q = 40, \alpha_1 = 1, \alpha_2 = 2, \beta = 0.25, \lambda = 0.8$ and $m = 9$. With this network data the right hand side of the inequality in (25) is 0.5411. In the first experiment we set $\theta = 0.01$, well below this critical value. In the second we set $\theta = 0.1$, and in the third we set $\theta = 1$ (exceeding the critical value). Three methods of estimating $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are considered:

Simulation: The assignment process was simulated over a sequence of 40000 days. The first 4000 simulations were discarded as a so called 'burn-in period' (see Gilks *et al.*, 1996) in order to allow the process to reach equilibrium.

Naive: In this method we estimated $\boldsymbol{\mu}$ by $\boldsymbol{f}^*$ and $\boldsymbol{\Sigma}$ by the conditional covariance matrix evaluated at SUE, $\boldsymbol{\Theta}^*$.

Approximation: Here $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ were estimated by $\boldsymbol{f}^*$ and $\widehat{\boldsymbol{\Sigma}}$ from (24) respectively.

For each method the estimated equilibrium mean and variance of the flow on route 1 are displayed in Table 1. (Route 2 results are omitted since they can be computed directly using the fact that $f_2 = q - f_1$.)

TABLE 1 HERE

These results are readily explained. In the case where $\theta = 0.01$, travellers are quite insensitive to changes in measured disutility and hence they take relatively little account of fluctuations in recent flow patterns when making route choices. The SUE flows on both links are almost equal, and the sequence of assigned route flows on route 1 evolves very much like a sequence of independent Binomial(40,0.5) random variables. It is therefore no surprise that the naive approximations to μ_1 and Σ_{11} are very close to the observed values (computed directly from the simulation). Notice that our approximation provides an estimate of variance indistinguishable (to two decimal places) from the naive one. When $\theta = 0.1$ travellers are considerably more sensitive to shifts in disutility. Travellers are more likely to react to recent fluctuations in flows, and hence the naive estimate of variance (which essentially ignores this variability) is significantly smaller than the observed value. Our approximation, on the other hand, is once more very close to the observed value. Finally, in the case where $\theta = 1$, travellers are very sensitive to even small changes in measured disutility. Travellers tend to react *en masse* to variations in recent traffic flows, producing a highly volatile assignment process in which the route 1 flow will sometimes go from 0 to 40 and back again to 0 over a sequence of 3 days.

(Such 'flip-flopping' behaviour in traffic assignment models is well known; see, for example, Horowitz, 1984, and Watling, 1999). In this situation the naive estimate of variance is an order of magnitude too small. Our approximation is also quite poor (as predicted by the preceding theory), markedly overestimating the observed value.

As a postscript to this example it is worth considering the usefulness of an estimate of variance in the $\theta = 1$ case. When the (marginal) equilibrium distribution of flows on route 1 is unimodal and non-skewed (as occurred when $\theta = 0.01$ and $\theta = 0.1$), the mean and variance are excellent summary statistics. For example, statistical calculations requiring only mean and variance (and based on the normal distribution courtesy of Theorem 1) show that the flow on route 1 will fall outside the prediction interval $(\hat{\mu}_{11} \pm 2\hat{\Sigma}_{11}^{1/2})$ only (approximately) 5% of the time. However, for $\theta = 1$ the equilibrium distribution was markedly bimodal. When the equilibrium distribution is multi-modal a multivariate normal approximation is clearly inappropriate. and the mean and variance do not constitute a satisfactory summary of this distribution. For instance, prediction intervals analogous to the one above cannot be computed without further information. To obtain a full description of the equilibrium distribution when $\theta = 1$ would require a lengthy simulation run, but this may well be necessary when attempting to learn about the properties of such a volatile assignment process.

2.3 A Probit Example

The shortcomings of logit route choice models are widely recognized, but these models continue to be used because of their mathematical tractability. In contrast, probit models in which the η_{ir} from Eqn. (2) are composed of sums of link based random variables (see Sheffi, 1985, for example) are more intuitively attractive but are generally far less easy to work with. Nonetheless, we are able to implement our approximation methodology for probit models by using a technique described by Daganzo (1979), and recently applied to traffic assignment by Clark and Watling (2000), for approximating the requisite Jacobian matrix $\boldsymbol{D}$. We illustrate the use of this technique using a five link network studied by Clark and Watling.

