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Reliable Network Design Problem: the case with uncertain demand and total travel time reliability

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ABSTRACT

This paper deals with Reliable Network Design Problem (RNDP) in which the main sources of uncertainty are variable demand and route choice. The objective is to maximize the network total travel time reliability (TTR) which is defined as the probability of the network total travel time to be less than a threshold. The paper presents a framework of stochastic network model with Poisson distributed demand and uncertain route choice. The travelers are assumed to choose their routes to minimize their perceived expected travel cost following Probit Stochastic Users' Equilibrium (SUE) condition. The paper presents an analytical method for approximating the first and second moments of the total travel time. These moments are then fitted with a log-normal distribution. The paper then tackles the design problem in which the analytical derivative of the TTR is derived using the sensitivity analysis of the equilibrated path choice probability. This derivative is then supplied to a gradient-based optimization algorithm to solve the RNDP. The algorithm is tested with a small network example.

Key words: Network Reliability, Network Design Problem, Stochastic Network, Sensitivity Analysis, Reliable Network Design

INTRODUCTION

Recently, the issue of network reliability has been a major focus in transport research. This is mainly due to the recognition of the importance of reliability of transport system. Early development in the area of network reliability has been on the evaluation of network reliability (with a broad definition), see for example (*1*), (*2*), (*3*), (*4*). Several measures for evaluating the network reliability have been proposed including the concept of travel time reliability (*5*), flow decrement reliability (*6*), vulnerability (*7*), capacity reliability (*8*) and connectivity reliability (*9*).

The source of uncertainty within the transport network can be classified into two main categories i) variations in the travel demand and ii) variations in the supply (*10*). The latter includes connectivity of nodes or links, variations of capacity or free-flow travel time, and others. The variations in the demand are mainly caused by travel demand variation from day-to-day and uncertainties from the traveler's behavior (e.g. route choice).

In the past, a common approach adopted for representing uncertainty within a network is the method of Monte Carlo simulation which is used to simulate different combinations of supply and demand states. This is rather a direct application of an existing model for evaluating network reliability. However, it is more complicated to deal with the variations in the demand side endogenously (for travel demand variation and travelers' behaviors uncertainty).

Some researchers have introduced the stochastic route choice model, which is one source of demand variation, with or without the demand uncertainty into network equilibrium models (*11*)- (*15*). Inouye (*16*) adapted an original Probit stochastic equilibrium model for evaluating travel time reliability by interpreting the error terms associated with the travel cost as the source of variation. Clark & Watling (*17*) proposed a stochastic network model assuming stochastic demand as well as stochastic route choice and defined an approach to approximate the probability density function (pdf) of the total travel time. The pdf of the total travel time within the network is then used to calculate the total travel time reliability which is defined as a probability of the total travel time to be less than a specified value.

Following the recent development of the stochastic network model with endogenous uncertainty, the natural progress is then to use the model to design or determine an optimal modification of the network. The original instance of this problem (without uncertainty) is the so called Network Design Problem (NDP). Several researchers have attempted to tackle the NDP under uncertainty. Yin & Ieda (*18*) analyzed the NDP with uncertain travel time. Waller & Ziliaskopoulos (*19*) studied the problem of a dynamic NDP dealing with demand uncertainty. Chen *et al* (*20*) introduced a mean-variance model for the determination of the optimal toll under the build-operate-transfer scheme under demand uncertainty.

The aim of this paper is to offer an alternative framework for the reliable network design problem (RNDP). The paper aims to develop a method for solving RNDP. The main uncertainty dealt with in this paper is the variability of travel demand and route choice. The problem formulation proposed allows the probability of travellers' route choice to change in response to the change of the network following the concept of Probit Stochastic User Equilibrium (SUE). The objective function of this RNDP is to maximize the probability of total travel time to be under a specified threshold, that is, total travel time reliability (*17*).

The paper is structured into four further sections. The next section explains the formulation, assumptions, and notation of the stochastic network model. Then, section 3 presents the formulation of RNDP and the solution algorithm including a method for calculating the total travel time reliability and its derivative with respect to the design parameters. The numerical results of a test with a small network are then discussed in section 4. Finally, the last section concludes the paper and discusses future research issues.

FRAMEWORK OF STOCHASTIC NETWORK MODEL

This section describes the framework of stochastic network model taking into account the demand and route choice uncertainties. The framework follows closely those proposed in (*17*) with an exception of the assumption that travelers choose their routes to minimize their perceived "expected travel time" rather than "travel time at expected demand".

Notation and assumptions

Define:

v_a	flow on link a $(a = 1,2,...,A)$, **v** the vector of flows across all links
q_w	mean demand on O-D movement w $(w = 1,2,...,W)$
q	W-vector of mean demands
R_w	index set of acyclic paths serving O-D movement w
δ_{ar}	indicator variable, equal to 1 if path r contains link a, 0 otherwise
$t_a(v_a)$	travel time on link a as a function of v_a $(a = 1,2,..., A)$
t(v)	vector of functions $t_a(v_a)$ $(a = 1,2,..., A)$.

The key statistical model assumptions are then:

- *The actual O-D demand on any day is independently distributed across inter-zonal movements, and for each movement w is distributed as a stationary Poisson random variable with constant mean* $q_w > 0$.
- *Conditional on the O-D movement w demand realised on any one day, drivers are assumed to choose independently between the alternative routes* $r \in R_w$ *with constant probabilities* p_r $(r \in R_w)$ *for each* $w = 1,2,...,W$.

Both assumptions together imply that for each $w = 1,2,...,W$, the *route* flows F_r $(r \in R_w)$ are random samples of a Poisson process with mean q_w and sampling rate p_r. It follows that the route flows F_r $(r \in R_w)$ are independent Poisson random variables with the mean value of $p_r q_w$ $(r \in R_w)$, for each $w = 1,2,....,W$.

Now, since the link flow random variables are related to the route flow random variables via the identities:

$$V_a = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} F_r \qquad (a = 1,2,..., A) \qquad (1)$$

then both assumption simply that the means of the link flows (1) are:

$$E[V_a] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} p_r q_w \qquad (a = 1,2,..., A) \qquad (2)$$

and the covariance

$$\mathrm{cov}[V_a, V_b] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} \delta_{br} p_r q_w \qquad (a = 1,2,..., A; b = 1,2,..., A). \qquad (3)$$

We then make the additional assumption: *"the variation in link flows across the network may be approximated by a multivariate Normal distribution (with means and covariances as given above)"*. The assumption of approximate multivariate Normal link flows is partially supported by the assumption of Poisson demands for movements with 'large' mean q_w, since the path flows F_r $(r \in R_w)$ are (as noted above) also independent Poisson random variables with means $p_r q_w$ $(r \in R_w)$. Then, for the (dominant) paths with large mean $p_r q_w$ (say, greater than 10), independent Normal approximations are supported for their flows, which clearly mix into multivariate Normal link flows. See (*21*) for a more detailed discussion of the validity of this assumption.

The formulation above establishes the analytical relationship between the random demand flows and the random link flows for a given vector of path choice probability (**p**). To complete the formulation of the stochastic model, we require knowledge of the route choice probabilities p_r $(r \in R_w; w = 1,2,..., W)$. In fact any rule for route choice assignment can be invoked to provide p_r (e.g. Wardrop's equilibrium or Stochastic equilibrium). In this paper we assume that the travellers choose their routes so as to minimise the perceived expected travel costs following the Probit Stochastic Users' Equilibrium (SUE).

Let C_r^w be a random route cost for route *r* and $E[C_r^w]$ denotes its means. Following the random utility theory, the perceived expected travel cost for route *r* can be defined as $\tilde{c}_r^w = E[C_r^w] + \varepsilon_r^w$ where ε_r^w is an error term associated with each path. Following the SUE condition (*22*), the probability of path *r* to be chosen can be defined as:

$$p_r = \Pr\left(E[C_r^w] + \varepsilon_r^w \le E[C_k^w] + \varepsilon_k^w \ \forall k \in R_w, r \ne k\right) = \Pr\left(\tilde{c}_r^w \le \tilde{c}_r^w \ \forall k \in R_w, r \ne k\right) \quad (4)$$

, where Pr(.) denotes probability.

From (4), the path cost distributions and hence their expected values are functions of path choice probabilities according to the assumptions made earlier. Thus, the condition shown in (4) defines the fixed point condition of the SUE. When ε_r^w is assumed to follow a normal distribution, (4) becomes the Probit SUE condition. However, in our stochastic model the fixed point condition is only applied to the path choice probability not to the path flow as the path flows are random variable. In the conventional SUE, p_r is used to assign a proportion of OD demand onto a certain route. On the other hand, in our stochastic model p_r is rather seen as a probability of route *r* being chosen at random.

