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Rasmussen, TK, Watling, DP orcid.org/0000-0002-6193-9121, Prato, CG et al. (1 more author) (2015) Stochastic user equilibrium with equilibrated choice sets: Part II – Solving the restricted SUE for the logit family. Transportation Research Part B: Methodological, 77. pp. 146-165. ISSN 0191-2615

https://doi.org/10.1016/j.trb.2015.03.009

Stochastic User Equilibrium with Equilibrated Choice Sets: Part II – Solving the Restricted SUE for the Logit Family

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Abstract

We propose a new class of path-based solution algorithms to solve the Restricted Stochastic User Equilibrium (RSUE), as introduced in Watling et al (2014). The class allows a flexible specification of how the choice sets are 'systematically' grown by considering congestion effects and how the flows are allocated among routes. The specification allows adapting traditional path-based stochastic user equilibrium flow allocation methods (designed for pre-specified choice sets) to the generic solution algorithm. We also propose a cost transformation function and show that by using this we can, for certain Logit-type choice models, modify existing path-based Deterministic User Equilibrium solution methods to fit the RSUE solution algorithm. The transformation function also leads to a two-part relative gap measure for consistently monitoring convergence to a RSUE solution. Numerical tests are reported on two real-life cases, in which we explore convergence patterns and choice set composition and size, for alternative specifications of the RSUE solution algorithm.

Keywords: Restricted Stochastic User Equilibrium; Solution Methods; Path-swapping; Convergence criteria; Gap function; Stochastic User Equilibrium.

1. Introduction

Within the family of Stochastic User Equilibrium traffic assignment models (SUE, Daganzo and Sheffi, 1977), the commonly-applied choice models are based on Random Utility Models (RUMs) in which the error terms are distributed according to some unbounded distribution (e.g. probit- and logit-type models). Such an assumption implies that, in the equilibrium state, some flow is assigned to all permitted routes, no matter how high their cost might be. In Deterministic User Equilibrium (DUE) models, on the other hand, routes that are only slightly more costly expensive than the equilibrium cost are unused. In a companion paper (Watling et al, 2014) we argued why neither of these extremes seems particularly realistic, and went on to consider two alternative model families for equilibrating choice sets within an SUE framework.

In the present paper we shall focus on one of these families, which we termed the Restricted Stochastic User Equilibrium (RSUE). In an RSUE solution the choice set is equilibrated via additional constraints on the costs of unused routes, while any of the standard range of RUMs distributes flow among the used routes. The constraints require the cost of an ununsed route to be no lower than some reference cost for that origin-destination (OD) movement, the reference cost defined by applying an operator Φ to the costs on the used routes. Each choice of Φ gives rise to a different model-type (RSUE(Φ**))** within the RSUE family, with a specific model form then defined by the choice of RUM for the used alternatives. Specifically, in the companion paper, we made a case for RSUE(min) and RSUE(max) as being natural model-types to consider.

In Watling et al (2014) we computed RSUE(min) and RSUE(max) solutions in small networks by enumerating all possible choice sets (i.e. all non-empty subsets of the available routes), but clearly such a strategy will be computationally infeasible for realistic-scale networks. The objective of the present paper is to address the challenge of developing RSUE solution methods for realistic-scale networks. Specifically, we focus on the expanding array of choice models for which closed-form expressions are available for the choice probabilities, such as C-logit (Cascetta et al, 1996; Zhou et al, 2012), path-size logit (Ben-Akiva and Bierlaire, 1999), generalised nested logit (Bekhor and Prashker, 2001), cross-nested logit (Bekhor et al, 2007, 2008), weibit (Castillo et al., 2008; Kitthamkesorn and Chen, 2014) and q-generalised logit (Nakayama, 2013). Such a focus is justified partly by our interest in developing efficient algorithms even for very large-scale networks, particularly as it obviates the need for computationally expensive simulation associated with other choice models (e.g. probit, mixed logit).

Since RSUE is formulated with two conditions, one concerned with the distribution of flow on used routes and one posing a cost restriction on the unused routes, it is reasonable to consider devising a solution algorithm which decomposes path generation and path loading. The path loading constitutes a sub-problem of loading traffic among a restricted set of routes according to a RUM, and this is a problem well-known in practical SUE applications. The advantage we hope to exploit in this way is that in the RSUE model, large parts of the universal choice set should remain unused for a given OD movement (the unused paths), and so it should avoid the ‘pressure’ in SUE algorithms of trying to assign and equilibrate very small amounts of flow to all available paths. This can be considered a ‘scaleability’ property of RSUE that it shares with DUE; adding new, high cost paths to a previously equilibrated network leaves the solution unchanged, unlike for SUE.

With these aims in mind, the particular contributions of the paper are as follows:

- we set out the RSUE formulation with a straightforward but important extension to multiple user classes and multiple vehicle types (section 2);
- we establish equivalence between (i) the closed-form, logit-type RSUE problem and (ii) a “DUE-like” problem utilising transformed travel cost functions (section 3);
- based on the equivalence in section 3, we propose a two-part gap measure for monitoring convergence to an RSUE choice set and RSUE flow distribution among used paths (section 4);

- we propose a generic solution algorithm for RSUE problems, consisting of a column generation step and a restricted master problem step, and identify alternative promising formulations of the restricted master problem based on (i) SUE-based flow distribution methods, and (ii) DUE-based path-swapping methods based on the transformed problem of section 3 (section 5):
- we present the results from numerical experiments, testing variants of these algorithms on the Sioux Falls network and on a much larger, multi-class real-life network (section 6);
- we discuss the implications of the findings from the numerical experiments (section 7), leading to conclusions and paths identified paths for further research (section 8).

2. Notation and Definitions

2.1. Notation

Consider a network as a directed graph composed of links a (a = 1, …, A) with non-negative flow f_a, and let $\mathbf{f}$ be the A-dimensional vector of link flows. We assume the actual flow-dependent (generalised) travel cost on link a to be a continuous function of the flow, and denote it by $t_a(\mathbf{f})$.

The network consists of M OD-pairs (Origin-Destination pairs), and the demand d_m for each OD-pair m composes a non-negative M-dimensional vector $\mathbf{d}$. For each OD-pair m, R_m is the set of all simple acyclic paths (routes) connecting the OD, and N_m is the number of paths in R_m. R refers to the joint set of all simple paths across OD-pairs, with dimension $N = \sum_{m \in M} N_m$.

We denote the flow on path r between OD-pair m as x_{mr} and the N-dimensional flow-vector on the universal choice set across all M OD-pairs as $\mathbf{x}$. The convex set G of demand-feasible non-negative path flow solutions G is given by:

$$G = \left\{ \mathbf{x} \in \mathbf{R}_{+}^{N} : \sum_{r=1}^{N_m} x_{mr} = d_m, m = 1, 2, ..., M \right\} \tag{1}$$

where $\mathbf{R}_{+}^{N}$ denotes the N-dimensional non-negative Eucledian space. It should be noted that x_{mr} refers to element number $r + \sum_{k=1}^{m-1} N_m$ in the vector **x.** The corresponding convex set of demand-feasible link flows is:

$$F = \left\{ \mathbf{f} \in \mathbf{R}_{+}^{A} : f_a = \sum_{m=1}^{M} \sum_{r=1}^{N_m} \delta_{amr} \cdot x_{mr}, a = 1, 2, ..., A, \mathbf{x} \in G \right\} \tag{2}$$

where δ_{amr} =1 if link a belongs to path r for OD-pair m and zero otherwise.

Let $c_{mr}(\mathbf{x})$ be the actual (generalised) cost on path r for OD-pair m. As links may be used by several paths within and across the OD-pairs, $c_{mr}(\mathbf{x})$ depends on the flow vector $\mathbf{x}$. Additionally, the cost $c_{mr}(\mathbf{x})$ is a positive value and may be a weighted sum of several attributes, such as e.g. travel time, travel distance, and congestion charge.

In vector/matrix notation, let **x** and **f** be column vectors, and define **Δ** as the A×N-dimensional link-path incidence matrix. Then the relationship between link and path flows may be written as $\mathbf{f} = \mathbf{\Delta x}$. We suppose that the travel cost on path r for OD-pair m is additive in the link travel costs of the utilised links:

$$c_{mr}(\mathbf{x}) = \sum_{a=1}^{A} \delta_{amr} \cdot t_a(\Delta \mathbf{x}) \qquad (r \in R_m;\ m = 1,2,\ldots,M;\ \mathbf{x} \in G) \tag{3}$$

Let $\mathbf{c}(\mathbf{x})$ be the vector of costs on all paths. The RSUE model distinguishes between used and unused paths, and consequently we let $\tilde{R}_m$ be the subset of R_m consisting of all utilised paths (non-zero flow) for OD-pair m (i.e. $\tilde{R}_m \subseteq R_m$).

The following notation is also used in the paper:

- l_a is the length of link a.

- L_{mr} is the length of path r for OD-pair m.
- Γ_{mr} is the set of links constituting path r for OD-pair m.
- $\Phi\left(\{c_{mr}(\mathbf{x}) : r \in \tilde{R}_m\}\right)$ is the mapping function used in the RSUE definition, specifies criterion to be fulfilled by unused paths.
- π_m is a free variable used in the DUE formulation, which equals the cost on the cheapest path between OD-pair m.
- z is the number of zones.
- V is the number of vertices in the network.
- $P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m)$ is the proportion of flow on OD-movement m that uses path r among the alternatives in the restricted set of utilised paths $\tilde{R}_m$ for OD-pair m.

2.2. Restricted Stochastic User Equilibrium

The choice probability function $P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m)$ is supposed given by a RUM separately for each OD-pair m:

$$P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) = \Pr\left(-\theta \cdot c_{mr}(\mathbf{x}) + \xi_{mr} \geq -\theta \cdot c_{ms}(\mathbf{x}) + \xi_{ms}, \forall s \neq r, s \in \tilde{R}_m\right) \quad \forall r \in \tilde{R}_m \tag{4}$$

where $\theta \geq 0$. The choice model hereby holds for any proper subset of the universal choice set. The flow on any used path is then the total demand multiplied by the choice proportion (4), whereas the flow on any unused path is zero by definition. The equilibrium path flows must then satisfy:

$$x_{mr} = \begin{cases} d_m \cdot P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) & \text{if } r \in \tilde{R}_m \\ 0 & \text{otherwise} \end{cases} \quad (r \in R_m;\ m \in M) \tag{5}$$

where

$$\tilde{R}_m = \{r : r \in R_m \text{ and } x_{mr} > 0\} \quad (m \in M) \tag{6}$$

The conditions (5) and (6) are necessary but not sufficient, as the restricted choice set is 'internally defined' and not necessarily the universal choice set. It should be noted that the RUM is supposed to be such that for any non-empty restricted choice set $\tilde{R}_m \subseteq R_m$ and for any cost vector **c**, the probability function has the properties:

$$\sum_{r \in \tilde{R}_m} P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) = 1 \tag{7}$$

and

$$P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) > 0 \qquad \forall r \in \tilde{R}_m \tag{8}$$

These properties imply that the OD demands are automatically satisfied (from (7)) and the path flows are non-negative (from (8)). Since we consider only RUMs in which these properties hold, then we shall not explicitly state them below as necessary conditions.

We introduce the mapping $\Phi(\cdot)$ that, for each OD-pair m,acts upon the costs of used paths, i.e. on $\{c_{mr}(\mathbf{x}) : r \in \tilde{R}_m\}$. We require that any unused path on OD-pair m has a cost greater than or equal to $\Phi(\{c_{mr}(\mathbf{x}) : r \in \tilde{R}_m\})$, i.e. for any unused path the following has to hold:

$$x_{mr} = 0 \quad \Rightarrow \quad c_{mr}(\mathbf{x}) \geq \Phi(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}) \qquad (r \in R_m;\ m = 1,...,M) \tag{9}$$

Bringing together these elements, the equilibrium conditions are defined as (for details, see Watling et al., 2014):

Definition: Restricted Stochastic User Equilibrium (RSUE(Φ))

Suppose that we are given a collection of continuous, unbounded random variables $\{\xi_{mr} : r \in R_m, m = 1,2,...,M\}$ defined over the whole choice set R_m; and that for any non-empty subsets $\tilde{R}_m$ of R_m (m = 1,2,…,M), probability relations $P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m)$ are given over $\tilde{R}_m$ (m =

1,2,…,M) by considering the relevant marginal joint distributions from{ $\xi_{mr} : r \in R_m, m = 1,2,...,M$ }. The route flow $\mathbf{x} \in G$ is a RSUE(Φ) if and only if for all $r \in R_m$ and m = 1,2,…,M:

$$x_{mr} > 0 \quad \Rightarrow \quad r \in \tilde{R}_m \quad \wedge \quad x_{mr} = d_m \cdot P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) \tag{10}$$

$$x_{mr} = 0 \quad \Rightarrow \quad r \notin \tilde{R}_m \quad \wedge \quad c_{mr}(\mathbf{x}) \geq \Phi\left(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}\right) \tag{11}$$

Note that the set of utilised paths $\tilde{R}_m$ is implicitly defined by the restrictions. In the companion paper (Watling et al., 2014) we introduced the RSUE(min) and RSUE(max) by letting $\Phi\left(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}\right)$ be $\min\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}$ and $\max\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}$, respectively.