The network topology is displayed in Figure 1. We assume that there is a single inter-zonal movement from zone 1 to 4, with a demand of $\zeta = 100$ trips. The link cost functions are given by

$$t_{\zeta a}(v_a) = \begin{cases} 1 + (v_a/\zeta)^2 & a = 1, 3 \\ 2 + v_a/\zeta & a = 2, 4, 5. \end{cases}$$

There are three available routes: route 1 comprising links 1 and 4, route 2 comprising links 2 and 5 and route 3 comprising links 1, 3 and 5. Traveller i has random disutility component η_{ir} for route r given by

$$\eta_{ir} = \sum_{a=1}^{5} \Delta_{ar} \epsilon_{ia}$$

where ϵ_{ia} is a link based normally distributed random variable with variance τ_a^2 (where $\tau_1^2 = \tau_3^2 = \tau_4^2 = 1$ and $\tau_2^2 = \tau_5^2 = 2^{-1}$).

FIGURE 1 ABOUT HERE

SUE was computed using the method of successive averages with 1 inner iteration and 10 million outer iterations (see Sheffi, 1985). This gave

$$\boldsymbol{f}^* = (54.3, 29.1, 16.6)^T \ .$$

The Jacobian matrix $\boldsymbol{D}$ for this problem is given approximately by (Clark and Watling, 2000):

$$\boldsymbol{D} = \begin{pmatrix} -0.276 & 0.150 & 0.126 \\ 0.150 & -0.208 & 0.058 \\ 0.126 & 0.058 & -0.184 \end{pmatrix} .$$

The simulation method was implemented with $40,000$ iterations in order to obtain reliable estimates

$$\boldsymbol{\mu} = \begin{pmatrix} 54.6 \\ 28.0 \\ 17.4 \end{pmatrix}$$

(which is closely approximated by $\boldsymbol{f}^*$) and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 26.4 & -16.8 & -9.6 \\ -16.8 & 21.3 & -4.5 \\ -9.6 & -4.5 & 14.2 \end{pmatrix} .$$

This was a computationally onerous task because of the need to compute shortest paths for every traveller on each day. In practice it will often be impossible to carry out such a substantial simulation run (particularly if the network is large), and so it is of interest to investigate the variability in estimates that will be obtained from using shorter simulation runs. We did this by obtaining results from 20 parallel implementations of the simulation approach, each based on only 2000 (and each having its own random number generation seed). The root mean squared errors over these 20 replications for the variances of the three routes were 4.9, 3.7 and 2.5; i.e. 18.6%, 17.2% and 17.7% respectively.

Our approximation methodology gives

$$\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} 25.7 & -16.5 & -9.2 \\ -16.5 & 21.3 & -4.7 \\ -9.2 & -4.7 & 13.9 \end{pmatrix} .$$

Note that the absolute errors in estimating the variance on the routes are 0.7, 0.0 and 0.3; i.e. 2.7%, 0.0% and 2.1% respectively. This is a clear improvement on the result obtained using a short simulation run, and provides a clear indication of the potential benefits of our approach.

3 Networks with Multiple O-D Movements

In this section we consider approximations of the equilibrium distribution mean and covariance matrix when the traffic network has multiple inter-zonal movements. Since Theorem 1 is valid in this case (as well as in the case of a single inter-zonal movement), so Corollary 1 applies and $\boldsymbol{f}^*$ remains the natural approximation to $\boldsymbol{\mu}$. Turning to $\boldsymbol{\Sigma}$, note that the conditional covariance matrix is given by

$$\text{var}(\boldsymbol{F}^{(n)}|\boldsymbol{U}^{(n-1)}) \equiv \boldsymbol{\Theta}(\boldsymbol{p}^{(n-1)})) = \begin{pmatrix} \boldsymbol{\Theta}_1(\boldsymbol{p}_1^{(n-1)}) & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Theta}_2(\boldsymbol{p}_2^{(n-1)}) & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_W(\boldsymbol{p}_W^{(n-1)}) \end{pmatrix}. \quad (26)$$