Although the condition expressed in (4) is different from its original formulation, it is still possible to apply any method for solving the Probit SUE, see (*23*) and (*25*) for example, to (4). The reason being is that $E[t_a(V_a)] = \sum_j b_{ja} E[V_a^j]$ (see the polynomial travel cost function adopted in this paper in (8) below) can be calculated using moment generating functions, see (*15*). Based on a property of the moment generating function, the mean link travel time is given as:

$$E[t_a(V_a)] = b_{0a} + \sum_{j=1}^{m} b_{ja} \left. \frac{\partial^j M_a(s)}{\partial^j s} \right|_{s=0} \tag{5}$$

where $M_a(s)$ is the moment generating function of travel time on the a^{th} link and is $\exp(E[V_a]\, s\,(1 + s/2))$, where $E[V_a] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} p_r q_w$ following (2). Thus, the mean path travel cost can be defined as:

$$E\left[C_r^w\right] = \sum_a \delta_{ar} E\left[t_a(V_a)\right] = \sum_a \delta_{ar} b_{0a} + \sum_{j=1}^{m} b_{ja} \left. \frac{\partial^j M_a(\exp(E[V_a]s(1+s/2)))}{\partial^j} \right|_{s=0} \tag{6}$$

, which is simply a function of expected link flows (hence a function of expected route flow). By using the reformulation of travel cost function as shown in (5) we can apply an algorithm for solving to Probit SUE to (4) directly using the expected value of the OD demands as the deterministic demand levels (just to obtain p_r). In summary, the system of equations comprising of (1)-(6) defines the equilibrium condition for the stochastic network model.

FORMULATION AND SOLUTION ALGORITHM OF RELIABLE NETWORK DESIGN PROBLEM

Formulation of RNDP

In deciding upon the optimal allocation of the budget for the network improvement, the network planner needs to take into account possible responses of road users to the change in the network. This users' response to the changes is the condition defined in (1)-(6) in the previous section. Let **s** be a vector of design parameters which are associated with the link travel time function, i.e. $t_a(V_a, s_a)$. Associated with the design parameter, let β_a denotes the cost per unit of change of s_a. $\sum_a S_a \beta_a$ is thus the total improvement cost. The objective function of the reliable network design problem (RNDP) considered in this paper is the total travel time reliability (TTR) which is defined as the probability of the total travel time, $T = \sum_a t_a(V_a,s_a)V_a$, to not exceed a pre-specified criterion (α). This can be expressed as $\Pr(T \le \alpha)$. This objective function is mainly developed from the planner's perspective. Several other possible objective functions can also be used (e.g. capacity reliability or expected net welfare gain). In addition to the objective function, a budget constraint is also included which is defined as $\sum_a S_a \beta_a \le \Psi$. Thus, the RNDP with stochastic network equilibrium condition can be defined as follows:

$$
\begin{aligned}
&\max_{(\mathbf{s},\mathbf{V})} \Pr\left(T(\mathbf{V},\mathbf{s}) \le \alpha\right) \\
&s.t. \\
&\sum_{a=1}^{A} s_a \beta_a && \forall a = 1,2,\ldots,A \\
&V_a = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} F_r && \forall a = 1,2,\ldots,A \\
&E[V_a] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} p_r q_w && \forall a = 1,2,\ldots,A \\
&\operatorname{cov}[V_a, V_b] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} \delta_{br} p_r q_w && \forall a = 1,2,\ldots,A; \forall b = 1,2,\ldots,A; a \ne b \\
&\mathbf{V} \sim \mathrm{MVN}\left(E[\mathbf{V}], \Sigma\right) \\
&p_r = \Pr\left(E\left[C_r^w\right] + \varepsilon_r^w \le E\left[C_k^w\right] + \varepsilon_k^w \ \forall k \in R_w, r \ne k\right) \\
&E\left[C_r^w\right] = \sum_a \delta_{ar} b_{0a} + \sum_{j=1}^{m} b_{ja} \left. \frac{\partial^j M_a(\exp(E[V_a]s(1+s/2)))}{\partial^j} \right|_{s=0} && \forall p \in R_w
\end{aligned} \tag{7}
$$

, where $\mathbf{V} \sim \mathrm{MVN}\left(E[\mathbf{V}], \Sigma\right)$ means that the random vector of link flows ($\mathbf{V}$) follows a multivariate normal distribution (MVN) with a mean vector of $E[\mathbf{V}]$ and variance-covariance matrix of Σ.

In order to develop a gradient-based optimization algorithm for solving (7), it is necessary to calculate (i) the approximated probability density function (pdf) of the total travel time (for the calculation of the objective function) and (ii) the derivative of the TTR with respect to the design variables. The next section discusses the main result from Clark & Watling (*17*) on an approach for estimating the pdf of the total travel time. Then, we will discuss the sensitivity analysis method which is used to define the derivative of the objective function with respect to the design variables.

Approximating probability density function of the total travel time

The link travel time function adopted in this paper can be written in a polynomial form as:

$$t_a(v_a) = \sum_{j=0}^{m} b_{ja} {v_a}^j \tag{8}$$

Note that the power-law forms of the commonly used Bureau of Public Roads functions are a special case of (8). Based on (8), the random variable for the total travel time on link *a* is defined as:

$$W_a = V_a t_a(V_a) = \sum_{j=0}^{m} b_{ja} V_a^{\ j+1} \tag{9}$$

where V_a is a random variable representing the flow on link *a*, and W_a is the total travel time on link *a* (throughout the paper the convention is used that a random variable is denoted by a capital letter). Our interest will be in the total travel time random variable *T* given by

$$T = \sum_{a=1}^{A} V_a t_a(V_a) = \sum_{a=1}^{A} W_a \tag{10}$$

In a general case, one can attempt to compute the first four moments of *T* about the mean of total travel time and then fit them to a flexible family of probability densities known as *'Johnson curves'* (*25*) according to the techniques described in (*26*). This family consists of distributions obtained by monotonic transformations of a MVN, with additional parameters incorporated to permit a flexible fit to observed data. In our case, we will fit *T* with the lognormal system (S_L) in which $\gamma + \delta \ln(X - \xi)$ ~ Nor(0,1) (for $X > \xi$) where Nor (0,1) denotes a Normal distribution with mean 0 and variance 1. Thus, S_L is a three-parameter system. We further assume that ξ which represents the minimum total travel time is zero. Thus, δ and γ can be estimated by using the following expressions:

$$\delta = \left(\ln\left(1 + \left(\frac{\sigma}{\mu} \right)^2 \right) \right)^{-\frac{1}{2}} \ ; \quad \gamma = \frac{1}{2\delta} - \delta \ln \mu \tag{11}$$

, where μ and σ are the mean and standard deviation of the total travel time (*T*). Thus, in fitting *T* with the lognormal system we only need to obtain the mean (μ_T) and the second moment about the mean of *T*, which is the variance ($\sigma^2 = E[T - \mu_T]^2$). We can derive that:

$$\begin{aligned} \mu &= E[T] = \sum_{a=1}^{A} E[W_a] \\ \sigma^2 &= E\left[T^2\right] - \left(E[T]\right)^2 = E\left[T^2\right] - \mu^2 \end{aligned} \tag{12}$$

, and then by Binomial expansion we obtain:

$$E\left[T^2\right] = E\left[\left(\sum_{a=1}^{A} W_a \right)^2 \right] = \sum_{a=1}^{A} E\left[W_a^2\right] + 2 \sum_{(\forall a \in A | a \neq b)} \ \sum_{(\forall b \in A | b \geq a+1)} E\left[W_a W_b\right] \tag{13}$$

Thus, from (12) and (13) in order to calculate the mean and variance of the total travel time we need to define $E[W_a]$ and $E[W_a W_b]$ which can be defined as follows, see (*17*) for the detail of the derivation:

$$E[W_a] = \sum_{i=0}^{m+1} \tilde{b}_{ia} E[(V_a - \mu_a)^i]$$

$$E[W_a W_b] = \sum_{i=0}^{m+1}\sum_{j=0}^{m+1} \tilde{b}_{ia}\tilde{b}_{jb} E[(V_a - \mu_a)^i (V_b - \mu_b)^j] \tag{14}$$

$$\tilde{b}_{0a} = \sum_{j=0}^{m} b_{ja}\mu_a^{\ j+1}\ ;\ \ \tilde{b}_{ia} = \sum_{j=i-1}^{m} b_{ja} \frac{(j+1)!}{i!(j+1-i)!}\mu_a^{\ j+1-i} \quad (i = 1,2,...,m+1).$$

From (14), T may be written as a sum of multivariate moments of $\mathbf{V}$. In order to compute the moments of $\mathbf{V}$, results due to (*27*) are applied, which allow the computation of appropriate bivariate Normal moments for any powers.