2.3. Extension to multiple user classes and vehicle types

The notation may be readily modified to include the more general case of (i) multiple user classes differring in their definition of travel cost and in the OD matrix, and (ii) multiple vehicle types differring in the contribution they make to the total traffic flow. In this case, m denotes a commodity which is a combination of OD movement, user class and vehicle type, so that M is the product of the number of OD movements, user classes and vehicle types. In order to reflect different contributions to traffic flow, we suppose that the demand d_m for commodity m is measured in equivalent passenger car units. The only modification required to the notation is then that $t_{am}(\mathbf{\Delta x})$ now denotes the travel cost on link a as perceived by commodity m when the total pcu route flows are $\mathbf{x}$. Thus, the route cost-flow functions are defined by:

$$c_{mr}(\mathbf{x}) = \sum_{a=1}^{A} \delta_{amr} \cdot t_{am}(\mathbf{\Delta x}) \qquad (r \in R_m;\ m = 1,2,\ldots,M) \ . \tag{11}$$

Under these changes, all the subsequent models and methods presented may be applied to this more general case. In the following we will refer to OD-pair m, but this may as well, in the case of multiple vehicle and/or user classes, refer to commodity m. In section 6.2 we perform tests using 19 user classes and 2 vehicle-types.

3. Equivalent User Equilibrium Transformation

We introduce a function transforming the actual path costs into transformed costs, in order to reformulate the RSUE problem in an equivalent form that resembles the DUE conditions. We explicitly show the transformation for the two logit-type choice models that we shall analyse in detail subsequently, the multinomial logit and path-size logit, although the same approach can be applied to other closed form models in this family (as cited in the introduction). This reformulation will subsequently be used both in section 4 (for creating a gap function) and section 5 (for testing some possibilities of RSUE solution algorithm inspired by DUE methods).

3.1. Multinomial Logit RSUE (MNL RSUE)

We show that the the MNL RSUE(Φ) can be transformed into an equivalent problem which has a similar form to DUE. The DUE is defined for all OD-pairs m (Patriksson, 1994):

$$x_{mr} > 0 \Rightarrow c_{mr}(\mathbf{x}) = \pi_m \qquad \forall r \in R_m \tag{12}$$

$$x_{mr} = 0 \Rightarrow c_{mr}(\mathbf{x}) \geq \pi_m \qquad \forall r \in R_m \tag{13}$$

The definition of DUE implicitly distinguishes between used and unused paths, and implies that the cost on all utilised paths are the same, namely equal to the cost of the lowest-cost alternative for OD-pair m. The DUE conditions can thus be rewritten as:

$$x_{mr} > 0 \Rightarrow r \in \tilde{R}_m \wedge c_{mr}(\mathbf{x}) = c_{ms}(\mathbf{x}) \qquad \forall s \in \tilde{R}_m \tag{14}$$

$$x_{mr} = 0 \Rightarrow r \notin \tilde{R}_m \wedge c_{mr}(\mathbf{x}) \geq \left(\pi_m = \min_{s\in\tilde{R}_m} c_{ms}(\mathbf{x}) = \max_{s\in\tilde{R}_m} c_{ms}(\mathbf{x})\right) \tag{15}$$

For the transformation of the MNL RSUE(Φ) into an equivalent set of conditions similar to conditions (14) and (15), we now introduce the transformed cost $\tilde{c}_{mr}(\mathbf{x})$ of path r as a function of three things: the actual generalised cost on path r, the flow on path r and parameter $\theta \geq 0$[1]:

[1] A similar transformation was used to equilibrate routes in the gradient projection algorithm using a reference path cost, proposed in Bekhor and Toledo (2005)

$$\tilde{c}_{mr}(\mathbf{x}) = x_{mr} \cdot \exp\left(\theta \cdot c_{mr}(\mathbf{x})\right) \tag{16}$$

We note that there is no physical meaning of the transformed route cost $\tilde{c}_{mr}(\mathbf{x})$; it is a purely mathematical construct. Equalising the transformed cost among the used paths induces the first MNL RSUE condition (10) to be fulfilled:

$$\tilde{c}_{mr}(\mathbf{x}) = \tilde{c}_{ms}(\mathbf{x}) \qquad \forall\left(r,s\right) \in \tilde{R}_m \tag{17}$$

Equation (16) can be rewritten to express the flow on path r:

$$x_{mr} = \frac{\tilde{c}_{mr}(\mathbf{x})}{\exp\left(\theta \cdot c_{mr}(\mathbf{x})\right)} \tag{18}$$

which allows us to deduce::

$$\begin{aligned}
P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) &= \frac{x_{mr}}{d_m} = \frac{x_{mr}}{\sum_{s \in \tilde{R}_m} x_{ms}} = \frac{\dfrac{\tilde{c}_{mr}(\mathbf{x})}{\exp\left(\theta \cdot c_{mr}(\mathbf{x})\right)}}{\sum_{s \in \tilde{R}_m} \dfrac{\tilde{c}_{ms}(\mathbf{x})}{\exp\left(\theta \cdot c_{ms}(\mathbf{x})\right)}} = \frac{\dfrac{\tilde{c}_{mr}(\mathbf{x})}{\exp\left(\theta \cdot c_{mr}(\mathbf{x})\right)}}{\sum_{s \in \tilde{R}_m} \dfrac{\tilde{c}_{mr}(\mathbf{x})}{\exp\left(\theta \cdot c_{ms}(\mathbf{x})\right)}} \\
&= \frac{\tilde{c}_{mr}(\mathbf{x}) \cdot \dfrac{1}{\exp\left(\theta \cdot c_{mr}(\mathbf{x})\right)}}{\tilde{c}_{mr}(\mathbf{x}) \cdot \sum_{s \in \tilde{R}_m} \dfrac{1}{\exp\left(\theta \cdot c_{ms}(\mathbf{x})\right)}} = \frac{\exp\left(-\theta \cdot c_{mr}(\mathbf{x})\right)}{\sum_{s \in \tilde{R}_m} \exp\left(-\theta \cdot c_{ms}(\mathbf{x})\right)}
\end{aligned} \tag{19}$$

This corresponds to the choice probability formulation of the MNL model with scale parameter θ, and shows that the solution to the problem (17) on the set of utilised paths is equal to the solution to the corresponding MNL SUE problem on the original costs $\mathbf{c}\left(\mathbf{x}\right)$. The opposite implication can also be shown: starting from the MNL choice probabilities and using the definition of the transformation function, (19) can be utilised to show that the system (17) arises from using the transformed costs $\tilde{\mathbf{c}}(\mathbf{x})$.

Using the second RSUE condition as in definition (11), the equivalence shown above yields the following transformed RSUE conditions, which are equivalent to the original RSUE conditions if for all $r \in R_m$ and $m \in M$:

$$x_{mr} > 0 \quad \Rightarrow \quad r \in \tilde{R}_m \quad \wedge \quad \tilde{c}_{mr}(\mathbf{x}) = \tilde{c}_{ms}(\mathbf{x}) \qquad \forall s \in \tilde{R}_m \tag{20}$$

$$x_{mr} = 0 \quad \Rightarrow \quad r \notin \tilde{R}_m \quad \wedge \quad c_{mr}(\mathbf{x}) \geq \Phi\left(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}\right) \tag{21}$$

These transformed RSUE conditions have some similarities with the DUE conditions (14) and (15): both contain a statement for used paths and a statement concerning non-used paths. Comparing equations (14) and (20), it can be seen that they both equalise costs on utilised paths. However, when distributing traffic between utilised paths, one operates on actual costs while the other operates on the transformed costs. The second condition is also quite similar, especially for the RSUE(max) and RSUE(min). The difference, however, is that while the criterion to be fulfilled by unused paths is implicitly defined by the DUE formulation, the RSUE needs an explicit definition of how the reference OD travel cost is related to the travel costs of used paths.

3.2. Path Size Logit RSUE (PSL RSUE)

The above transformation of the RSUE(Φ) was based on the MNL model, and hence it does not account for correlations across alternatives. This disadvantage is critical in a route choice application, as paths typically overlap with other considered paths on segments. The Path Size Logit (PSL) model (Ben-Akiva and Bierlaire, 1999) overcomes this deficiency, and has been applied with success in various route choice studies (e.g., Bekhor and Prato, 2009; Frejinger et al., 2009; Ramming, 2001; Ben-Akiva et al., 2012). In the following it is shown that by applying the transformed cost-function (22) we can write the PSL RSUE as a DUE-like system similar to equations (20) and (21).[2]

Consider the following cost transformation:

[2] A similar approach to that described can be applied for other closed-form logit-type models, where the modification from the MNL model consists of altering the deterministic term of the utility function, by replacing the expression $c_{mr}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{mr}\right)$ in (22) with the corresponding expression for the selected choice model.

$$\tilde{c}_{mr}(\mathbf{x}) = x_{mr} \cdot \exp\left(\theta \cdot \left(c_{mr}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{mr}\right)\right)\right) \tag{22}$$

where $\theta \geq 0$, $\beta_{PS} \leq 0$ and $c_{mr}(\mathbf{x})$ is the general MNL cost-function applied in section **Error! Reference source not found.**. Let PS_{mr} be defined as in Ben-Akiva and Bierlaire (1999):

$$PS_{mr} = \sum_{a \in \Gamma_{mr}} \frac{l_a}{L_{mr}} \cdot \frac{1}{\sum_{k \in \tilde{R}_m} \delta_{amk}} \tag{23}$$

where L_{mr} and l_a are measures of impedance, and can either be measured as distance or cost ($l_a = c_a(\mathbf{x})$ and $L_{mr} = c_{mr}(\mathbf{x})$). It is noted that using cost as a measure of impedance implies that PS_{mr} is dependent not only on the composition of the choice set, but also on the flows on the paths in the choice set; thus, when solving for equilibrium, the PS_{mr}-factors need to be updated in every iteration of a solution algorithm, even if no additional paths are added to the choice set. It should be also noted that the allocation of flow at equilibrium may also vary between using cost or length as measure of impedance (Zhou et al., 2012). In the following, it is assumed that the impedance is equal to the flow-dependent cost, but the derivation is the same if using distance.

Expression (22) can be rewritten to express the flow on path r:

$$x_{mr} = \frac{\tilde{c}_{mr}(\mathbf{x})}{\exp\left(\theta \cdot \left(c_{mr}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{mr}\right)\right)\right)} \tag{24}$$

The condition on the transformed costs (17) should hold in equilibrium (as in the MNL case), and combining (17) and (24) with the choice probability for OD-pair m yields the following:

$$\begin{aligned} P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) &= \frac{x_{mr}}{d_m} = \frac{x_{mr}}{\sum_{s \in \tilde{R}_m} x_{ms}} = \frac{\dfrac{\tilde{c}_{mr}(\mathbf{x})}{\exp\left(\theta \cdot \left(c_{mr}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{mr}\right)\right)\right)}}{\sum_{s \in \tilde{R}_m} \dfrac{\tilde{c}_{ms}(\mathbf{x})}{\exp\left(\theta \cdot \left(c_{ms}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{ms}\right)\right)\right)}} = \\ &\frac{\tilde{c}_{mr}(\mathbf{x}) \cdot \dfrac{1}{\exp\left(\theta \cdot \left(c_{mr}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{mr}\right)\right)\right)}}{\tilde{c}_{mr}(\mathbf{x}) \cdot \sum_{s \in \tilde{R}_m} \dfrac{1}{\exp\left(\theta \cdot \left(c_{ms}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{ms}\right)\right)\right)}} = \frac{\exp\left(-\theta \cdot \left(c_{mr}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{mr}\right)\right)\right)}{\sum_{s \in \tilde{R}_m} \exp\left(-\theta \cdot \left(c_{ms}(\mathbf{x}) + \beta_{PS} \cdot \ln\left(PS_{ms}\right)\right)\right)} \end{aligned} \tag{25}$$

This corresponds to the PSL choice probabilities with scale parameter θ and path size parameter β_{PS}. The term $\ln(PS_{mr})$ ranges from $-\infty$ to 0, where 0 arises when path r shares no links with any other path, and the value then decreases with increasing overlap with other paths in the choice set. As $\beta_{PS}<0$, the path-size term for path r increases with decreasing 'distinctiveness' of path r (i.e. with increasing overlap with some other path).

The equivalence shown above yields equivalence between the PSL RSUE on the PSL costs $c_{mr}(\mathbf{x})+\beta_{PS}\cdot\ln\left(PS_{mr}\right)$ ((10) and (11)) and the DUE-like system (20) and (21) when $\tilde{c}_{mr}(\mathbf{x})$ is defined by (22).