where $\boldsymbol{\Theta}_i(\boldsymbol{p}_i^{(n-1)})$ is the multinomial covariance matrix from Eqn. (13) for the i'th inter-zonal movement only, and $\boldsymbol{0}$ represents appropriately sized matrices with all elements zero. We define

$$\boldsymbol{\Theta}^* = \boldsymbol{\Theta}\left(\{\text{diag}(\boldsymbol{\Gamma}\boldsymbol{q}_0)\}^{-1}\zeta^{-1}\boldsymbol{f}^*\right)$$

to be this conditional covariance matrix computed at the SUE probability vector (as before). It is worth noting, in passing, that the flows on two routes servicing different inter-zonal movements are conditionally uncorrelated. This in turn means that if $\boldsymbol{\Theta}^*$ is used as a naive estimate of $\boldsymbol{\Sigma}$ then it would appear as if route flows corresponding to separate inter-zonal movements are unconditionally uncorrelated. However, there are certainly circumstances in which intuition suggests that significantly non-zero correlations between such flows will exist. For instance, if two routes servicing different inter-zonal movements share road links (with these common links perhaps constituting the majority of both routes) then heavy congestion on these links on day $n-1$ will tend to produce relatively low flows on both routes on day n (and hence a positive correlation). Our approximation $\widehat{\boldsymbol{\Sigma}}$ can successfully estimate the unconditional covariances between routes servicing different inter-zonal movements.

Employing the extended notation introduced in Eqn. (26), it is straightforward to show that Theorem 2 generalizes to the multiple inter-zonal movement as follows.

Theorem 3: *Under the assumptions of Theorem 1,*

$$\zeta^{-1}\mathbf{\Sigma} = \zeta^{-1}\mathbf{\Theta}^* + s(\lambda)^{-2}\zeta^{-1}\boldsymbol{Q}_0\boldsymbol{D}\left\{\sum_{i=0}^{\infty}\boldsymbol{M}^i(\boldsymbol{B}\mathbf{\Theta}^*\boldsymbol{B}^T)(\boldsymbol{M}^i)^T\right\}\boldsymbol{D}^T\boldsymbol{Q}_0^T + O(\zeta^{-1/2} + \lambda^{m-1}). \quad (27)$$

where $\boldsymbol{Q}_0 = \mathrm{diag}(\mathbf{\Gamma}\boldsymbol{q}_0)$.

Hence a natural approximation to $\mathbf{\Sigma}$ is provided by

$$\widehat{\mathbf{\Sigma}} = \mathbf{\Theta}^* + s(\lambda)^{-2}\left\{\boldsymbol{Q}_0\boldsymbol{D}\boldsymbol{B}\mathbf{\Theta}^*(\boldsymbol{Q}_0\boldsymbol{D}\boldsymbol{B})^T + \boldsymbol{Q}_0\boldsymbol{D}\boldsymbol{M}\boldsymbol{B}\mathbf{\Theta}^*(\boldsymbol{Q}_0\boldsymbol{D}\boldsymbol{M}\boldsymbol{B})^T\right\}, \quad (28)$$

following the same type of reasoning as in section 2.2.

We illustrate the use of our methodology for multiple inter-zonal movements using the traffic network with topology as displayed in Figure 2. In this example there are two inter-zonal movements, the first from zone 1 to zone 5 and the second from zone 3 to zone 5. In both cases the travel demand is 50. We decompose this demand so that $\zeta = 50$ and $\boldsymbol{q}_0$ is a vector with appropriately places zeros and ones. The path-link incidence matrix is

$$\mathbf{\Delta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the first two routes service the inter-zonal movement 1, and the last two routes service inter-zonal movement 2. Each link has cost function

$$t_{\zeta a}(v_a) = 5 + 2.5(v_a/\zeta)^2 .$$

The disutility was computed with $m = 5$ and $\lambda = 0.5$ and a logit route choice model with logit parameter 0.35 was employed.