Let $q_{ab}(i,j) = E[(V_a - \mu_a)^i(V_b - \mu_b)^j]$. This is simply the problem of finding the bivariate normal moments. When $i+j$ is an odd number, $q_{ab}(i,j)$ is equal to zero (*27*). For the case of even number of $i+j$, either both i and j must be odd or even numbers. Pearson and Young (*27*) provided general formulas for computing $q_{ab}(i,j)$ for these two cases:

$$q_{ab}(2x,2y) = \left(\sqrt{\left(\sigma_a^2\right)^i\left(\sigma_b^2\right)^j}\right)\frac{(2x)!(2y)!}{2^{x+y}}\sum_{u=0}^{x}\left[\frac{(2r_{ab})^{2u}}{(x-u)!(y-u)!2u!}\right]$$

$$q_{ab}(2x+1,2y+1) = \left(\sqrt{\left(\sigma_a^2\right)^i\left(\sigma_b^2\right)^j}\right)\frac{r_{ab}(2x)!(2y)!}{2^{x+y}}\sum_{u=0}^{x}\left[\frac{(2r_{ab})^{2u}}{(x-u)!(y-u)!(2u+1)!}\right] \tag{15}$$

, where $r_{ab} = \mathrm{cov}(V_a, V_b)/\sqrt{\sigma_a^2\sigma_b^2}$, and σ_a^2 is the variance of V_a.

The expression of covariance and variance of variable link flows is provided in (2) and (3). Note that $E[(V_a - \mu_a)^k]$ can be considered as a special case of $E[(V_a - \mu_a)^i(V_b - \mu_b)^j]$ for the case of $k > 2$ by setting $i = 2$ and $j = k$-2 which is associated with $q_{aa}(2, k\text{-}2)$.

In summary, the process of approximating the pdf of total travel time (T) is to calculate (15)-(11) (in that order).

Analytical derivative of the total travel time reliability

As discussed earlier, the second information we need to devise an optimization algorithm is the derivative of the objective function with respect to the design parameter. In (7), one of the complications is the presence of the fixed-point condition representing the SUE condition. In this paper, the strategy is to reformulate (7) to an implicit programming problem in which the path choice probability will be defined as an equilibrium path choice probability (p^*_r). This equilibrated path choice probability will be a function of the design parameters ($\mathbf{s}$) capturing the fixed point condition, $p^*_r(\mathbf{s})$.

Recall that the total travel time distribution is approximated by the log-normal system with the first and second moments of the distribution as defined in the previous section. The TTR (given the criteria of α) can then be defined solely as a function of $\mathbf{s}$ as follows:

$$\psi(\mathbf{s}) = \Pr(T(\mathbf{s}) \le \alpha) = \Pr(\gamma(\mathbf{s}) + \delta(\mathbf{s})\ln T(\mathbf{s}) \le \gamma(\mathbf{s}) + \delta(\mathbf{s})\ln\alpha) = \Phi(\gamma(\mathbf{s}) + \delta(\mathbf{s})\ln\alpha) = \int_{-\infty}^{\gamma(\mathbf{s})+\delta(\mathbf{s})\ln\alpha} \varphi(y)dy$$

, where $\varphi(y)$ defines the pdf of Nor(0,1).

Define $x = \gamma(\mathbf{s}) + \delta(\mathbf{s})\ln\alpha$; the derivative of ψ with respect to s can then be defined using the chain rule:

$$\frac{\partial\psi(\mathbf{s})}{\partial s} = \frac{\partial\psi(\mathbf{s})}{\partial x}\sum_{\forall r}\left[\frac{\partial x}{\partial p_r^*}\frac{\partial p_r^*}{\partial s} + \frac{\partial x}{\partial s}\right] \tag{16}$$

The sensitivity analysis of the equilibrated path choice probability with respect to the design parameter ($\partial p^*_r/\partial s$) can be derived using the result in (*28*). This will be revisited later on. Firstly, we will deal with $\partial x/\partial p^*_r$. The derivation of $\partial x/\partial s$ is similar to that of $\partial x/\partial p^*_r$ with some final adjustment which will also be discussed later.

Since x is equal to $\gamma(\mathbf{s}) + \delta(\mathbf{s})\ln\alpha$, $\partial x/\partial p^*_r$ can then be defined as:

$$\frac{\partial x}{\partial p_r^*} = \frac{\partial x}{\partial\gamma}\frac{\partial\gamma}{\partial p_r^*} + \frac{\partial x}{\partial\delta}\frac{\partial\delta}{\partial p_r^*}\text{, in which } \frac{\partial x}{\partial\gamma} = 1 \text{ and } \frac{\partial x}{\partial\delta} = \ln\alpha\,. \tag{17}$$

Thus, we are left with only the derivative of γ and δ with respect to p^*_r. Recall the definition of γ and δ from (11), we can then define the derivative of both γ and δ as:

$$\frac{\partial\gamma}{\partial p_r^*} = \frac{\partial\gamma}{\partial\delta}\frac{\partial\delta}{\partial p_r^*} + \frac{\partial\gamma}{\partial\mu}\frac{\partial\mu}{\partial p_r^*}\,;\quad \frac{\partial\delta}{\partial p_r^*} = \frac{\partial\delta}{\partial z}\frac{\partial z}{\partial p_r^*} \tag{18}$$

, where $z = \dfrac{\sigma}{\mu}$, $\dfrac{\partial\gamma}{\partial\delta} = -\dfrac{1}{2\delta^2} - \ln\mu$, $\dfrac{\partial\gamma}{\partial\mu} = \dfrac{\delta}{\mu}$, and $\dfrac{\partial\delta}{\partial z} = \dfrac{\left(\ln\left(1+z^2\right)\right)^{-\frac{3}{2}}}{\left(1+z^2\right)}z$. Again we can use the chain rule to define the derivative of z with respect to p^*_r .

$$\frac{\partial z}{\partial p_r^*} = \frac{\partial z}{\partial\sigma}\frac{\partial\sigma}{\partial p_r^*} + \frac{\partial z}{\partial\mu}\frac{\partial\mu}{\partial p_r^*} = \frac{1}{\mu}\frac{\partial\sigma}{\partial p_r^*} - \frac{\sigma}{\mu^2}\frac{\partial\mu}{\partial p_r^*} \tag{19}$$

Thus, the only terms we need to define for calculating (18) and (19) are $\partial\sigma/\partial p^*_r$ and $\partial\mu/\partial p^*_r$.

Firstly, let's look at the derivative of the standard deviation of the total travel time with respect to the equilibrated path choice probability, $\partial\sigma/\partial p^*_r$. The information available is the second moment of the total travel time around its mean (variance of the total travel time) which is defined in (12). Therefore, we can define $\partial\sigma/\partial p^*_r$ through the chain rule as:

$$\frac{\partial\sigma}{\partial p_r^*} = \frac{\partial\sqrt{\sigma^2}}{\partial\sigma^2}\frac{\partial\sigma^2}{\partial p_r^*} = \frac{1}{2\sigma}\frac{\partial\sigma^2}{\partial p_r^*} \tag{20}$$

, and we already showed that $\sigma^2 = E[T^2] - \mu^2$. Thus, we can define the derivative of the variance with respect to the equilibrated path choice probability as:

$$\frac{\partial\sigma^2}{\partial p_r^*} = \frac{\partial E\left[T^2\right]}{\partial p_r^*} - \frac{\partial\mu^2}{\partial p_r^*} = \frac{\partial E\left[T^2\right]}{\partial p_r^*} - 2\mu\frac{\partial\mu}{\partial p_r^*} \tag{21}$$

The last term of (21), $\partial\mu/\partial p^*_r$, will be addressed later. Now, we will focus on the first term, $\partial E[T^2]/\partial p^*_r$. As described earlier, $E[T^2]$ can be defined as (13). Thus, its derivative can be simply defined as:

$$\frac{\partial E\left[T^2\right]}{\partial p_r^*}=\sum_{a=1}^{A}\frac{\partial E\left[W_a^2\right]}{\partial p_r^*}+2\sum_{a=1}^{A}\sum_{b=a+1}^{A}\frac{\partial E\left[W_aW_b\right]}{\partial p_r^*} \tag{22}$$

Since W_a^2 is simply a special case of W_aW_a, we will focus only on finding $\partial E\,[W_aW_b]/\partial p^*_r$.