4. Measuring Convergence to a RSUE Solution

Traditionally, it has been a challenge to measure convergence for SUE models, and in practice most applications of SUE do not directly measure proximity to equilibrium. A common approach is to use measures of stability, such as change in link flows between iterations (e.g., Liu et al., 2009; Zhou et al., 2012); Chen et al (2014) used the mean square error of the generalised cost of routes within a pre-defined choice set for a paired combinatorial logit model application. In contrast, in the present section we devise an explicit gap function to measure convergence to RSUE. This utilises the introduced transformation functions derived in section 3 for MNL (equation (16)) and PSL (equation (22)) choice models, as well as the derived knowledge that these transformed costs are equal on used paths at equilibrium, in order to propose a novel two-part consistent convergence measure for the proximity to a RSUE(min) or RSUE(max) solution for the MNL/PSL cases. The first part of the measure concerns the convergence to fulfil the SUE conditions among utilised paths (RSUE condition (10)), whereas the second part measures to what degree the criteria on unutilised paths are fulfilled (RSUE condition (11)). The first part is thus 'conditional' on the choice set, and the second part measures the 'convergence' of the composition of the choice sets. It is important to ensure convergence of both measures – obtaining convergence among the used routes does not imply overall convergence if there are additional attractive routes not in the choice set.

The first part is based on applying (to the RSUE transformed costs) the relative gap measure of DUE (Rose et al., 1988). For iteration n, the measure can be computed as:

$$\mathrm{Rel.gap}_n^{\mathrm{Used}} = \frac{\sum_{m=1}^{M}\sum_{r\in\tilde{R}_m} x_{mr,n}\cdot\left(\tilde{c}_{mr}(\mathbf{x}_n)-\tilde{c}_{m,\min}(\mathbf{x}_n)\right)}{\sum_{m=1}^{M}\sum_{r\in\tilde{R}_m} x_{mr,n}\cdot\tilde{c}_{mr}(\mathbf{x}_n)} = \frac{\sum_{m=1}^{M}\sum_{r\in\tilde{R}_m} x_{mr,n}\cdot\left(x_{mr,n}\cdot\exp\left(\theta\cdot c_{mr}(\mathbf{x}_n)\right)-x_{\min\left(x_{mr,n}\cdot\exp\left(\theta\cdot c_{mr}(\mathbf{x}_n)\right)\right)}\cdot\exp\left(\theta\cdot c_{m,\min}(\mathbf{x}_n)\right)\right)}{\sum_{m=1}^{M}\sum_{r\in\tilde{R}_m} x_{mr,n}\cdot x_{mr,n}\cdot\exp\left(\theta\cdot c_{mr}(\mathbf{x}_n)\right)} \tag{26}$$

where $\tilde{c}_{m,\min}(\mathbf{x}_n)$ is the minimum of the transformed cost on paths utilised between OD-pair m for iteration n, and $\tilde{c}_{mr}(\mathbf{x}_n)$ and $c_{mr}(\mathbf{x}_n)$ depend on the selected choice model. As the transformed costs are equal in the RSUE solution, the proposed relative gap is zero at an exact RSUE, and is positive otherwise.

The second part of the convergence measure captures that there may exist unused paths which violate the second RSUE condition. The measure computes how close the costs on the cheapest unused path are to fulfilling the criteria describing unused paths in the RSUE definition. For the RSUE(min) model, this corresponds to investigating whether any unused path is cheaper than the cheapest used path, whereas for the RSUE(max) it requires to determine whether there exists any unused path which is cheaper than the most expensive used path. The measure becomes zero at convergence and it is based on actual costs rather than transformed costs. For the RSUE(min) it is defined as:

$$\mathrm{Rel.Gap}_n^{\mathrm{Unused}} = \frac{\sum_{m=1}^{M} d_m\cdot\left(\min_{\forall r\in R_m, x_{mr}>0}\left(c_{mr}(\mathbf{x}_n)\right)-\min_{\forall r\in R_m}\left(c_{mr}(\mathbf{x}_n)\right)\right)}{\sum_{m=1}^{M} d_m\cdot\min_{\forall r\in R_m, x_{mr}>0}\left(c_{mr}(\mathbf{x}_n)\right)} \tag{27}$$

For the RSUE(max) model it is expressed as:

$$\mathrm{Rel.Gap}_n^{\mathrm{Unused}} = \frac{\sum_{m=1}^{M} d_m \cdot \left(\max_{\forall r \in R_m, x_{mr}>0} \left(c_{mr}(\mathbf{x}_n) \right) - c_{mr,k}(\mathbf{x}_n) \right)}{\sum_{m=1}^{M} d_m \cdot \max_{\forall r \in R_m, x_{mr}>0} \left(c_{mr}(\mathbf{x}_n) \right)} \tag{28}$$

where k refers to the amount of used paths for the corresponding OD-pair m in iteration n and $c_{mr,k}$ $(\mathbf{x}_n)$ is the cost on the current k-th shortest path on the network between OD-pair m.

The computation of the second part of the convergence measure requires for the RSUE(min) the result of a shortest path search based on updated travel costs, and for the RSUE(max) model, the results of a k-th shortest path search based on updated costs. These searches are quite computationally demanding, but the computation of the convergence measure does not induce significant extra calculation time, as these searches are anyhow part of the column generation step in the subsequent iteration of our solution algorithm (see Section 5.2).

The first part of the convergence measure is zero if the choice set consists of only one route for a given OD-pair. This is the case at the end of the first iteration of many traditional SUE solution algorithms based on column generation (e.g., Sheffi and Powell, 1982). The algorithm has (probably) not converged at the end of iteration 1 though, as the distribution of flow to this one route will cause other routes to be attractive. This will be caught by a non-zero value of the second part of the measure, which highlights the need to evaluate both parts of the gap measure. It should be noted that the proposed measures of convergence are not only valid for the solution methods proposed in the present study, but they can also be applied to all other solution methods solving for a closed-form Logit-type RSUE(min) or RSUE(max).

5. RSUE solution methods

In this section, we set out a generic solution algorithm for solving RSUE(Φ) problems, and propose possible approaches to adopt in the two important steps of the algorithm. In doing so, we consider evidence for the likely convergence and computational attractiveness of different variants of the algorithm (learning from experience with DUE and SUE algorithms), these variants to be tested in detail in the numerical experiments of section 6.

5.1. Solving in the space of link- or path flows

The first key question to address is to whether to devise a solution algorithm in the space of link or path flows, and how we can learn from the experience with SUE and DUE methods. Link-based formulations are attractive in avoiding path enumeration, which is potentially computationally demanding. In the case of DUE, while several path-based algorithms have been proposed (see Perederieieva et al., 2015, for a recent paper testing several variants), link-based formulations retain an attraction due to the fact that we can only hope at best for DUE uniqueness in the space of link flows, not path flows. On the other hand, SUE solutions are unique in path and link flows under mild conditions (Cantarella, 1997). Consequently, a one-to-one mapping exists between path and link flows for SUE solutions, making link- or path-based methods in this sense equally attractive. Many link- and path-based solution methods have been proposed for SUE (e.g., Sheffi and Powell, 1982; Damberg et al., 1996; Bekhor and Toledo, 2005; Zhou et al., 2012; Akamatsu, 1996; Bell et al., 1997; Leurent, 1997; Maher and Hughes, 1997).

The RSUE conditions, unlike the DUE and SUE, do not provide uniqueness in link flows under the usual sufficient conditions, nor in path flows, as in general the set of used paths may not be uniquely defined by the RSUE conditions (see Watling et al., 2014). Consequently, a given link-flow allocation cannot be mapped to an RSUE path-flow solution via a one-to-one relation, and in any case, the RSUE conditions may be satisfied by multiple link-flow solutions. Furthermore, the definition of the RSUE conditions clearly distinguishes between used and unused paths, and specifies a criterion to be fulfilled among used paths. As a consequence, the definition of the conditions necessitates the consideration of paths, which has led us to pursue the proposal of a path-based solution algorithm for RSUE.

5.2. RSUE(*Φ*) solution algorithm

We propose an iterative approach to solve for RSUE(Φ) solutions. One iteration of the proposed generic solution algorithm consists of four steps, namely the Column generation phase, the Restricted master problem phase, the Network loading phase and the Convergence evaluation phase.

<table>
<tr><td>Algorithm</td><td colspan="3"></td></tr>
<tr><td>Step 0</td><td colspan="3">Initialisation. Iteration n=1. Perform deterministic all-or-nothing assignment for all $m \in M$ OD-pairs and obtain the flow vector for all utilised paths $\mathbf{X}_n$.
Perform network loading, compute link travel costs $t_a(\mathbf{f}_n)$ on all network links $a \in A$, and compute generalised path travel costs $c_{mr}(\mathbf{X}_n)$. Set n=2.</td></tr>
<tr><td rowspan="2">Step 1</td><td colspan="3">Column generation phase. Let $k_{m,n-1}$ denote the current number of distinct paths in the choice set of used paths for OD-pair m=1...M in iteration n-1.</td></tr>
<tr><td>For RSUE(min):
For each origin, perform a shortest path search to all destinations based on actual link travel costs $t_a(\mathbf{f}_{n-1})$. If for any OD-pair m=1...M a new distinct path i is generated, add</td><td>For RSUE(Φ):
For each OD-pair m∊M, based on actual link travel costs $t_a(\mathbf{f}_{n-1})$, check for a new route to add to the choice set $\tilde{R}_{m,n}$ by applying some path</td><td>For RSUE(max):
Perform a $k_{m,n-1}$-shortest path search for each OD-pair (m=1...M) based on actual link travel costs $t_a(\mathbf{f}_{n-1})$.
If for any OD-pair m=1...M a new distinct path i is generated among the $k_{m,n-1}$ generated</td></tr>
</table>

	it to the choice set $\tilde{R}_{m,n}$ with flow $x_{mi,n-1}$=0.	generation method which supports the fulfilment of the Φ operator. If for any OD- pair m=1...M a new distinct path i is generated, add it to the choice set $\tilde{R}_{m,n}$ with flow $x_{mi,n-1}$=0; if several routes possible, add only the shortest one.	paths, add it to the choice set $\tilde{R}_{m,n}$ with flow $x_{mi,n-1}$=0; if several new distinct paths are generated, add only the shortest one.
Step 2	Restricted master problem phase. Given the choice sets $\tilde{R}_{m,n}$ for all m=1...M, apply the selected inner assignment component and averaging scheme to find the new flow solution $\mathbf{X}_n$.		
Step 3	Network loading phase. Perform the network loading to obtain $\mathbf{f}_n$ from $\mathbf{X}_n$, compute the link travel costs $t_a(\mathbf{f}_n)$, the generalised path travel costs $\mathbf{C}(\mathbf{X}_n)$ and (if relevant/included) the Path Size factors.		
Step 4	Convergence evaluation phase. If the gap measure consisting of the sum of $\mathrm{Rel.Gap}_n^{\mathrm{Used}}$ and $\mathrm{Rel.Gap}_n^{\mathrm{Unused}}$ is below a pre-specified threshold ξ, Stop. Else, set n=n+1 and return to Step 1		

It should be noted that the path flow vector is denoted by **X** rather than **x.** This is to emphasise that in practical implementations it is not possible/practical to operate with the vector **x**, as this requires enumerating the universal choice set for all OD-pairs to obtain its dimension. Rather, the dimension of the flow vector is not pre-specified in practical implementations, but is allowed to increase as the algorithm progresses. The same occurs for the path cost vector **c**(**x**), which we have denoted **C**(**X**) to highlight that this might grow as the algorithm progresses. The elements x_{mr} and c_{mr} thus refer to the vectors **X** and **C**, respectively.

5.3. Column generation phase

The choice set is "systematically" grown based on rules for the generation of new alternatives. The search for new alternatives may be performed in various ways, but in the solution algorithm we have proposed a single shortest path search for the RSUE(min) and a k-shortest path search for the RSUE(max). Basing the search for new paths to introduce to the choice set on the actual costs allows us to attempt to fulfil condition (11) on unused paths.

For choices of the operator Φ other than min or max, alternative path generation techniques may be applied, such as variations of shortest path algorithms (e.g., Akgün et al., 2000; Hunt and Kornhauser, 1997; Lombard and Church, 1993; Van der Zijpp and Fiorenzo-Catalano, 2005), application of heuristic rules (e.g., Ben-Akiva et al., 1984; Azevedo et al., 1993; De la Barra et al., 1993), branch and bound algorithms (Hoogendoorn-Lanser et al., 2006; Prato and Bekhor, 2006), biased random walk algorithm (Frejinger et al., 2009), and breadth first search with network reduction (Rieser-Schüssler et al., 2013). Some of these alternative approaches may also be attractive to apply for the RSUE(min) and RSUE(max). It is however essential that the column generation approach adopted should ensure that the condition (10) on unused paths are fulfilled upon termination of the algorithm.

It should be noted that the proposed approach for the Column generation phase for the RSUE(min) and RSUE(max) induces the RSUE condition (11) to be fulfilled; For the RSUE(min) there is no non-included path which is deterministically shorter than the once already included in the choice set when the algorithm above terminates. The condition (11) is fulfilled for the RSUE(max) model when the algorithm terminates, since then there is no non-included path that is deterministically shorter than the longest path already included in the choice set.

5.4. The Restricted master problem phase

The allocation of flow between the alternatives in the choice sets (the Restricted master problem phase) can be performed by deploying either DUE allocation methods using the transformed costs or SUE allocation methods.