FIGURE 2 ABOUT HERE

The mean and covariance matrix computed from a 40000 day simulation run were

$$\boldsymbol{\mu} = \begin{pmatrix} 21.7 \\ 28.3 \\ 28.3 \\ 21.7 \end{pmatrix}$$

and

$$\mathbf{\Sigma} = \begin{pmatrix} 16.5 & -16.5 & -3.0 & 3.0 \\ -16.5 & 16.5 & 3.0 & -3.0 \\ -3.0 & 3.0 & 16.5 & -16.5 \\ 3.0 & -3.0 & -16.5 & 16.5 \end{pmatrix} .$$

The SUE approximation of the mean is

$$
\boldsymbol{f}^* = \begin{pmatrix} 28.3 \\ 21.7 \\ 21.7 \\ 28.3 \end{pmatrix}
$$

while our estimate of the covariance matrix is

$$
\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} 17.1 & -17.1 & -3.3 & 3.3 \\ -17.1 & 17.1 & 3.3 & -3.3 \\ -3.3 & 3.3 & 17.1 & -17.1 \\ 3.3 & -3.3 & -17.1 & 17.1 \end{pmatrix} .
$$

The conditional covariance matrix was

$$
\boldsymbol{\Theta}^* = \begin{pmatrix} 12.3 & -12.3 & 0 & 0 \\ -12.3 & 12.3 & 0 & 0 \\ 0 & 0 & 12.3 & -12.3 \\ 0 & 0 & -12.3 & 12.3 \end{pmatrix} .
$$

Our methodology has performed well in this example; the largest error in estimating a single route flow variance is only 3.6%. This compares very favourably with the naive estimate $\boldsymbol{\Theta}^*$ where the corresponding error is 26.1%. It is particularly interesting to note that $\widehat{\boldsymbol{\Sigma}}$ has provided a reasonable estimate of the covariance between the flows on routes 2 and 3. These routes service different inter-zonal movements but there is a correlation of 0.43 between the route flows. This is a result predicted by the preceding discussion, since these routes share link 3 in common.

4 Conclusions

The work in this paper provides a methodology for computing an approximation to the equilibrium covariance matrix in the important class of Markov traffic assignment models where the traveller learning mechanism is described by a linear filter with exponentially decreasing weights. This covariance matrix, in combination with a standard SUE assignment, allows the full equilibrium distribution of the process to be approximated. The methodology described has the potential to be of significant practical benefit, since our approximations of Eqns (11) and (24) (or (28)) can be computed efficiently in comparison to existing methods. In comparison, if a Monte Carlo simulation approach were instead adopted, long and highly time-consuming runs would typically be required in order to obtain reliable results; as shown in section 2.3, shorter simulation runs are liable to produce highly variable (and hence unreliable) estimates. As an avenue for further research, it would be valuable to extend our methodology to cope with more general learning mechanisms. However, this appears to be a non-trivial problem because it is the structure of the linear learning filter with exponentially decreasing weights that allows the fundamental decomposition of Eqn. (19) to be employed.

On a wider level, the research reported has significance for those developing complex computer microsimulation models, incorporating interacting sub-models of traveller choice behaviour and vehicular movement, such as the TRANSIMS program (e.g. Nagel and Barrett, 1997; Smith, 2001). The Markovian environment described here is in principle applicable at levels of detail down to the individual traveller, as well as to choice mechanisms other than route selection, and has the advantage of providing a formal mathematical framework within which to analyse the phenomenon model-users refer to as 'feedback'.

Acknowledgements

This research was carried out during the visit of the second author to the University of Western Australia, funded by an Australia Research Council Small Grant, and the later visit of the first author to the University of Leeds, partially funded by EPSRC grant GR/M79493. The support of an Advanced Fellowship AF/1997 from the UK Engineering and Physical Sciences Research Council is also gratefully acknowledged.

References

M.G.H. Bell and C. Cassir, *Reliability of Transport Networks*, Research Studies Press, Baldock, UK (2000).

M. Ben-Akiva, A. De Palma and I. Kaysi, "Dynamic Network Models and Driver Information Systems," *Transportation Research* **25A**, 251–266 (1991).