From the expansion of $E\,[W_aW_b]$ shown in (14), we can define its derivative as:

$$\frac{\partial E\left[W_aW_b\right]}{\partial p_r^*}=\sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\frac{\partial\left[\widetilde{b}_{ia}\widetilde{b}_{jb}E\left[\left(V_a-\mu_a\right)^i\left(V_a-\mu_a\right)^j\right]\right]}{\partial p_r^*} \tag{23}$$

Define Ω_{ab}^{ij} as $\widetilde{b}_{ia}\widetilde{b}_{jb}E\left[\left(V_a-\mu_a\right)^i\left(V_a-\mu_a\right)^j\right]$, we can then derive:

$$\frac{\partial\Omega_{ab}^{ij}}{\partial p_r^*}=\widetilde{b}_{ia}\widetilde{b}_{jb}\,\frac{\partial E\left[\left(V_a-\mu_a\right)^i\left(V_a-\mu_a\right)^j\right]}{\partial p_r^*}+E\left[\left(V_a-\mu_a\right)^i\left(V_a-\mu_a\right)^j\right]\left(\widetilde{b}_{ia}\,\frac{\partial\widetilde{b}_{jb}}{\partial p_r^*}+\widetilde{b}_{jb}\,\frac{\partial\widetilde{b}_{ia}}{\partial p_r^*}\right) \tag{24}$$

Following (14), we can define:

$$\frac{\partial\widetilde{b}_{0a}}{\partial p_r^*}=\sum_{j=0}^{m}\left[b_{ja}\left(j+1\right)\mu_a^j\,\frac{\partial\mu_a}{\partial p_r^*}\right] \tag{25}$$

$$\frac{\partial\widetilde{b}_{ia}}{\partial p_r^*}=\sum_{j=0}^{m}\left[b_{ja}\,\frac{\left(j+1\right)!}{i!\left(j+1-i\right)!}\left(j+1-i\right)\mu_a^{(j-1)}\,\frac{\partial\mu_a}{\partial p_r^*}\right]\quad i=1,\ldots,m+1 \tag{26}$$

where, μ_a is defined in (2), and thus $\partial\mu_a/\partial p^*_r = \sum\delta_{ar}\delta_{rw}q_w$ in which δ_{ar} and δ_{rw} denote the dummy variables taking value of 1 if link a is relevant to route r and if route r is relevant to OD pair w respectively (and 0 otherwise). Now return to (24). We still need to define $\partial E[(V_a\text{-}\ \mu_a)^i(V_b\text{-}\ \mu_b)^j]/\partial p^*_r$ to complete the formulation. Recall that $E[(V_a\text{-}\ \mu_a)^i(V_b\text{-}\ \mu_b)^j]$ can be defined following (15), we can define $\partial E[(V_a\text{-}\ \mu_a)^i(V_b\text{-}\ \mu_b)^j]/\partial p^*_r$ into two cases:

(i) i and j are both even numbers

Let A = $\sqrt{\left(\left(\sigma_a^2\right)^{2l}\left(\sigma_b^2\right)^{2s}\right)}$ and B = $\sum_{u=0}^{s}\left(\frac{\left(2r_{ab}\right)^{2u}}{\left(l-u\right)!\left(s-u\right)!2u!}\right)$

$$\frac{\partial E\left[\left(V_a-\mu_a\right)^i\left(V_b-\mu_b\right)^j\right]}{\partial p_r^*}=\frac{2l!2s!}{2^{l+s}}\left[A\sum_{u=0}^{s}\left(\frac{4u\left(2r_{ab}\right)^{2u-1}}{\left(l-u\right)!\left(s-u\right)!2u!}\frac{\partial r_{ab}}{\partial p_r^*}\right)+B\,\frac{\partial A}{\partial p_r^*}\right] \tag{26}$$

$$\frac{\partial A}{\partial p_r^*}=\frac{1}{2\sqrt{\sqrt{\left(\left(\sigma_a^2\right)^{2l}\left(\sigma_b^2\right)^{2s}\right)}}}\left(2s\left(\sigma_a^2\right)^{2l}\left(\sigma_b^2\right)^{2s-1}\frac{\partial\sigma_b^2}{\partial p_r^*}+2l\left(\sigma_b^2\right)^{2s}\left(\sigma_a^2\right)^{2l-1}\frac{\partial\sigma_a^2}{\partial p_r^*}\right)$$

$$\frac{\partial r_{ab}}{\partial p_r^*} = \frac{\left(\frac{\partial \operatorname{cov}(V_a,V_b)}{\partial p_r^*}\sqrt{\sigma_a^2\sigma_b^2}\right) - \left(\frac{1}{2\sqrt{\sigma_a^2\sigma_b^2}}\left(\sigma_a^2\frac{\partial \sigma_b^2}{\partial p_r^*} + \sigma_b^2\frac{\partial \sigma_a^2}{\partial p_r^*}\right)\operatorname{cov}(V_a,V_b)\right)}{\sigma_a^2\sigma_b^2}$$

, where $\partial\sigma^2_a/\partial p^*_r = \sum\delta_{ar}\delta_{rw}q_w$ and $\partial\operatorname{cov}(V_a,V_b)/\partial p^*_r = \sum\delta_{ar}\delta_{br}\delta_{rw}q_w$.

(ii) i and j are both odd numbers

Let A = $\sqrt{\left(\left(\sigma_a^w\right)^{2l+1}\left(\sigma_b^w\right)^{2s+1}\right)}$ and B = $\sum_{u=0}^{s}\left(\frac{(2r_{ab})^{2u}}{(l-u)!(s-u)!(2u+1)!}\right)$

$$\frac{\partial E\left[(V_a-\mu_a)^i(V_b-\mu_b)^j\right]}{\partial p_r^*} = \frac{(2l+1)!(2s+1)!}{2^{l+s}}\left[A\left[\sum_{u=0}^{s}\left(\frac{4u(2r_{ab})^{2u-1}}{(l-u)!(s-u)!(2u+1)!}\frac{\partial r_{ab}}{\partial p_r^*}\right)+\frac{\partial r_{ab}}{\partial p_r^*}B\right]+r_{ab}B\frac{\partial A}{\partial p_r^*}\right]$$

$$\frac{\partial A}{\partial p_r^*} = \frac{1}{2\sqrt{\sqrt{\left(\left(\sigma_a^2\right)^{2l+1}\left(\sigma_b^2\right)^{2s+1}\right)}}}\left((2s+1)\left(\sigma_a^2\right)^{2l+1}\left(\sigma_b^2\right)^{2s}\frac{\partial\sigma_b^2}{\partial p_r^*}+(2l+1)\left(\sigma_b^2\right)^{2s+1}\left(\sigma_a^2\right)^{2l}\frac{\partial\sigma_a^2}{\partial p_r^*}\right) \quad (27)$$

, and $\partial r_{ab}/\partial p^*_r$ is as defined above.

At this point, we completed the derivation of the $\partial E\ [W_aW_b]/\partial p^*_r$ which can then be used to find $\partial E\ [T^2]/\partial p^*_r$. Return to (21), we still need to define $\partial\mu/\partial p^*_r$ to complete the calculation. Following (14), we can define:

$$\frac{\partial\mu}{\partial p_r^*} = \sum_{a=1}^{A}\sum_{i=0}^{m+1}\left[\tilde{b}_{ia}\frac{\partial E\left[(V_a-\mu_a)^i\right]}{\partial p_r^*} + E\left[(V_a-\mu_a)^i\right]\frac{\partial\tilde{b}_{ia}}{\partial p_r^*}\right] \quad (28)$$

Note that $\partial\tilde{b}_{ia}/\partial p_r^*$ is already defined in (25) and (26). Thus, we only need to define $\partial E\ [(V_a -\mu_a)^i]/\partial p^*_r$ to complete the calculation. We also can use the process to calculate $E\ [(V_a -\mu_a)^i(V_a -\mu_a)^i(V_b -\mu_b)^j]$ to find $E[V_a -\mu_a)^i]$ with some remedy. If $i = 0$, then $E\ [(V_a -\mu_a)^0] = 1$, and hence $\partial E\ [(V_a -\mu_a)^0]/\partial p^*_r = 0$. On the other hand, if $i = 1$, then $E\ [(V_a -\mu_a)^1] = 0$, and again $\partial E\ [(V_a -\mu_a)^1]/\partial p^*_r = 0$.