Numerous path-based DUE solution algorithms which equilibrate path costs on used routes are available, such as the method of successive averages All-or-Nothing (MSA AoN, applied in e.g. Bekhor and Toledo (2005)), the Path Equilibrator (Dafermos and Sparrow, 1969), the Disaggregate Simplicial Decomposition (DSD, Larsson and Patriksson, 1992), the Gradient Projection (GP, Jayakrishnan et al., 1994; Chen et al., 2002b), the Social Pressure (Kupiszewska and van Vliet, 1998), the Projected Gradient (Florian et al., 2009) and the slope-based multipath flow update (Kumar and Peeta, 2010) methods. Components of these DUE solution algorithms may be modified to fit into Step 2 by applying them to the transformed costs rather than the actual costs in the inner direction finding step. Thereby the transformed costs are equilibrated, which corresponds to the first RSUE condition (10) being fulfilled. The GP and DSD algorithms are considered to be efficient DUE solution algorithms and a comparison showed better performance of the GP (Chen et al., 2002b). It should be noted that this might be partly due to the generation of smaller choice sets in the GP, that allows for column dropping of unused paths. This advantage of the GP would not transfer to the RSUE case, since the use of the transformed cost would not allow any already included paths to be dropped from the choice set in later iterations.

Another branch of path-based DUE solution algorithms are the algorithms based on path-swapping, which usually swap traffic to/from paths based on cost differences. The swapping has a direct and a plausible behavioural interpretation, namely flow should be swapped to cheap paths and away from costly paths. The algorithms proposed in the literature are differentiated by swapping between pairs of paths (Han, 2007; Carey and Ge, 2012), all paths (Mounce and Carey, 2011), or to only one path (Mounce and Carey, 2011; Nie, 2003).

Among the algorithms tested in Carey and Ge (2012), a pairwise path-swapping algorithm was found to provide stable and fast convergence. The algorithm swaps flow from the most expensive path to the cheapest path, the second-most expensive path to the second-cheapest path, etc., and performs one network loading per iteration. We utilise the introduced cost transformation function to adapt this approach to fit the generic RSUE solution algorithm proposed above by letting Step 2 be comprised of the following.

Step 2	
Step 2.1	For each path r in the choice set for OD-pair m for iteration n, compute the transformed cost $\tilde{c}_{mr}(\mathbf{X}_{n-1})$ according to equation (16) or (22) (dependent on selection of choice model).
Step 2.2	Rank all paths for OD-pair m in ascending order of $\tilde{c}_{mr}(\mathbf{X}_{n-1})$: $\tilde{c}_{mp_1}(\mathbf{X}_{n-1}) \le \tilde{c}_{mp_2}(\mathbf{X}_{n-1}) \le \cdots \le \tilde{c}_{m\bar{p}}(\mathbf{X}_{n-1})$. Pair the paths as $\left(p_1, \bar{p}\right), \left(p_2, \bar{p}-1\right)$, ...: for odd number of paths, the path $\left(\bar{p}+1\right)/2$ will not be paired.
Step 2.3	For each pair $\left(p_i, p_j\right)$, compute the swapping-factor: $$\Gamma\left(p_i, p_j\right) = \frac{\tilde{c}_{mp_j}(\mathbf{X}_{n-1}) - \tilde{c}_{mp_i}(\mathbf{X}_{n-1})}{\sqrt{\left(\tilde{c}_{mp_j}(\mathbf{X}_{n-1})\right)^2 + \left(\tilde{c}_{mp_i}(\mathbf{X}_{n-1})\right)^2}}$$ For each pair $\left(p_i, p_j\right)$, perform the swap: $$x_{mp_i,n} = x_{mp_i,n-1} + \gamma_n \cdot \Gamma\left(p_i, p_j\right) \cdot x_{mp_j,n-1}$$ $$x_{mp_j,n} = x_{mp_j,n-1} - \gamma_n \cdot \Gamma\left(p_i, p_j\right) \cdot x_{mp_j,n-1}$$ For odd number of paths, set $$x_{m,(\bar{p}+1)/2,n} = x_{m,(\bar{p}+1)/2,n-1}$$

As the path-swapping algorithm equalizes the transformed costs, we know from section **Error! Reference source not found.** that the RSUE condition (10) is also fulfilled upon termination. We note that the pairwise path-swapping algorithm by Carey and Ge (2012) is especially attractive for Logit-type choice models. This is due to a special characteristic of the Logit-type choice models, namely the Independence from Irrelevant Alternatives (IIA) property. This induces the ratio of the choice probabilities between two paths to not depend on any other path, thereby allowing SUE to be defined as a condition on pairs of routes. Basing the solution methods on pairwise path-swapping makes good sense in this framework by only considering the two paths between which flow is swapped in the swapping process.

Using the cost transformation in combination with a path-based DUE cost equilibration is however not the only option. Components of path-based SUE solution algorithms can also be adapted fit Step 2 of the proposed generic solution algorithm. Among these are modified versions of two of the most promising path-based DUE solution methods, namely the DSD and GP methods. These have been modified to apply to SUE with an MNL choice model (Damberg et al., 1996; Bekhor and Toledo, 2005) and a Cross-Nested Logit model (Bekhor et al., 2008; Zhou et al., 2012). While the latter three references use a pre-defined choice set (which in the adaptation to the RSUE solution algorithm would be defined by the routes generated by the Column generation phase), it is worth noting that the algorithm proposed by Damberg et al. (1996) allows augmenting the choice set. Damberg et al. (1996) noted that this augmentation can be achieved by following one of many strategies. Indeed, even though the RSUE model had not then been defined, one of their suggested strategies generates a solution fulfilling the MNL RSUE(min) conditions! Their suggestion fits as one instance of the generic RSUE(min) solution algorithm we propose. Regarding the computational attractiveness of the SUE adaptiations of the GP and DSD algorithms, Bekhor and Toledo (2005) found that they perform similarly when applied to a small grid network and the

Sioux Falls network. Bell et al. (1997) also formulated a path-based SUE solution algorithm which augments the choice set; again, even though RSUE had not then been defined, this algorithm also automatically produces an MNL RSUE(min) solution!

The averaging scheme of Step 2 involves weighing the current solution with the found auxiliary solution by using a step-size γ_n. To avoid potentially obtaining negative path flows, the step-size should be chosen such that $0 \leq \gamma_n \leq 1$. The step-size can be determined in various ways, for example pre-determined (such as in the Method of Successive Averages, MSA, and its variants), as well as various versions of the Armijo (1966) line search method, such as those described by Chen et al. (2012, 2013), Xu et al. (2012) and Zhou et al. (2012). Bekhor et al. (2007) compare the MSA, Armijo's approximation method and the computation of the exact optimal step-size, and find that Armijo's approximation performs best in terms of computation time until convergence. MSA is known for requiring many iterations before convergence, because the auxiliary flow pattern generated at each iteration contributes equally to the final solution (e.g., Bekhor et al., 2007). Liu et al. (2009) test different alternative pre-defined averaging schemes, and introduce the method of successive weighted averages (MSWA). While being pre-defined, the MSWA allows giving higher weight to auxiliary flow patterns from later iterations, and the step-size γ_n at iteration n is defined as the following.

$$\gamma_n = \frac{n^d}{1^d + 2^d + ... + n^d} \tag{29}$$

where $d \geq 0$ is a real number. Increasing the value of d moves more flow towards the auxiliary solution. The MSA is a special case of the MSWA, namely when d=0. Liu et al. (2009) also introduce the self-regulated averaging method, which has been applied with success in various SUE models (e.g., Yang et al., 2013; Xu et al., 2013; Kitthamkesorn and Chen, 2013, 2014; Chen et al., 2014; Yao et al., 2014). This updates the step-size in each iteration according to the development of the 'residual error' (the difference between the current solution and auxiliary solution) across the current and previous iteration.

5.5. Convergence of proposed solution methods

To obtain convergence, both RSUE conditions have to be fulfilled. Condition (11) on unused routes will always be fulfilled upon termination of our generic algorithm, as Step 1 requires that if additional attractive routes (violating condition (11)) exist, these will be added to the choice set. The other RSUE condition (10) is fulfilled if the flow is allocated among the paths in the restricted choice sets so that the flow solution fulfil the 'SUE' condition among these paths. To our knowledge, no proof of convergence has been given for the standard DUE version of the pairwise path-swapping algorithm. The proposed solution algorithm based on pairwise path-swapping may thus be seen as a heuristic only. However, Carey and Ge (2012) provide numerical evidence of promising convergence in the DUE case, which matches with our own experience of applying these methods in an RSUE setting to the transformed costs (section 6). Some of the other path-based DUE and SUE solution algorithms (such as the SUE DSD proposed by Damberg et al. (1996)) have been proven to converge under certain assumptions, and thus applying e.g. the Damberg et al. (1996) algorithm for the Restricted master problem phase (Step 2) will guarantee convergence to an RSUE solution.

5.6. Computational performance of the algorithms

For large-scale applications, it is important that the algorithms adopted are computationally attractive. The path searches (Column generation phase) and the network loading are usually by far the most expensive components in practical implementations. The number of path searches and network loadings needed per iteration, as well as the complexity of the path searches, varies across alternative algorithms. Regarding the Column generation phase, the RSUE(min) is far less computationally expensive per iteration than the RSUE(max), for several reasons. Firstly, searching for only one shortest path is far less computationally demanding than searching for k shortest paths. Secondly, single shortest path algorithms are available to identify simultaneously the shortest path from an origin to all destinations, whereas the available k-shortest path algorithms find only the k shortest paths between a single origin and destination. In general, per iteration of the Column generation phase we require z searches, each with calculation complexity $O(V \cdot \log(V) + A)$, for the RSUE(min), while z^2 searches, each with calculation complexity $O(k_m \cdot V \cdot (V \cdot \log(V) + A))$, are needed for the RSUE(max) (Cormen et al., 2009).

Regarding the number of network loadings, the proposed solution algorithm performs one network loading as part of Step 3. However, it should be noted that some of the proposed approaches for the Restricted master problem phase may also require one or several network loadings. The pairwise path-swapping algorithm with pre-determined step-size is attractive by not requiring any network loadings in the Restricted master problem phase, whereas algorithms such as that of Han (2007) require an additional network loading per OD-pair per iteration in order to determine the step-size. The DSD algorithm proposed by Damberg et al. (1996) is potentially highly time-consuming per iteration, as it iterates the inner assignment step until full convergence among the choice set before the Column generation phase is re-evaluated. This requires numerous time-consuming network loadings (as well as Armijo line searches).

The calculation complexity thus varies greatly between specifications of the generic solution algorithm. Comparison of convergence speed across algorithms therefore needs to be based on computation time, not only the number of iterations.

6. Numerical experiments

We have evaluated several different versions of the proposed generic solution algorithm numerically on two case-studies, namely the well-known Sioux Falls network and the large-scale Zealand network.

6.1. Sioux Falls network

The Sioux Falls network contains 76 links and 528 OD-pairs between which there is a non-zero demand[3]. We present the application of several instances of the solution algorithm with the focus on the MNL model. Firstly, the RSUE(min) as well as the RSUE(max) problem are addressed by applying the cost transformation to solve the Restricted master problem phase by using the pairwise path-swapping algorithm introduced in section 5.4 All assignments used θ=0.1 and were done using MATLAB.

6.1.1. MNL RSUE(min) and MNL RSUE(max)

Figure 1 reports the proposed two-part convergence measure for the path-swapping MNL RSUE(min) and the MNL RSUE(max)[4] solution algorithms with d=0. As can be seen, the MNL RSUE(min) as well as the MNL RSUE(max) solution algorithms both provided promising convergence patterns. It is noteable that after only a small number of iterations (around 7), the equilibrium choice sets for the MNL RSUE(min) had converged (with a few exceptions), with the remaining effort then expended on obtaining a good flow allocation among these routes. As might be expected, the MNL RSUE(max) model requires more iterations to converge the choice set (as indicated by the gap measure on unused routes). As may be expected, the gap measure of used routes (measuring continuously-changing allocations of flow across the choice set) shows a smoother, monotonic convergence than the gap measure for unused routes (which is intended to be sensitive to discrete changes to the composition of the choice set).

[Insert Figure 1 about here]

[3] See Bar-Gera (2013) for a detailed description of the network structure, performance and demand. Note that the travel cost functions are constituted solely by the travel time.

[4] It is important to note that the gap for unused paths of the RSUE(max) cannot be directly compared to that of the RSUE(min), as the measures are computed differently. They should only be compared for different algorithms for either RSUE(min) alone, or for RSUE(max) alone.

Figure 2 illustrates the development of the minimum, average and maximum size of the path choice sets per OD movement, as the solution algorithm progresses[5]. The difference in the conditions to be satisfied in the RSUE(min) and RSUE(max) models would be expected to result in larger choice sets in the latter. The average choice set size of the MNL RSUE(max) was approximately 7.4 from iteration 50 and onwards, which is larger than the largest choice set generated by the MNL RSUE(min) algorithm (7 paths). The choice sets had an average size of 2.5 from iteration 5 onwards when solving for the MNL RSUE(min) flow solution.

[Insert Figure 2 about here]

Figure 3 shows a typical example of the generated choice set for one OD-pair. The choice set of the MNL RSUE(min) consisted of three paths, whereas the choice set of the MNL RSUE(max) contained 10 paths (3 of which are the paths of the MNL RSUE(min)).