E. Cascetta, "A Stochastic Process Approach to the Analysis of Temporal Dynamics in Transportation Networks," *Transportation Research* **23B**, 1–17 (1989).

G.C. Cantarella, and E. Cascetta, "Dynamic Processes and Equilibrium in Transportation Networks: Towards a Unifying Theory," *Transportation Science* **29**, 305–329 (1995).

S.D. Clark and D.P. Watling, "Probit Based Sensitivity Analysis for General Traffic Networks," *Transportation Research Record* **1733**, 88–95 (2000).

C.F. Daganzo and Y. Sheffi, "On Stochastic Models of Traffic Assignment," *Transportation Science* **11**, 253–274 (1977).

C.F. Daganzo, *Multinomial Probit Theory and Its Application to Demand Forecasting.* Academic Press, New York (1979).

G.A. Davis and N.L. Nihan, "Large Population Approximations of a General Stochastic Traffic Assignment Model," *Operations Research* **41**, 169–178 (1993).

R.M.H. Emmerink, K.W. Axhausen, P. Nijkamp and P. Rietveld, "Effects of Information in Road Networks with Recurrent Congestion," *Transportation* **22**, 21–53 (1995).

R. Emmerink and P. Nijkamp, *Behavioural and Network Impacts of Driver Information Systems*, Ashgate, Aldershot, UK (1999).

W.R. Gilks, S. Richardson and D.J. Spiegelhalter *Markov Chain Monte Carlo in Practice*, Chapman & Hall, London (1996).

S. Hanson and J. Huff (1988). "Repetition and Day-to-Day Variability in Individual Travel Patterns," in *Behavioural Modelling in Geography and Planning*, R.C. Golledge and H. Timmermans, (eds.), Croom Helm, Kent, UK, 1998.

M.L. Hazelton, "Some Remarks on Stochastic User Equilibrium," *Transportation Research* **32B**, 101–108 (1998).

M.L. Hazelton, "Day-to-Day Variation in Markovian Traffic Assignment Models, *Transportation Research* **36B**, 61–72 (2002).

M.L. Hazelton, S. Lee and J.W. Polak, "Stationary States in Stochastic Process Models of Traffic Assignment: a Markov Chain Monte Carlo Approach.," in *Proceedings of the 13th International Symposium on Transportation and Traffic Theory*, J.-P. Baptiste, (ed.), 341–357. Pergamon Press, London, 1996.

M.L. Hazelton, "Some Remarks on Stochastic User Equilibrium," *Transportation Research* **32B**, 101–108 (1998).

J.L. Horowitz, "The Stability of Stochastic Equilibrium in a Two Link Transportation Network," *Transportation Research* **8B**, 13-28 (1984).

Y. Iida, T. Akiyama and T. Uchida, "Experimental Analysis of Dynamic Route Choice Behaviour," *Transportation Research* **26B**, 17–32 (1992).

H.S. Mahmassani and R. Jayakrishnan, "System Performance and User Response under Real-time Information in a Congested Traffic Corridor," *Transportation Research* **25A**, 293–308 (1991).

R. Mohammadi, "Journey Time Variability in the London Area," *Traffic Engineering and Control* **38**, 250–257 (1997).

F.O. Montgomery and A.D. May, "Factors Affecting Travel Times on Urban Radial Routes," *Traffic Engineering and Control* **28**, 452–458 (1987).

K. Nagel and C. Barrett, "Using Microsimulation Feedback for Trip Adaptation for Realistic Traffic in Dallas," *International Journal of Modern Physics* **8**, 505–525 (1997).

B. Ran, and D.E. Boyce, *Dynamic Urban Transportation Network Models*, Springer-Verlag, Berlin (1996).

Y. Sheffi, *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*, Prentice-Hall. Englewood Cliffs, NJ (1985).

J.P. Smith, "TRANSIMS Feedback Modelling," presented at TRB 80th annual meeting, Washington, January 7-11, 2001.

A.W. van der Vaart, *Asymptotic Statistics*. Cambridge University Press, Cambridge (1998).