For the case where $i > 1$, we can define $E[V_a -\mu_a)^i]$ in the form of $E\ [(V_a -\mu_a)^i(V_a -\mu_a)^{i-1}(V_a -\mu_a)^1]$. With this reformulation, the approach employed to calculate $E\ [(V_a -\mu_a)^i(V_a -\mu_a)^i(V_b -\mu_b)^j]$ can then be applied. Note that we will always have the case of both i and j are odd numbers since j is always 1. Therefore, we only need to find the derivative for the case (ii) in (27). We now complete the formulation of $\partial\mu/\partial p^*_r$. In summary, by calculating (16)-(28) we can compute $\partial x/\partial p^*_r$ analytically. Given that $\partial p^*_r/\partial s$ can be evaluated, we can calculated $\partial x/\partial p^*_r\partial x/\partial p^*_r$. Recall that p^*_r is defined as the path choice probability at the equilibrium condition defined by the equation system of (1)-(6). To deal with $\partial p^*_r/\partial s$, the technique of sensitivity analysis will be employed which will be discussed in the next section.

For $\partial x/\partial s$, the same derivation from (17)-(23) for $\partial x/\partial p^*_r$ can be employed by replacing the derivative with respect to p^*_r to the derivative with respect to s. Recall the modified (23) (now for $\partial x/\partial s$) :

$$\frac{\partial E[W_a W_b]}{\partial s} = \sum_{i=0}^{m+1}\sum_{j=0}^{m+1} \frac{\partial\left[\tilde{b}_{ia}\tilde{b}_{jb}E\left[(V_a-\mu_a)^i(V_a-\mu_a)^j\right]\right]}{\partial s} \tag{29}$$

In this case, $E\ [(V_a\ -\mu_a)^i(V_a\ -\mu_a)^i(V_b\ -\mu_b)^j]$ is not a function of s. Define Ω_{ab}^{ij} as $\tilde{b}_{ia}\tilde{b}_{jb}E\left[(V_a-\mu_a)^i(V_a-\mu_a)^j\right]$, the derivative in (29) can be expressed as:

$$\frac{\partial \Omega_{ab}^{ij}}{\partial s} = E\left[(V_a-\mu_a)^i(V_a-\mu_a)^j\right]\left(\tilde{b}_{ia}\frac{\partial \tilde{b}_{jb}}{\partial s} + \tilde{b}_{jb}\frac{\partial \tilde{b}_{ia}}{\partial s}\right) \tag{30}$$

Following (14), we can define:

$$\frac{\partial \tilde{b}_{0a}}{\partial s} = \sum_{j=0}^{m}\left(\frac{\partial b_{ja}}{\partial s}\mu_a^{j+1}\right) \tag{31}$$

$$\frac{\partial \tilde{b}_{ia}}{\partial p_r^*} = \sum_{j=0}^{m}\left[\frac{\partial b_{ja}}{\partial s}\frac{(j+1)!}{i!(j+1-i)!}\mu_a^{\ j+1-i}\right] \quad i=1,\ldots,m+1 \tag{32}$$

Similarly, (28) can be modified to:

$$\frac{\partial \mu}{\partial s} = \sum_{a=1}^{A}\sum_{i=0}^{m+1}\left[E\left[(V_a-\mu_a)^i\right]\frac{\partial \tilde{b}_{ia}}{\partial s}\right] \tag{33}$$

, and again we can use (31) and (32) to compute (33).

In summary, $\partial x/\partial s$ can be computed using (17)-(23) and (30)-(33). This completes the derivative of the derivative of the total travel time reliability with respect to the design parameters.

Sensitivity analysis of equilibrated path choice probability

The equilibrated path choice probability can be defined as follows:

$$p_r(\mathbf{s}) = \Pr\left(E\left[C_r^w(p_r(\mathbf{s}))\right] + \varepsilon_r^w \le E\left[C_k^w(p_r(\mathbf{s}))\right] + \varepsilon_k^w\ \forall k \in R_w, r \ne k\right).$$

Define the gap function:
$\Theta(\mathbf{p}(\mathbf{s}),\mathbf{s}) \equiv \mathbf{p}(\mathbf{s}) - \mathbf{P}(\mathbf{p}(\mathbf{s}),\mathbf{s})$ and for any given $\mathbf{s}$ and equilibrated path choice probability, $\mathbf{p}^*(\mathbf{s})$, $\Theta(\mathbf{p}^*(\mathbf{s}),\mathbf{s})$ = 0. Assuming the differentiability of all functions involved and treating s as variable, a first-order series expansion of $\Theta(\mathbf{p}(\mathbf{s}),\mathbf{s})$ around ($\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0$) is:

$$\Theta(\mathbf{p},\mathbf{s}) \approx \Theta(\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0) + \nabla_{\mathbf{p}}\Theta\big|_{(\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0)}(\mathbf{p}-\mathbf{p}^*(\mathbf{s}_0)) + \nabla_{\mathbf{s}}\Theta\big|_{(\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0)}(\mathbf{s}-\mathbf{s}_0),$$

where $\nabla_{\mathbf{p}}\Theta\big|_{(\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0)}$ (denoted as $\mathbf{J}_1$) and $\nabla_{\mathbf{s}}\Theta\big|_{(\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0)}$ (denoted as $\mathbf{J}_2$) are the Jacobian matrices of Θ with respect to $\mathbf{p}$ and $\mathbf{s}$ respectively both evaluated around ($\mathbf{p}^*(\mathbf{s}_0),\mathbf{s}_0$). We can approximately solve the equilibrium condition $\Theta(\mathbf{p}(\mathbf{s}),\mathbf{s}) = 0$ for $\mathbf{p}(\mathbf{s})$ following $0 \approx 0 + \mathbf{J}_1(\mathbf{p}-\mathbf{p}^*(\mathbf{s}_0)) + \mathbf{J}_2(\mathbf{s}-\mathbf{s}_0)$; and hence $\lim_{\mathbf{s}\to\mathbf{s}_0}\frac{(\mathbf{p}-\mathbf{p}^*(\mathbf{s}_0))}{(\mathbf{s}-\mathbf{s}_0)} \equiv \frac{\partial\mathbf{p}^*}{\partial\mathbf{s}} = -\mathbf{J}_1^{-1}\mathbf{J}_2$.

Recall that $\mathbf{J}_1$ is the Jacobian of the fixed-point condition with respect to the path choice probability. This can be defined as:

$$\frac{\partial\Theta}{\partial p_r} = \sum_{\forall k}\left[\frac{\partial\Theta}{\partial E[C_k]}\sum_a\left(\Delta_{ak}\frac{\partial E[t_a(V_a)]}{\mu_a}\frac{\partial\mu_a}{\partial p_r}\right)\right] = \sum_{\forall k}\left[\frac{\partial\Theta}{\partial E[C_r]}\sum_a\left(\Delta_{ak}\frac{\partial E[t_a(V_a)]}{\mu_a}\sum_{w=1}^{W}(\Delta_{wr}q_w)\right)\right]$$

$$\frac{\partial\Theta}{\partial s_a} = \sum_{\forall k}\left[\frac{\partial\Theta}{\partial E[C_k]}\sum_e\left(\Delta_{ek}\sum_{j=0}^{2}\frac{\partial E[t_e(V_e)]}{\partial b_{je}}\frac{\partial b_{je}}{\partial s_a}\right)\right] \tag{34}$$

Similarly to the case of SUE, any technique for computing the derivative of the Probit path choice probability can be applied with the modified link cost functions representing the expected travel costs. The technique for finding the derivative of path choice probability can be found in (*29*) and those who are interested in the implementation of this approach should be referred to (*28*).