[Insert Figure 3 about here]

This example of used paths highlights an important comparative issue with an alternative strategy of solving for a traditional path-based SUE based on a fixed, pre-specified master choice set based on free-flow travel times, since these paths are avoided by the RSUE model in this case. The path 1-3-4-5-9-8-7-18-20 (path 3) carried 7% and 16% of the total demand between the selected OD-pair in the MNL RSUE(max) and MNL RSUE(min) solutions, respectively. This path was however the 41st shortest path when based on free flow travel time (34 minutes as opposed to 22 minutes of travel time for the shortest). Consequently, this obviously attractive path would not have been generated to the choice set if a k-shortest path approach with k<41 had been used for pre-generating the path choice sets.

[5] Iterations 1-50 reported. Minimum and maximum choice set size does not change after iteration 50, and the mean choice set size only changes marginally.

6.1.2. Alternative approaches to the restricted master problem phase

This section presents the results of an evaluation of four alternative possible approaches for the Restricted master problem phase for the MNL RSUE(min) problem. The first two alternatives exploited the RSUE transformation proposed in section 3, by applying alternative path-based DUE solution algorithms to the transformed cost functions: (i) a pairwise path-swapping algorithm (MNL Path Swap) as described in section 5; (ii) an all-or-nothing algorithm assigning all traffic to the path with the lowest transformed cost (MNL AoN). The third alternative tested was: (iii) an algorithm using the MNL probability formula to obtain an auxiliary solution (MNL Inner Logit). Each of these three alternatives were tested with two MSWA averaging strategies, namely the MSA (d=0) and one which 'trusted' the auxiliary solution more (d=2). The final alternative tested was: (iv) the SUE DSD solution algorithm (MNL DSD, Damberg et al., 1996). As aforementioned, the MNL DSD iterates within the Restricted master problem phase until a converged solution is found for the 'conditional' choice set. In the application of the MNL DSD algorithm, the Restricted master problem phase was terminated (and new possible attractive paths were identified) when the relative gap on used routes reached 0.01%.

Figure 4 and Figure 5 illustrate the convergence of the algorithms. The relative gap for the choice set composition converged within the first 4-15 iterations (the MNL AoN with d=2 required 30 iterations), indicating that the final choice sets were generated within the first iterations. The convergence of the distribution of flow (which is conditional on the choice set) also converged fast for most algorithms. The MNL DSD algorithm required only a very few iterations to converge to a very low value of the relative gap measures. The MNL Inner Logit with d=2 was the second fastest (in terms of number of iterations), followed by MNL Path Swap with d=2. However, both required considerably more iterations than the MNL DSD algorithm to converge. The MNL AoN, the MNL

Inner Logit and MNL Path Swap path-swapping algorithms with d=0 required additional iterations to reach convergence. The worst performance was seen by the MNL AoN approach with d=2.

[Insert Figure 4 about here]

[Insert Figure 5 about here]

The calculation complexity however differed between algorithms, and especially the MNL DSD algorithm was notably more demanding in requiring iterations in the Restricted master problem phase until near-convergence for the 'conditional' choice set. This aspect has also to be considered in the evaluation of algorithms, and Figure 6-Figure 7 show the convergence as a function of the computation time rather than the number of iterations.

[Insert Figure 6 about here]

[Insert Figure 7 about here]

The MNL DSD was not so attractive when considering computation time, as the final choice sets were identified after 41 minutes, whereas these were determined within 2-9 minutes for the remaining algorithms. This means that while the distribution of flow among the available choice set converged fast to an MNL solution (due to the line search, see the dotted line in Figure 7), the overall convergence of the MNL DSD was slow. Looking at the overall convergence, the MNL Inner Logit with d=2 thus converged the fastest to the final choice sets with a relative gap among the used routes of less than 0.01%.

None of the four alternative algorithms required simulation, and each algorithm thus converges to the same solution for repeated applications, given the same initial conditions. The RSUE conditions however do not induce uniqueness, and the algorithms tested may therefore converge to different solutions which are all MNL RSUE(min) solutions. The non-uniqueness lies within the generation of the choice sets, which is also illustrated by the varying average and maximum choice set size across algorithms (Table 1).

[Insert Table 1 about here]

Using d=0 generated quite similar average choice set sizes across the algorithms (2.39-2.59), whereas the average choice set size varied more when d=2 (3.03-4.01). The larger choice sets when d=2 were due to the larger oscillations of flow in the initial iterations, which were a consequence of the larger step-size moving a larger share of the flow to the auxiliary solution. This caused larger changes in the costs of the links, which induced more distinct routes to be generated. The MNL DSD solution algorithm generated small choice sets. This was (also) a consequence of the flow solution found in the Restricted master problem phase – the “inner equilibration” before considering adding additional routes to the choice sets induced less flow fluctuations between iterations.

6.2. Zealand network

Two versions of the proposed RSUE(min) solution algorithm were applied to the large-scale Zealand network. Approximately 2.5 million people live in the area covering 9200 km^2, and the digitised road network representation consists of 12,015 links and 429 zones. The demand matrix applied contained a total of 3.2 million trips across 19 different user classes and two vehicle types (car and lorry) to be assigned to the road network[6]. The study area consists of urban as well as rural areas, and the congestion level is spatially distributed as well as distributed across road type classifications (see Figure 8). This is, thus, an ideal case in which to test an RSUE-based approach; DUE is unattractive as it will not be able to generate realistic levels of path dispersion in the uncongested parts of the network, and SUE is computationally unattractive as it effectively aims to disperse small amounts of traffic to all routes in an enormous network.

[Insert Figure 8 about here]

[6] The Zealand network is a subset of the network to be used in the Danish National Model, currently under development at DTU Transport.

The two tested versions of the algorithm were differentiated by the approach to the Restricted master problem phase. One used the pairwise path-swapping approach presented in section 5.4 (Path Swap), while the other deployed an approach where the auxiliary solution is found by applying the corresponding logit choice probability formula directly (Inner Logit). The algorithms have been implemented in the Traffic Analyst traffic assignment module for ArcGIS (Rapidis, 2013), and both accommodated the use of the MNL and the PSL models.

In the application we specified θ=0.2 and the step-size constant d=2 (an initial test comparing d=0 to d=2 found best performance in terms of convergence speed when d=2, as we also reported earlier in the Sioux Falls application). The generalised travel costs were constituted by a weighted sum of free flow travel time, congested travel time, travel distance, and travel (monetary) cost.

The calculation time per iteration was approximately one minute for both solution algorithms when using a computer with a 3.2 GHz Intel Zeon CPU and 32 GB RAM. Figure 9 illustrates the convergence pattern as a function of the iteration number for the MNL choice model.

[Insert Figure 9 about here]

The figure indicates that the applied algorithms seem able to maintain the convergence efficiency noted in the Sioux Falls case when applied to a much larger-scale network. The distribution of flow among used routes seemed to converge better for the Path Swap algorithm in the first 50 iterations, but the relative gap on used routes reached a very low level of 0.001% at iteration 50 for both algorithms. From this iteration the relative gap among used routes was at the same level for the two algorithms, although with some fluctuations for the Path Swap algorithm. It is however important to remember that this measure of convergence is conditional on the choice set composition. The corresponding gap measure converged fast for both algorithms. The measure quickly reached zero, but for the Path Swap algorithm it repeatedly increased slightly from zero as the algorithm iterated (the redistribution of the flow caused new routes to be attractive). It was not until iteration 129 that no extra routes were added to any of the choice sets for the algorithm based on path-swapping. The relative gap did however never grow large in these increases from zero, indicating that it was only for a few OD-pairs that new routes were attractive. This is also indicated by the development of the average choice set size, which was almost constant from iteration 10 onwards. The choice sets were thus (generally) generated within the first few iterations.

We applied the algorithms to the original OD-matrices scaled by different factors ranging from 1 to 2. The number of iterations needed to obtain convergence as well as the size of the choice sets generated showed the expected to increase with increasing scale-factor. However, both algorithms converged within a reasonable number of iterations for all demand levels tested, and overall both algorithms seems to be robust towards the general congestion level in the network it is applied to.

The converged solution generated was not the same for the two algorithms, as the composition of the choice sets varied between them. The Inner Logit generated, on average, slightly smaller choice sets (Table 2). This was probably a consequence of a more ‘equal’ distribution of flow between the paths in the initial iterations (smaller oscillations), as indicated by a lower relative gap on used routes during the first few iterations. A high share of the OD-pairs only contain one or two paths, and an average choice set size of 2.5-3 routes may seem small. However, this should be seen in light of the network composition; the case-study area includes, in addition to urban areas, large rural areas in which there is no congestion and only one or two relevant alternatives. An analysis of the spatial distribution of the choice set size showed that the choice sets generated for trips conducted in rural areas were considerably smaller than those generated for urban trips.

Finally, we note that while identifying a flow solution that fulfils the traditional SUE conditions would require dispersion of traffic across millions of routes for each OD-pair, the solution algorithms proposed found converged solutions satisfying the RSUE(min) conditions with a relative modest (and computationally attractive) maximum size of the generated choice sets of 11 routes (Table 2). Monitoring the memory usage supports the computational feasibility of the path-based solution algorithms in modern computers; the large-scale Zealand applications used approximately 12 GB of RAM.

[Insert Table 2 about here]

We also implemented and tested the corresponding algorithms for the PSL choice model. An analysis of the results showed that, in general, the convergence pattern as well as choice set composition and size were similar to the corresponding results obtained when using the MNL model. In the PSL application we tested different values of β_{PS} (ranging from –25 to 0) and evaluated the results by comparing the link flows obtained with corresponding real life observed link flow counts (for 1169 links distributed across the network). The evaluation was done using the coefficient of determination (R^2) obtained from a linear regression of the modelled flows as a function of the observed flows (using the Path Swap algorithm). In general very high correspondence was found, with R^2=0.9444 when β_{PS} =0 (MNL case), with slightly declining R^2 with increasing negative value of β_{PS} until R^2=0.9404 when β_{PS} =-25.

Last, we performed a qualitative, disaggregate evaluation of the choice set composition and flow distribution for one OD-pair within the study area. Both algorithms generated the same five distinct routes shown in Figure 10 for the MNL as well as the PSL choice models. The trip was a commuting trip and the size and composition of the choice sets seems plausible from our local knowledge of the network; one alternative (Path 3) used motorways as far as possible, one alternative avoided motorways but rather used uncongested minor local roads (Path 2) and three alternatives were versions of the lowest cost route using a combination of motorways and minor local roads.

[Insert Figure 10 about here]

The generalised route costs and flow shares for the MNL and the PSL choice models can be seen in Table 3 (results from two different β_{PS} values reported).

[Insert Table 3 about here]

In the MNL case, the three routes with the lowest generalised costs (paths 1,4 and 5) attracted 78% of the traffic, whereas path 2 (which has almost no overlap with other used paths) only attracted 9.7% of the flow, despite being only 4% more expensive than the cheapest path. In the PSL case, accounting for path overlap changed the distribution of flow between the path as well as the path costs (through redistribution of flows for all OD-pairs). While paths 1, 3 and 5 highly overlap, path 2 is the most distinct path, and thus was the one that attracted flow from the other paths when the PSL results are compared to the MNL case. The share on path 2 ranged from 15.2% when β_{PS} =-3 to as much as 30.9% when β_{PS} =-8.5. This highlights the need for aggregate as well as disaggregate analysis when evaluating the models; while on an aggregate level it was difficult to choose between the models (and in fact the MNL performed a little better in reproducing link counts), accounting for path overlapping (by choice of a suitably-calibrated β_{PS} value) can have a major influence on the distribution of flow among paths. Such a disaggregate-level calibration would, however, require more informative data than link counts alone.

7. Discussion & Future Research Directions

We demonstrated that the k-shortest path approach in the Column generation phase for the RSUE(max) was feasible for the Sioux Falls network, but its calculation complexity led us to not pursue to apply it to the Zealand network. We have supported this by performing some initial large-scale tests of the k-shortest path search algorithm of Yen (1971), for which we found computation times per OD-pair that would cause infeasibly long calculation times in an iterative algorithm on large-scale networks. Consequently, while it is possible in principle to apply the proposed k-shortest path approach to any size of network, more research is needed to deduce a sufficiently efficient choice set generation approach to the RSUE (max) for very large networks. An alternative computational approach that might be pursued in the future could be to reduce the 'solution space' through techniques proposed for network aggregation (e.g. Connors and Watling, 2015). Alternatively, it could also be interesting to investigate other formulations of the operator $\Phi\left(\{c_{ms} : s \in \tilde{R}_m\}\right)$, which allows application of efficient choice set generation algorithms, while posing stricter requirements on the costs of unused paths than the RSUE(min) does.

Our experiments show that the efficiency of a step-size strategy varies across networks and demands, and more research could be put into how to systematically choose a suitable step-size strategy based on the characteristics of a network. While the DSD algorithm of Damberg et al (1996) provided slow overall convergence, the line search strategy caused the inner assignment problem to be solved very fast. Therefore, a promising possibility to test may be to implement the line search strategy in the Inner Logit and Path Swap algorithms (i.e. only one line search step per iteration, rather than an equilibration as in the DSD algorithm). Also, it would be interesting to study the effect on convergence speed of performing the line search and network loading per OD-pair ('one-at-a-time'), rather than line search across all OD-pairs before performing the network loading ('all-at-once'). In tests with other models, such a strategy was found to work well for the GP algorithm, especially when only a few OD-pairs require path flow updating (Chen and Jayakrishnan, 1998; Chen et al., 2002a). Alternatively, the self-regulated averaging method has also been successfully implemented in recent SUE applications (Xu et al, 2012; Kitthamkesorn and Chen, 2014), and the effect of using this averaging method would also be an interesting future research direction.