J.G. Wardrop, "Some Theoretical Aspects of Road Traffic Research," *Proceedings, Institution of Civil Engineers* **II**(1), 325–378 (1952).

D.P. Watling, "Stability of the Stochastic Assignment Problem: a Dynamical Systems Approach," *Transportation Research* **33B**, 281–312 (1999).

D.P. Watling, "A Second Order Stochastic Network Equilibrium Model I: Theoretical Foundation," *Transportation Science*, **36**, 149-166 (2002).

TABLE 1.

	Mean	Variance	Mean	Variance	Mean	Variance
	$\theta = 0.01$		$\theta = 0.1$		$\theta = 1$	
Simulation	20.35	10.08	20.55	10.30	21.06	112.60
Naive	20.07	10.00	20.67	9.99	21.00	9.98
Approximation	20.07	10.00	20.67	10.32	21.00	137.12

FIGURE 1.

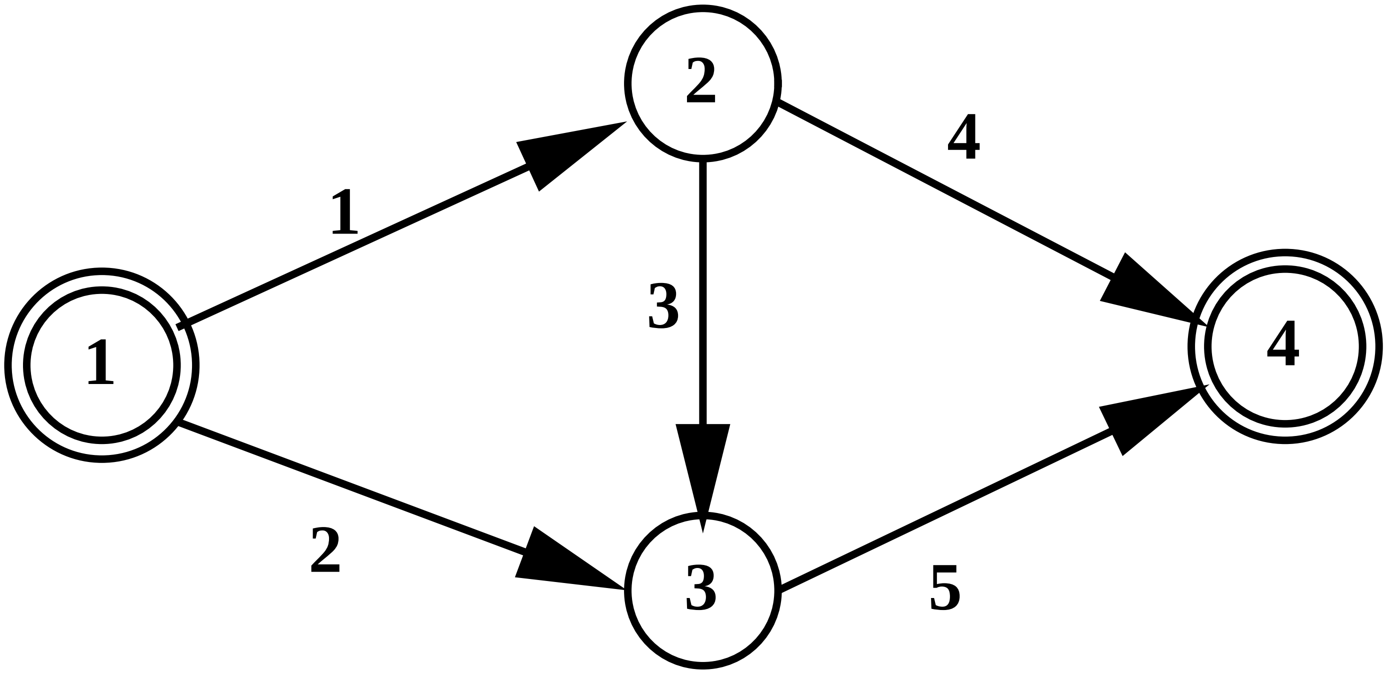

FIGURE 2.