In the practical implementation, s_a is directly associated with the change of the parameters in the original BPR link cost function of the form $t_a(V_a,s_a) = \zeta_a(s_a) + \eta_a(s_a)[V_a/\theta(s_a)]^4$. Therefore, when calculating $\partial b_{je}/\partial s_a$, the relationship between the original BPR link travel cost function and the (approximated) polynomial travel cost function, as defined in (8), must be considered. In the numerical test which will be presented in the next section, only the quadratic link cost function, $t_a(V_a) = b_{0a} + b_{1a}V_a + b_{2a}(V_a)^2$, will be adopted to reduce the complexity of the calculation. Note that this does not pose any restriction on applying the method to a travel cost function with a higher order. The second-order Taylor series expansion of $f(x)$ about a point $x = u$ is given by:

$$f(x) \approx f(u) + f'(x-u) + \frac{1}{2}f''(x-u)^2$$

$$f(x) \approx f(u) - uf'(u) + \frac{u^2}{2}f''(u) + [f'(u) - uf''(u)]x + \frac{f''(u)}{2}x^2.$$

Thus, the relationships between the parameters in the quadratic link cost function and the BPR function can be made:

$$b_{0a} = t_{a,BPR}(u) - ut'_{a,BPR}(u) + \frac{u^2}{2}t'_{a,BPR}(u) = \xi_a + \frac{3\eta_a}{\theta_a}u^4$$

$$b_{1a} = t'_{a,BPR}(u) - ut''_{a,BPR}(u) = -\frac{8\eta_a}{\theta_a^4}u^3 \tag{35}$$

$$b_{2a} = \frac{t''_{a,BPR}(u)}{2} = \frac{6\eta_a}{\theta_a^4} u^2$$

In summary with the information about the derivative of the objective function with respect to the design parameter (including the effect of the equilibrium condition), one can apply any of the fruitful available optimization routines to solve the RNDP stated in (7). The next section presents numerical results from a test with a small network.

ILLUSTRATIVE EXAMPLE

This section tests the algorithm developed in the previous section with a small five link network. We adopt the 'fmincon' solver within MATLAB in which the derivative of the objective function with respect to the design parameters is supplied to the solver in addition to the derivative of the constraints. The algorithm inside the 'fmincon' solver is the sequential quadratic programming (SQP) (*30*).

Figure 1 shows the five-link test network with the BPR link cost function. The network comprises of five links and a single origin-destination pair. This network has been used in previous studies, e.g. (*17*) and (*31*).

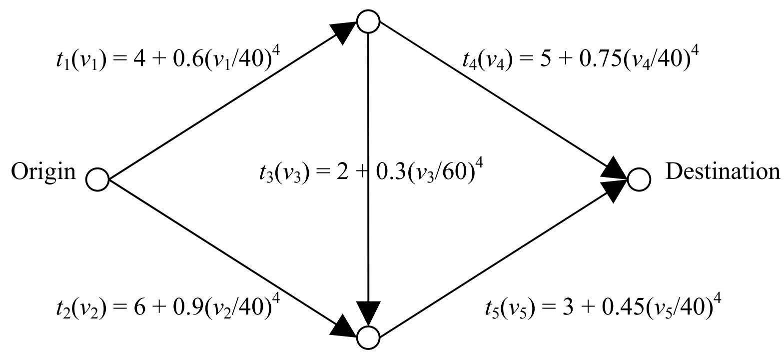

FIGURE 1 Test network, O-D demand *q* = 100.

The quadratic link cost function is constructed using the Taylor series expansion around a vector of link flow at the conventional Probit SUE condition, v_a, (see the previous section). Table 1 shows the coefficients of the quadratic link cost functions. Using (5) the modified link cost function representing the expected link travel time can be defined as:

$$E[t_a(V_a)] = b_{0a} + b_{1a}\mu_a + b_{2a}(\mu_a + \mu_a^2) = b_{0a} + (b_{1a} + b_{2a})\mu_a + b_{2a}\mu_a^2 \qquad (36)$$

It should be noted that the link cost parameters adopted in approximating the pdf of the total travel time using (8)-(15) should be the coefficients for the quadratic link cost function as shown in Table 1.

TABLE 1 Base SUE Link Flows and Quadratic Travel Time Coefficients

link number (a)	v_a	b_{0a}	b_{1a}	b_{2a}
1	55.48	10.6637	-0.3203	0.00433
2	44.52	10.1417	-0.2481	0.00418
3	12.39	2.0016	-0.0004	0.00002
4	43.10	8.0320	-0.1876	0.00326
5	56.90	8.5289	-0.2591	0.00342

Monte Carlo simulation is employed to simulate an empirical frequency distribution of the total travel time. Figure 2 shows the empirical frequency distribution of the total travel time with 4000 Monte Carlo draws of the Poisson O-D demand. The mean and std as calculated from the empirical frequency distribution are 1277.30 and 275.14 respectively. The analytical mean and variance based on the approach described earlier are 1306.42 and 279.96 which are relatively close.

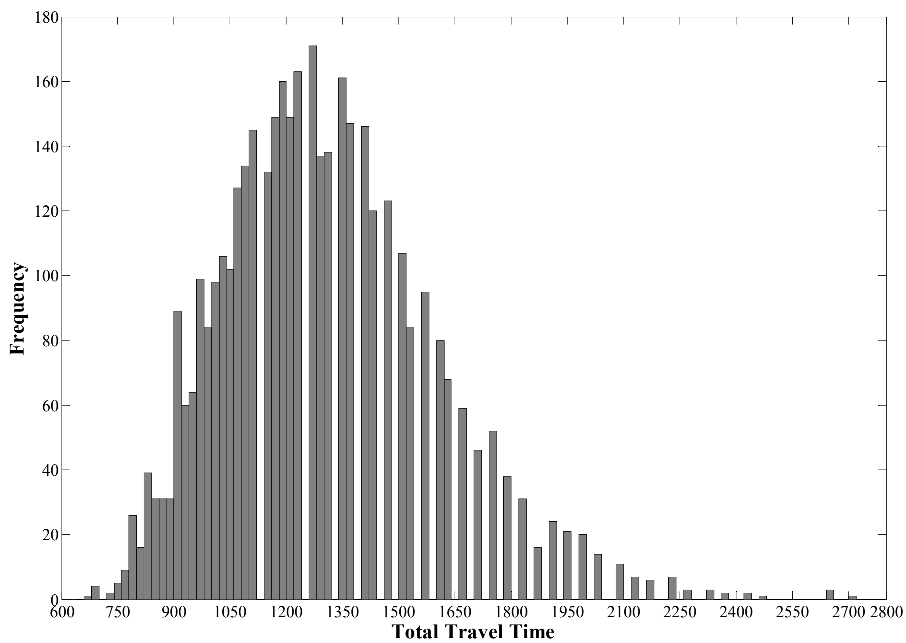

FIGURE 2 Empirical distribution of total network travel time based on 4000 Monte Carlo draws from the Poisson demand distribution (q = 100)

Figure 3 illustrates the travel time distribution as the capacity of link 1 was gradually increased. The figure shows that in addition to the expected shift of the mean of the distribution changes in the dispersion and shape of the distribution is also observed. Particularly, one can observe the subtle changes on the right-tail of the distribution whereas the left-tails of the distributions are more stable. These results imply the concept of “spare capacity” in which a network with higher spare capacity is more capable of dealing with uncertain demand (especially a high level demand).

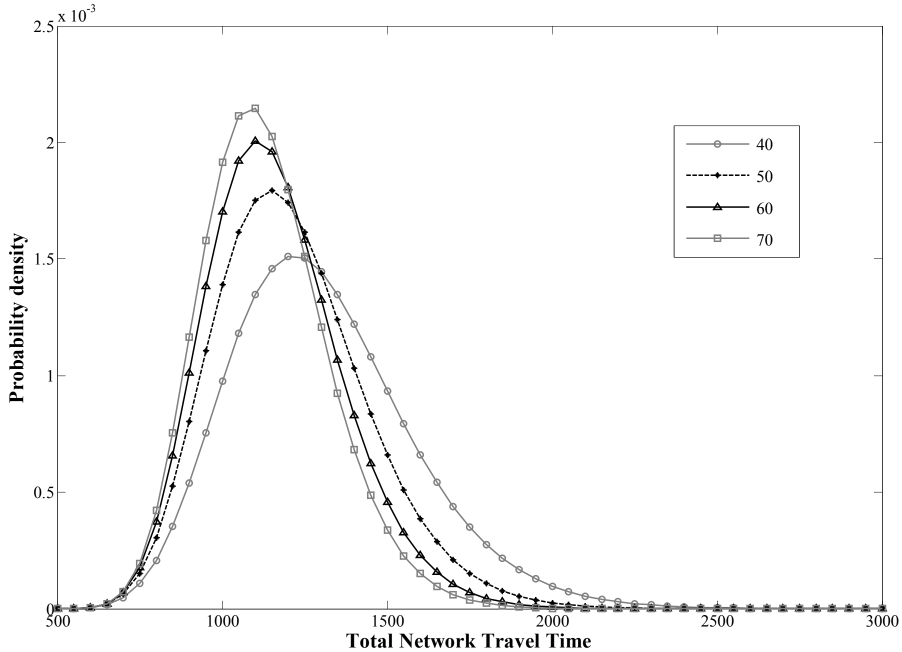

FIGURE 3 Analytical distribution of total network travel time for a range of changes on the capacity of link 1.