Consistent evaluation of convergence has traditionally been a challenge for conventional SUE solution algorithms, and so we believe our two-part RSUE gap measure to be particularly attractive. From our experience and qualitative evaluation of the results, we have found that using 0.01% as a threshold for the sum of the two parts of the gap measure seems to produce reasonably converged solutions. We note that Boyce et al. (2004) also proposed a threshold of 0.01% for the DUE relative gap measure.

We compared the modelled link flows to real-life observed link flows for the Zealand application, and found remarkably high correspondence. This was despite the parameters not having been properly estimated to the RSUE, which would possibly generate even higher correspondence. An interesting finding is that the correspondence decreases with increasing weight on the path size correction term. This does however not mean that no correction for path overlapping should be done, as this might lead to poor fit at a disaggregate level to the distribution of flow between the paths or to the set of routes used in real-life (as highlighted by our study of the Zealand example). A further analysis of this could e.g. compare generated and corresponding observed paths to evaluate whether the real-life observed paths are represented in the choice sets and how large a flow-share these are assigned.

8. Conclusions

The paper addressed the problem of applying our recently-proposed RSUE(Φ) model framework to networks of varying size. Our results have demonstrated the applicability, convergence and scalability of different variants of the RSUE(min) and RSUE(max) solution algorithms on the well-known Sioux Falls network as well as the large-scale Zealand network.

The Sioux Falls application compared several different approaches to allocate flow between routes, and found that the algorithms, in general, converged reliably to an RSUE solution. The algorithms were analysed based on the number of iterations they required, but also importantly (given their different per-iterations requirements) on computation time. It was seen that the strategy for step-size determination highly influences the convergence speed, as would be anticipated.

Two promising algorithms were tested for the RSUE(min) on the Zealand network, one utilising a cost transformation function to apply a DUE-based solution algorithm based on pairwise path-swapping and the other an adaptation of an SUE-based algorithm utilising the closed-form choice probabilities directly to find the auxiliary solution.These converged fast to fulfil the underlying RSUE conditions and were efficient in generating the choice sets within the first few iterations. The equilibrated non-universal choice sets were reasonable and computationally attractive in size. We found that the RSUE solutions generated provided a good aggregate-level fit to real-life observed link flows, for both the MNL and the PSL choice models.

Acknowledgements

The financial support of the Danish Council for Strategic Research to the project "Analyses of activity-based travel chains and sustainable mobility" (ACTUM) is gratefully acknowledged. We also thank three anonymous referees for their insightful comments, which helped us to improve an earlier draft of this paper.

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Tables and Figures

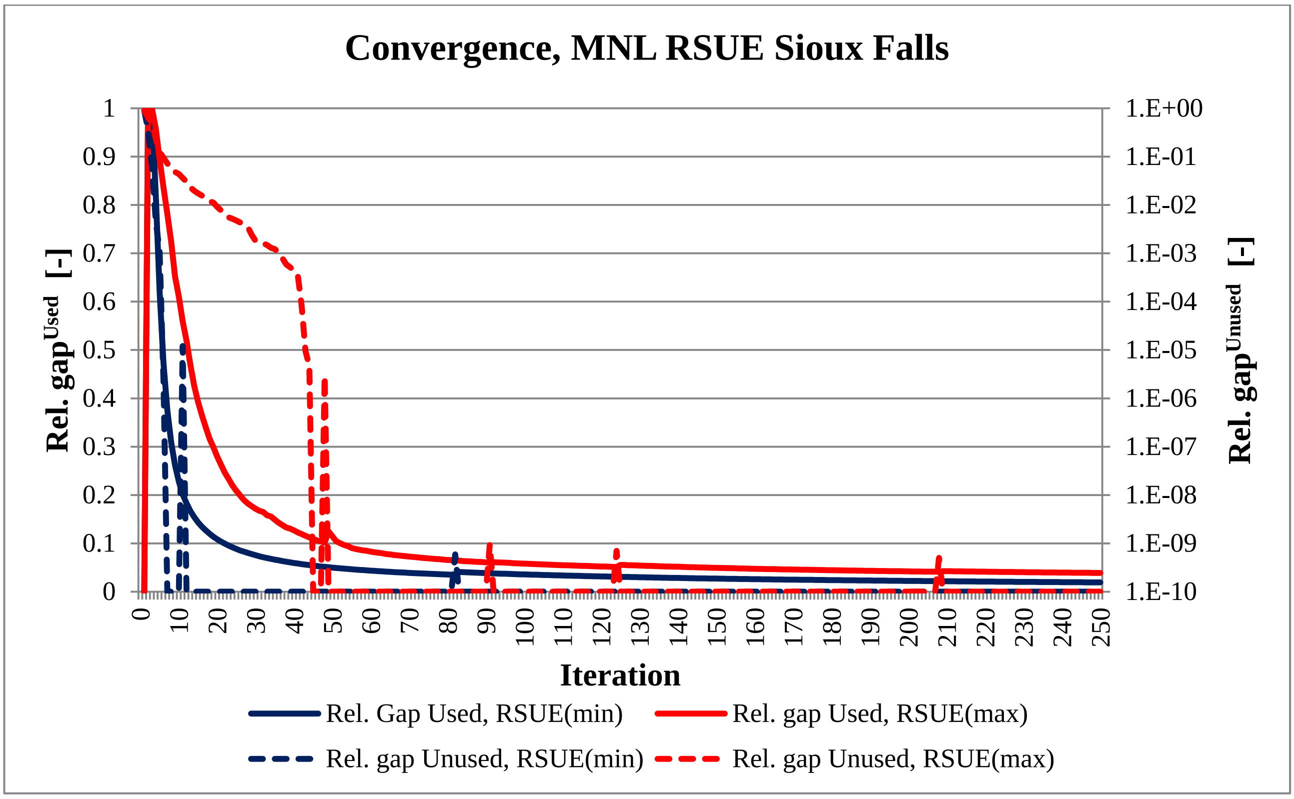


Figure 1. Convergence of pairwise path-swapping algorithm with MSA step-size, for MNL RSUE(min) and MNL RSUE(max) models, Sioux Falls network.

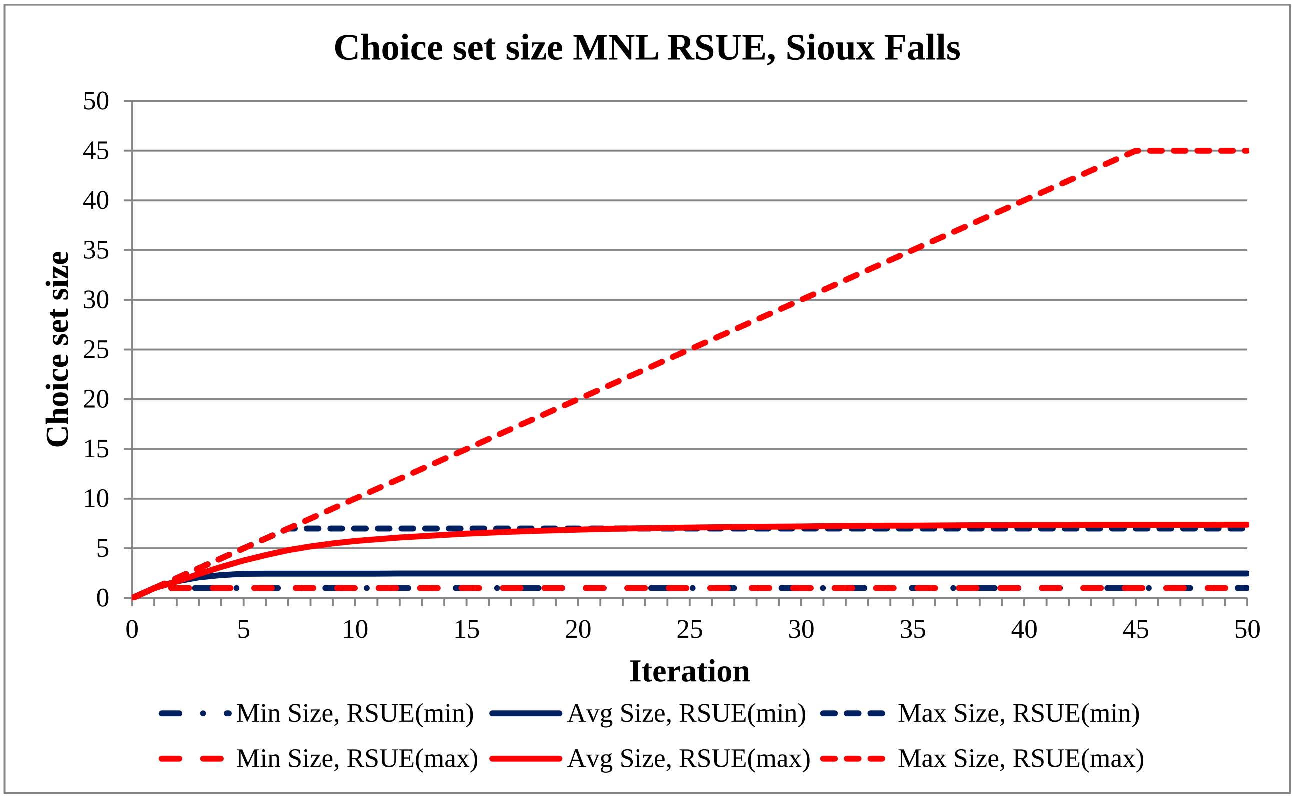
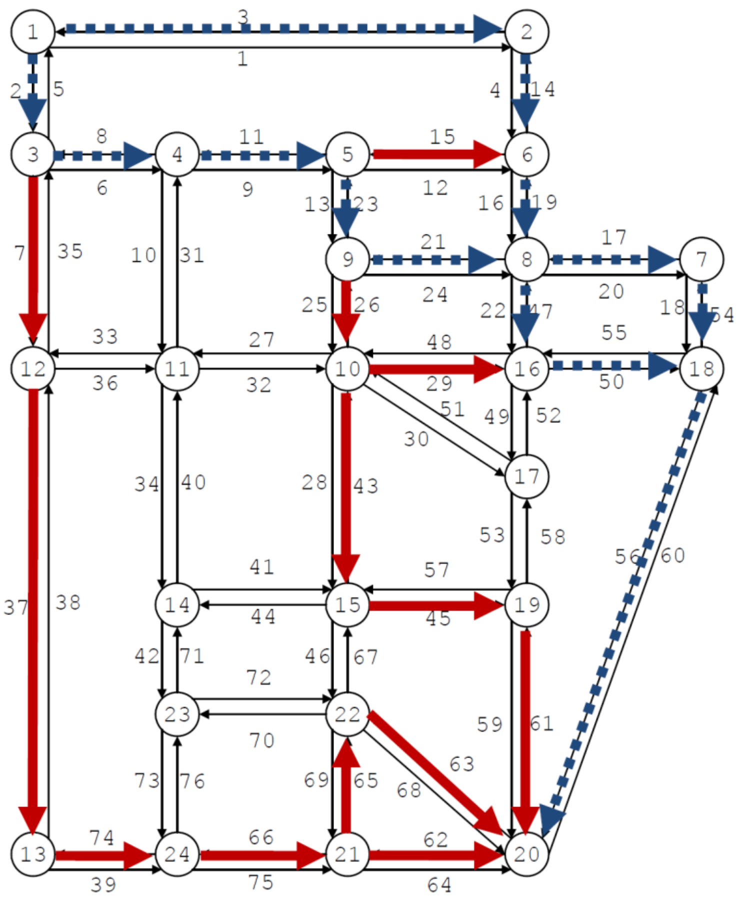


Figure 2. Change in choice set size per OD across iterations for pairwise path-swapping algorithm with MSA step-size, for MNL RSUE(min) and MNL RSUE(max) models, Sioux Falls network.



Path	Node sequence
1	1-2-6-8-7-18-20
2	1-2-6-8-16-18-20
3	1-3-4-5-9-8-7-18-20
4	1-3-4-5-6-8-7-18-20
5	1-3-4-5-6-8-16-18-20
6	1-3-4-5-9-8-16-18-20
7	1-3-4-5-9-10-16-18-20
8	1-3-4-5-9-10-15-19-20
9	1-3-12-13-24-21-22-20
10	1-3-12-13-24-21-20

Figure 3. Example of utilised path choice set 1 OD-pair, Sioux Falls. Left: Dashed links are links used in RSUE(min), dashed+continuous links are links used in RSUE(max). Right: Definition of generated paths – paths 1-3 generated in RSUE(min), paths 1-10 generated in RSUE(max).