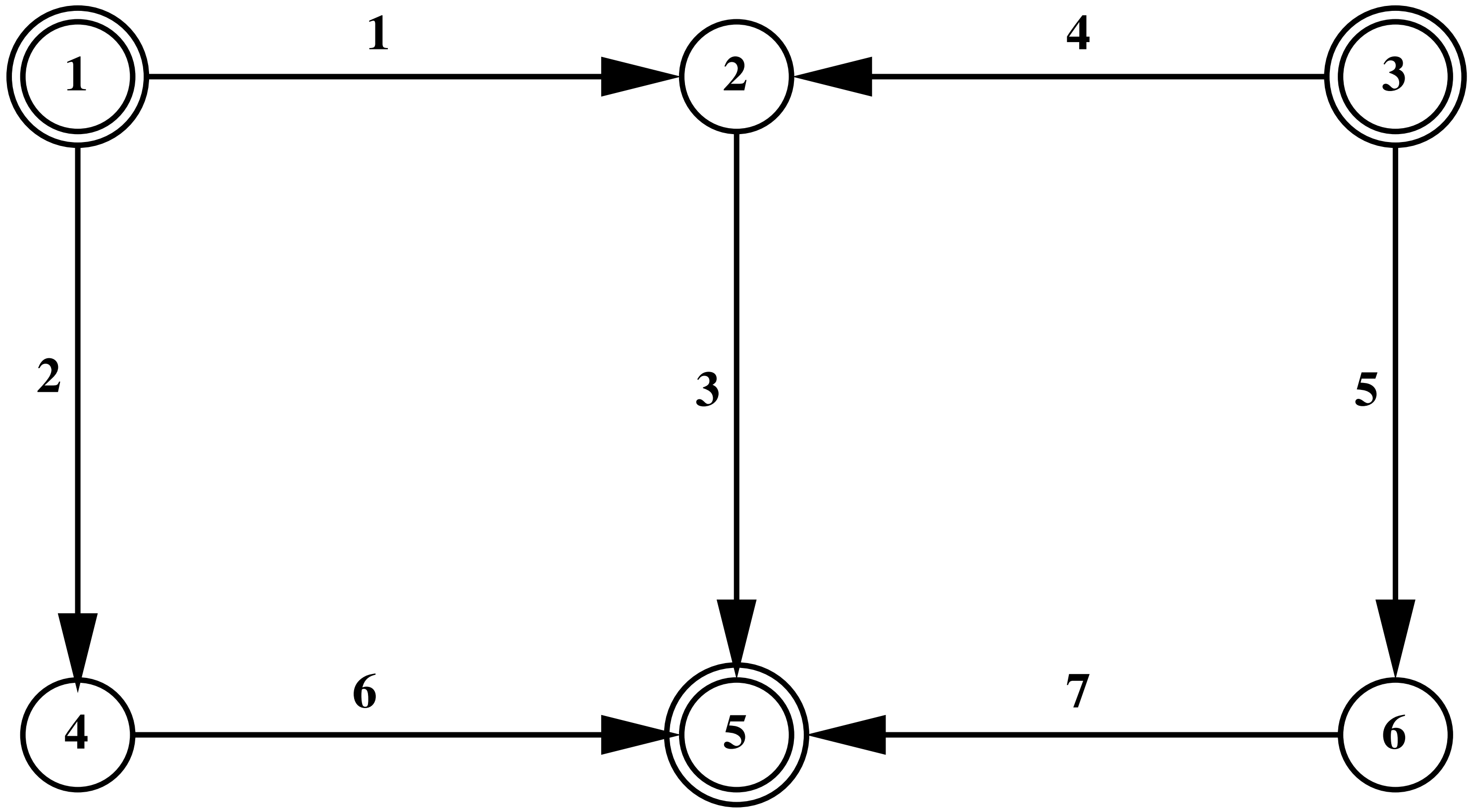

TABLE AND FIGURE LEGENDS

- TABLE 1: Estimates of the equilibrium mean and variance of flows on route 1 in a two route network.
- FIGURE 1: Network topology for probit example.
- FIGURE 2: Network topology for example with two inter-zonal movements.