Figure 4 demonstrates the effect of the link capacity on the TTR ($TT \leq 1{,}500$). The capacity of each link is increased by 1-30 units. The figure shows that the TTR is most sensitive to the change on link 1, 5, 2, 4, and 3 in that order. Links 1 and 5 are related to two of the three paths in the network and hence covers a high volume of traffic. On the other hand, the capacity change on link 3 almost has no impact on the TTR. Link 3 is related to the path with the least path choice probability, and hence related to a small volume of traffic.

The next test involves directly the RNDP. The design parameters considered in this test is the link capacity (θ). The budget constraint is set to be 50 and the cost per a unit change of link capacity is 2. In this test, the capacities of links 1, 2, 4, and 5 are the design parameters. We decided to drop link 3 due to its small influence on the TTR. Interestingly, the optimal result found by the 'fmincon' only involves the improvement of the capacities on links 1 and 5 with the magnitude of 16.02 and 8.98 giving the probability of 0.97 for the total travel time to be less than 1,500.

Figure 5 illustrates the sensitivity of the objective function with the change in the budget constraint. Six levels of the budget constraint are tested: 60, 50, 40, 30, 20, and 10. As shown in the figure, one can notice the declining rate of the improvement over the total travel time reliability as the budget increases.

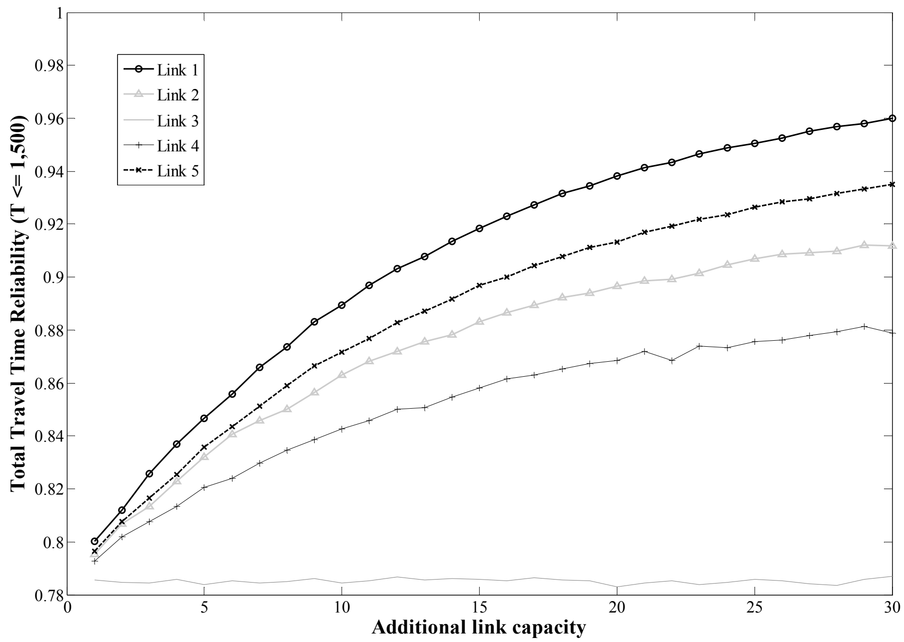

FIGURE 4 Effects on the total travel time reliability ($TT \leq 1{,}500$) as the capacity of each link is increased.

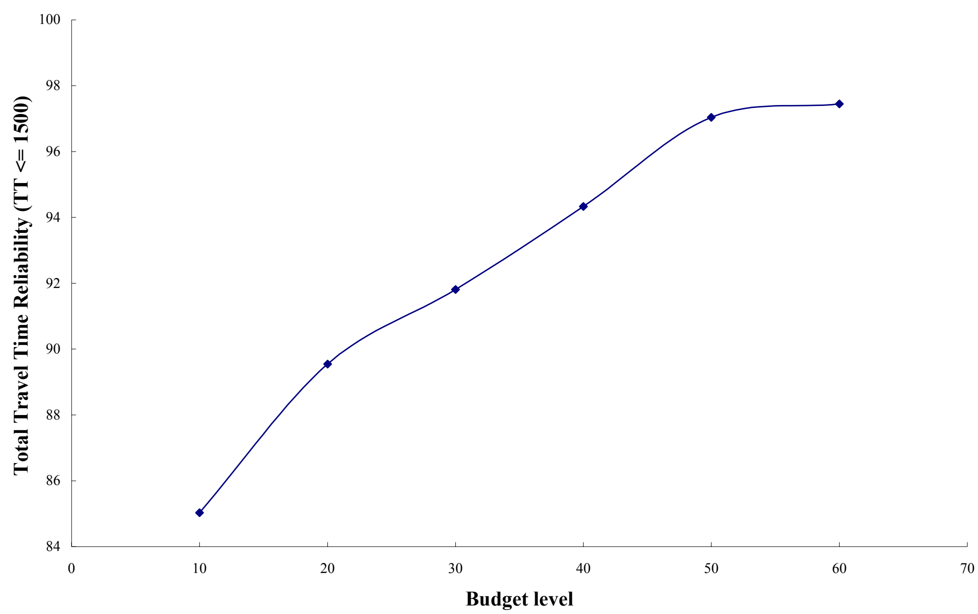

FIGURE 5 Sensitivity of the improvement of the total travel time reliability as the level of budget constraint varies

CONCLUSIONS

The paper presented a stochastic network model that includes two sources of uncertainties: demand and route choice uncertainty. The OD demands are assumed to follow Poisson distribution and the travelers' route choices are assumed to follow Probit SUE in which the drivers consider the perceived expected travel times. Then, the formulation of the reliable network design problem (RNDP) was presented with an objective function of maximizing the total travel time reliability (TTR) with a budget constraint. TTR is defined as the probability of the network total travel time to be less than a specified threshold.

An approximation method for deriving the first and second moments (mean and variance) of the total travel time was explained. These moments are then used to fit the total travel time with the Johnson distribution using the log-normal system. With the approximated pdf of the total travel time, the TTR for a given threshold can be calculated. The paper then derived an analytical derivative of the TTR with respect to the design parameters (e.g. link capacity). To complete the derivative, the sensitivity analysis method for calculating the derivative of equilibrated Probit path choice probability with respect to the design parameters was described.

The analytical derivative of the TTR is then supplied to the 'fmincon' solver in MATLAB to solve the RNDP. The algorithm was tested with a five-link network. The tests were also conducted to investigate the changes of the total travel time distribution as the link capacities varied. The results showed that the link capacity change did not affect only the location of the total travel time distribution but also its dispersion and shape (especially a strong effect of the right-tail of the pdf was found). Additional tests on the influence of the capacity improvement on each link on the TTR were also conducted. The result showed that links which are related to a higher traffic volume and number of paths are likely to have more influence on the TTR. The algorithm was then employed to solve the test problem of RNDP successfully. Further investigation on the sensitivity of the TTR with the budget constraints was also made. The results suggested a gradually decreasing trend of the marginal gain in the TTR improvement as the budget level increased.

The main reason for choosing the TTR as the main objective function of the RNDP is due to our view that TTR is an extension of the travel time reliability and is appropriate for the analysis with demand uncertainty. However, it is also envisaged that the modeling framework and algorithm proposed in this paper can also be applied to other indices. This possibility will be explored in our future research. In addition, further research will attempt to apply the algorithm with a larger scale network to show its practicality and investigate the discrete network design problem.

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link number (*a*)	v_a	b_{0a}	b_{1a}	b_{2a}
1	55.48	10.6637	-0.3203	0.00433
2	44.52	10.1417	-0.2481	0.00418
3	12.39	2.0016	-0.0004	0.00002
4	43.10	8.0320	-0.1876	0.00326
5	56.90	8.5289	-0.2591	0.00342

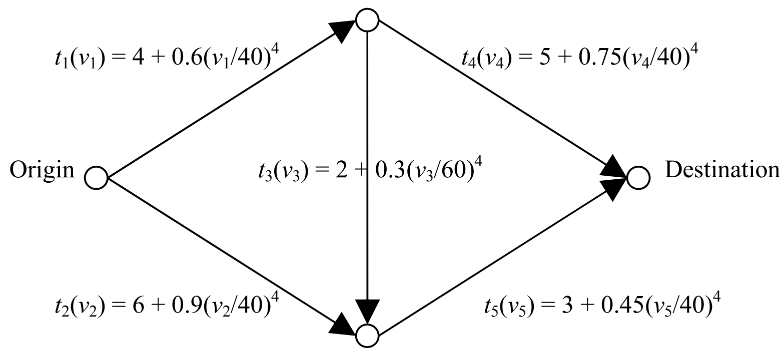

FIGURE 1 Test network, O-D demand *q* = 100.