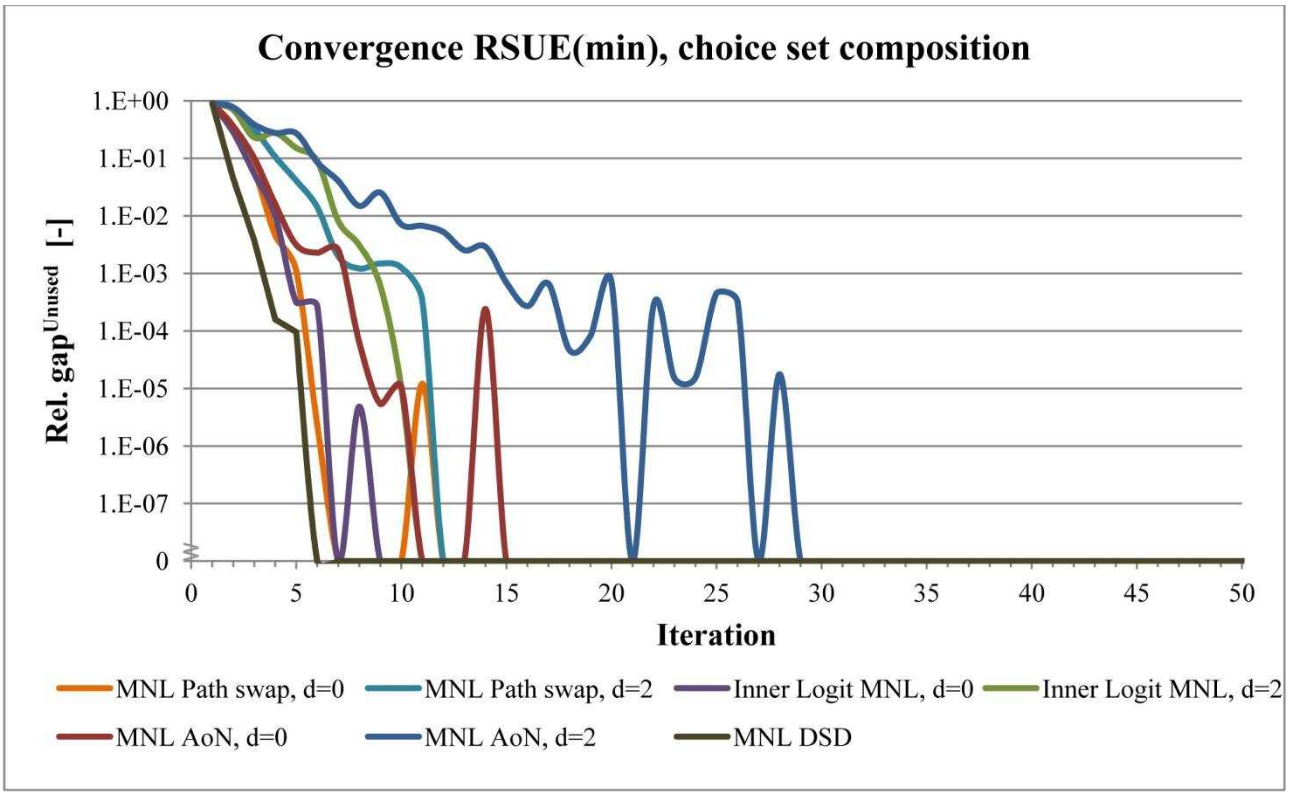


Figure 4. Relative gap measure for convergence of choice set composition as function of iteration number, Sioux Falls application. Notice the log-scale on the vertical axis.

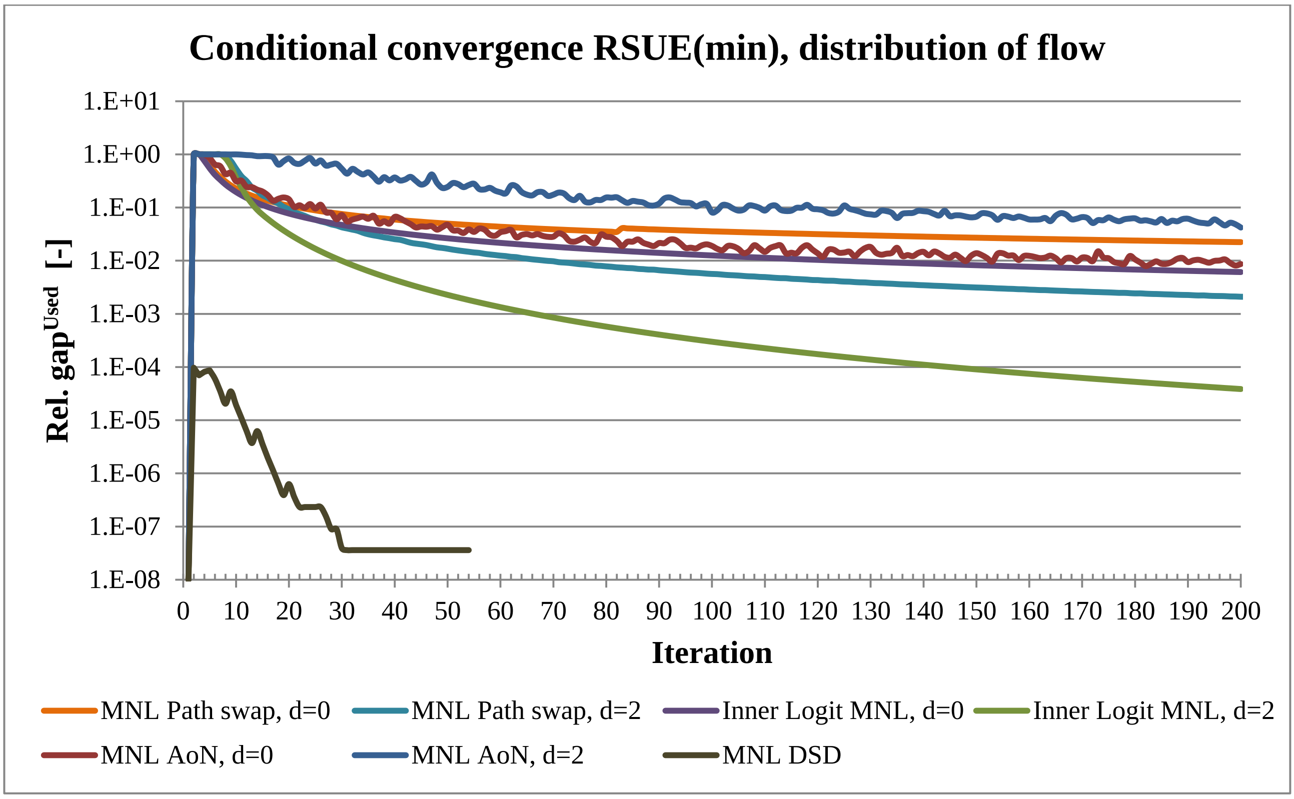


Figure 5. Relative gap measure for convergence of flow distribution among routes in the choice set as function of iteration number, Sioux Falls application. Notice the log-scale on the vertical axis.

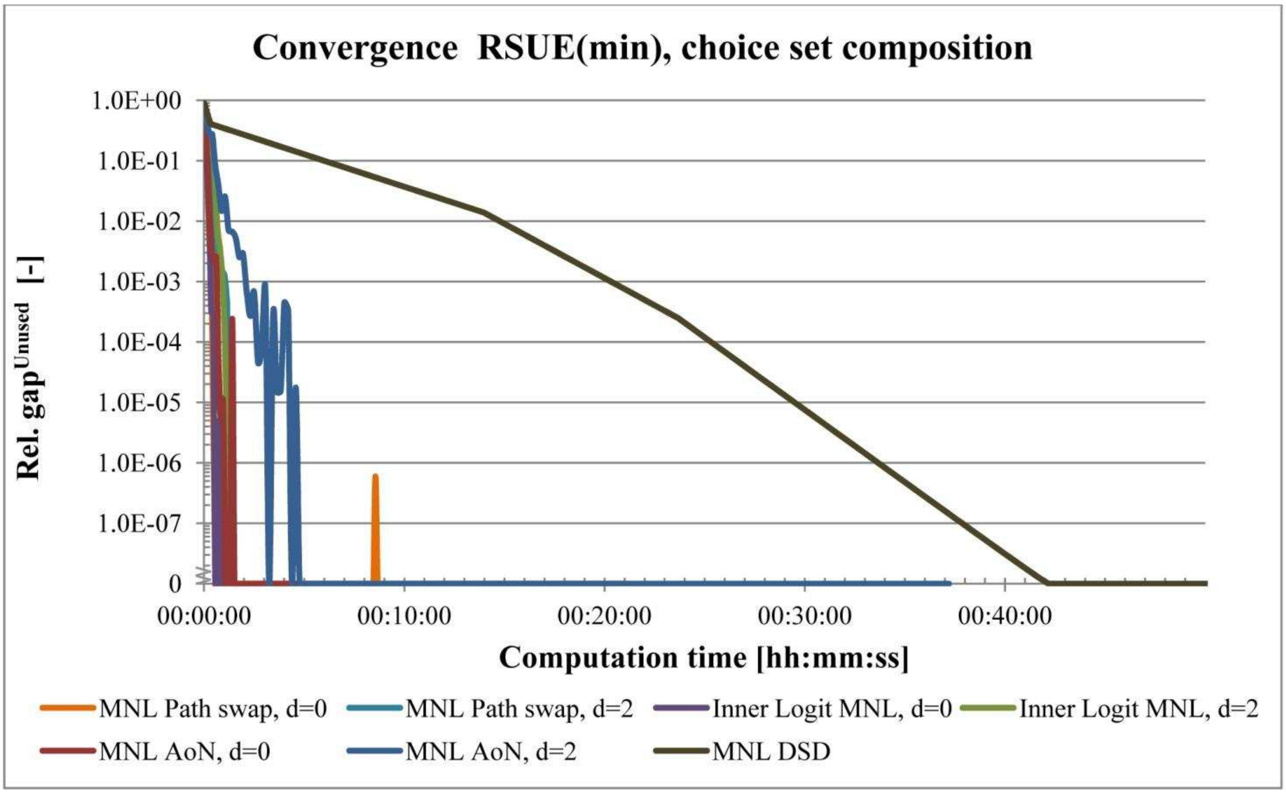


Figure 6. Relative gap measure for convergence of choice set composition as function of computation time, Sioux Falls application. Notice the log-scale on the vertical axis.

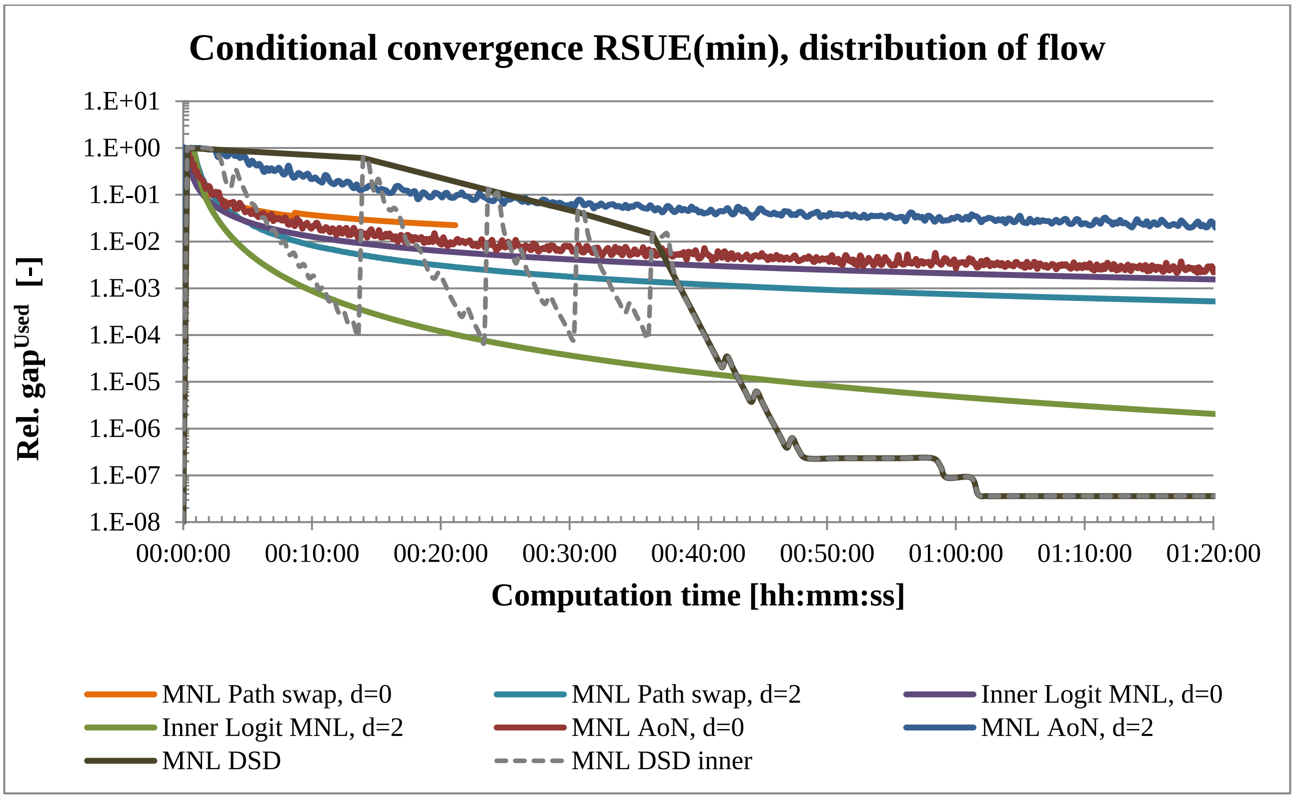


Figure 7. Relative gap measure for convergence of flow distribution among routes in the choice set as function of computation time, Sioux Falls application. Notice the log-scale on the vertical axis.