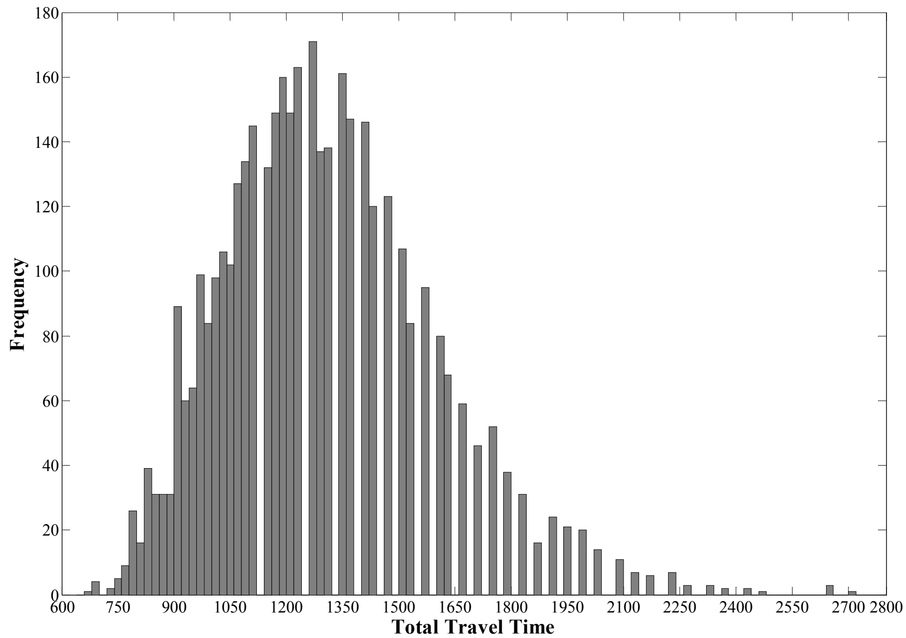

FIGURE 2 Empirical distribution of total network travel time based on 4000 Monte Carlo draws from the Poisson demand distribution (q = 100)

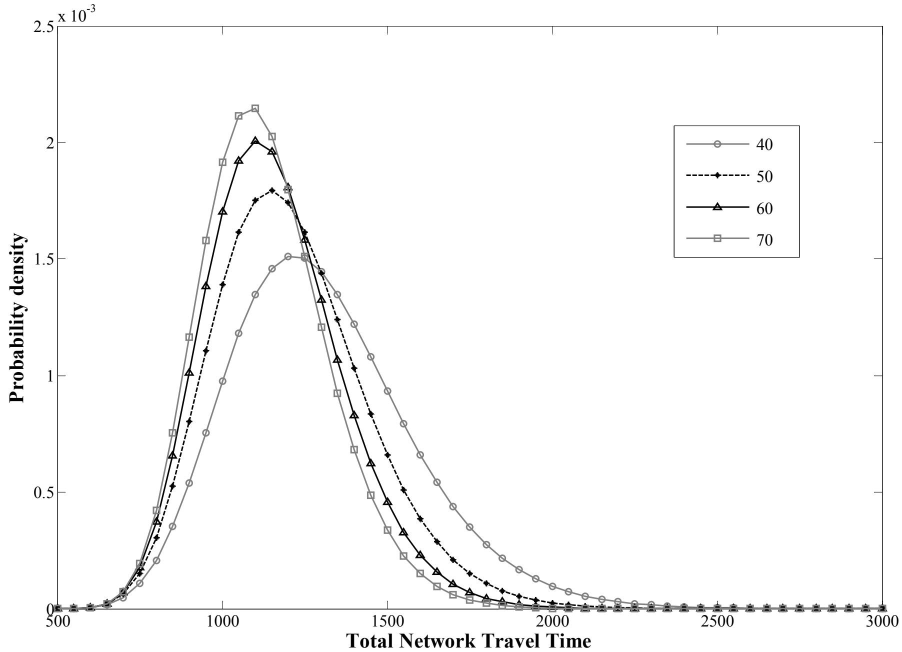

FIGURE 3 Analytical distribution of total network travel time for a range of changes on the capacity of link 1.

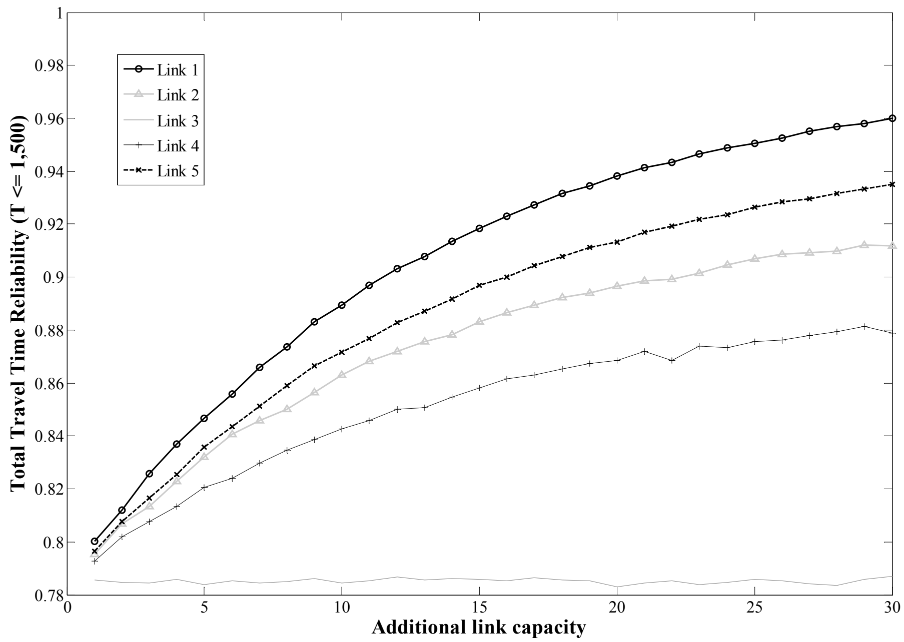

FIGURE 4 Effects on the total travel time reliability (*TT* ≤ 1,500) as the capacity of each link is increased.

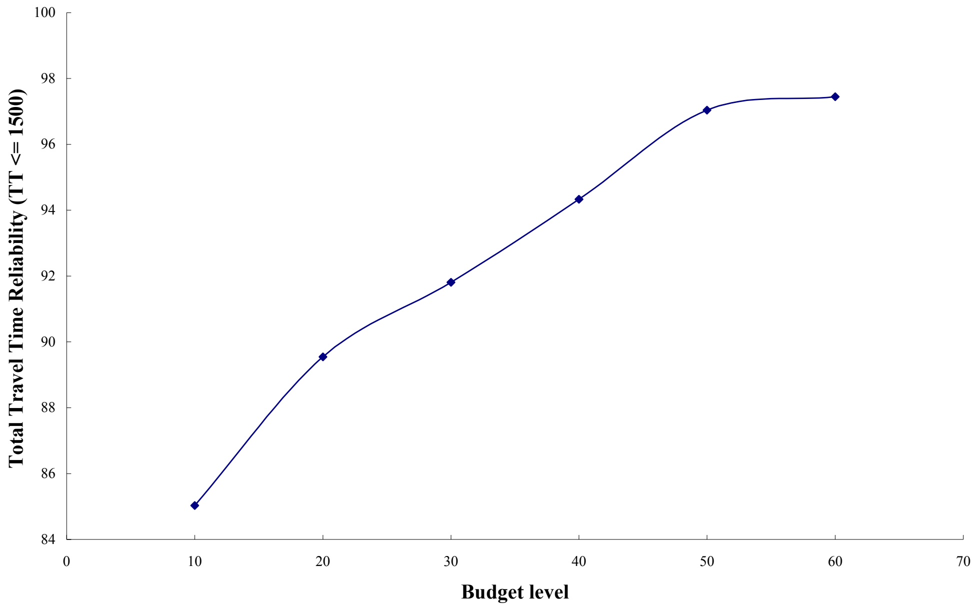

FIGURE 5 Sensitivity of the improvement of the total travel time reliability as the level of budget constraint varies