Table 1. Choice set size characteristics, Sioux Falls.

Method	Step size	Choice set size		
	d	Min.	Avg.	Max.
AoN	0	1	2.59	7
	2	1	4.01	10
Path Swap	0	1	2.46	7
	2	1	3.03	7
Inner Logit	0	1	2.39	6
	2	1	3.37	9
DSD	-	1	2.05	4

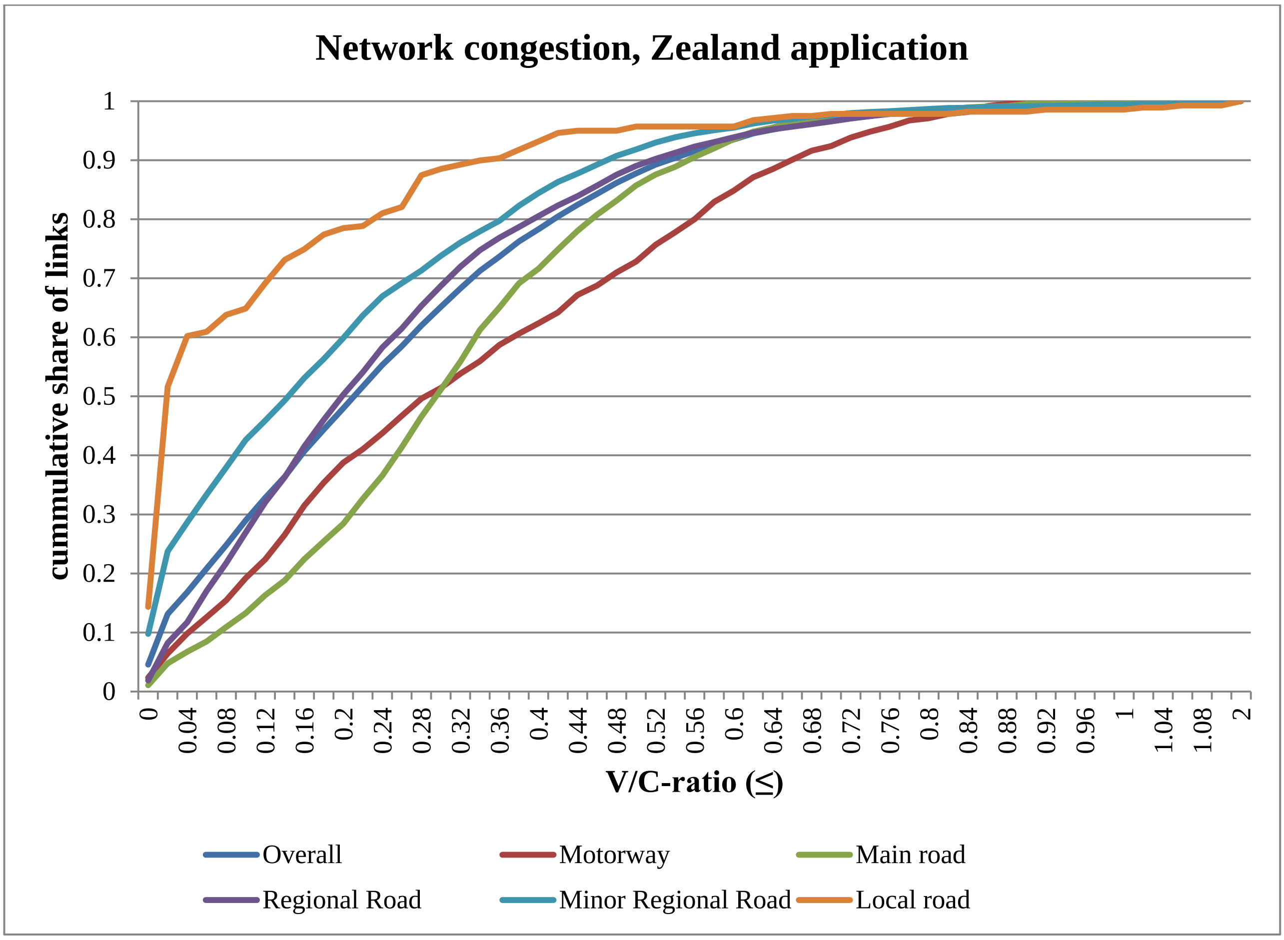


Figure 8. Network congestion by road type classifications. Cummulative share of links as function of Volume/Capacity-ratio.

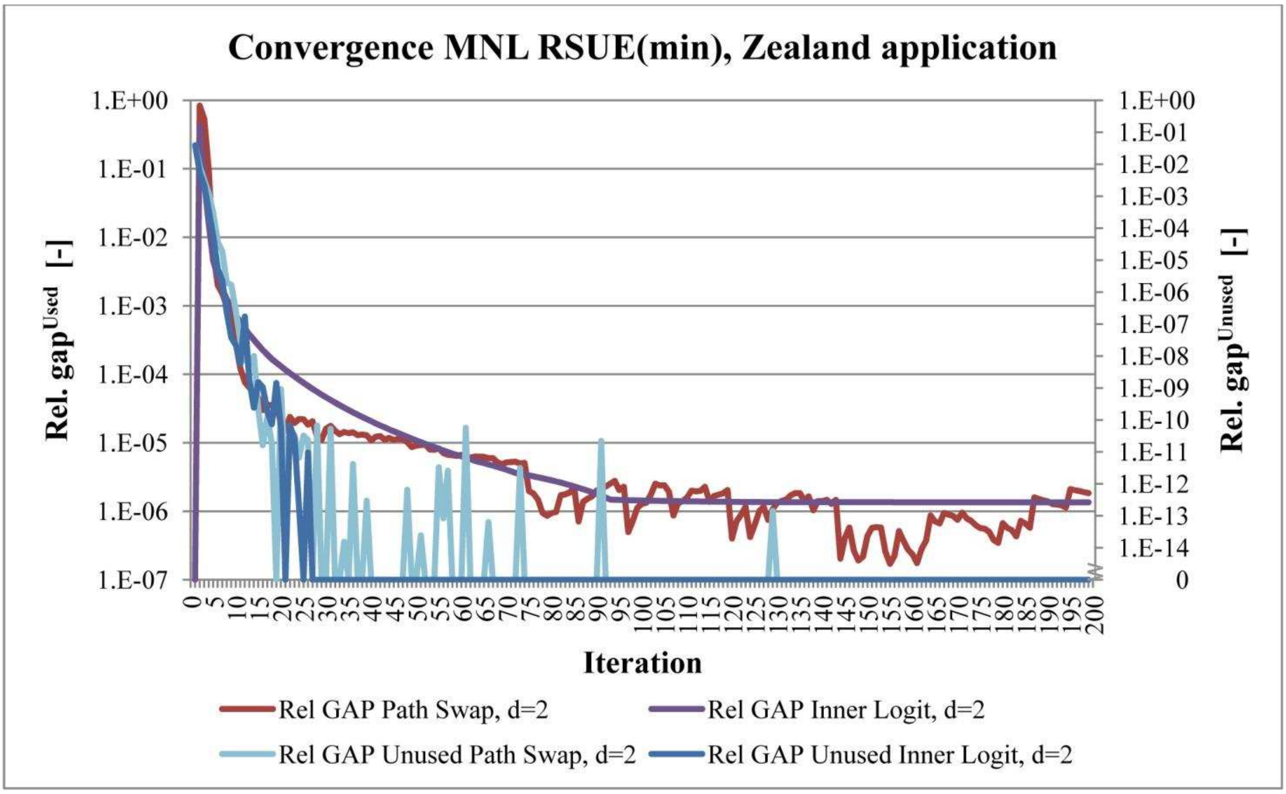


Figure 9. Relative gap measures, Zealand network application. Notice the log-scale on the vertical axis.

Table 2. Average choice set size and distribution of number of paths in the choice sets, Zealand application.

	Avg. size	Choice set size										
		1	2	3	4	5	6	7	8	9	10	11
Inner Logit d=2	2.45	479,286	452,449	357,576	200,260	92,046	30,175	8,019	1,131	256	0	2
Path Swap d=2	2.71	444,808	396,939	327,525	222,882	132,667	68,354	21,366	5,502	749	400	8

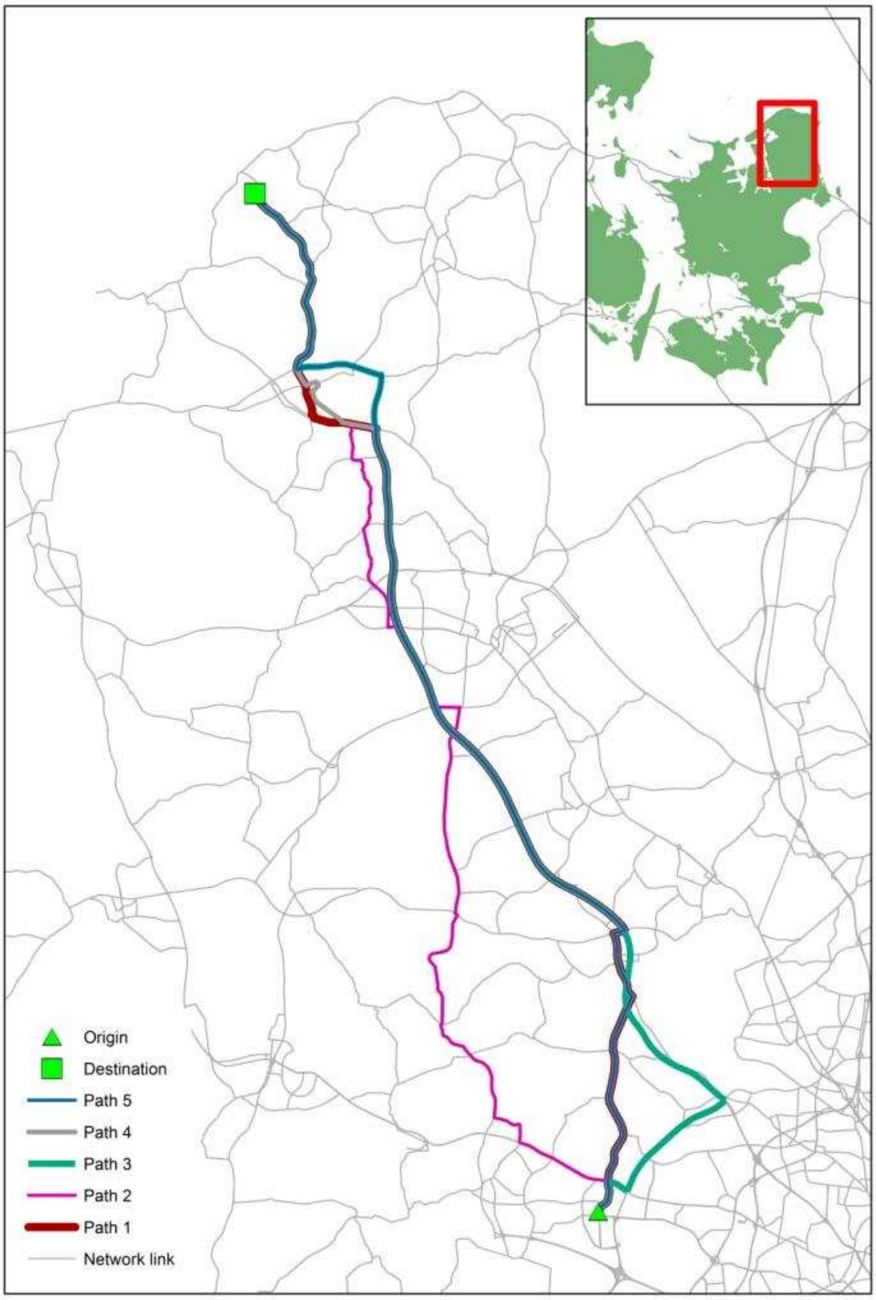


Figure 10. Illustration of generated choice set, 1 OD relation Zealand application.

Table 3. Generalized costs and flow distribution for various choice models, 1 OD relation Zealand application.

	MNL		PSL β_{PS} =-3			PSL β_{PS} =-8.5		
PathID	Gen. Cost [-]	Flow share [%]	Gen. Cost [-]	PS term [-]	Flow share [%]	Gen. Cost [-]	PS term [-]	Flow share [%]
1	126.16	22.7	126.07	0.30	21.0	125.90	0.30	16.2
2	130.40	9.7	130.49	0.78	15.2	130.65	0.78	30.9
3	129.19	12.4	129.18	0.48	14.9	129.18	0.48	18.4
4	125.28	27.1	125.20	0.28	23.7	125.05	0.28	16.6
5	125.08	28.1	125.00	0.29	25.1	124.87	0.29	18.0