# Stochastic User Equilibrium with Equilibrated Choice Sets: Part I – Model Formulations under Alternative Distributions and Restrictions

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**Abstract**

The aim of this paper is to remove the known limitations of Deterministic and Stochastic User Equilibrium (DUE and SUE), namely that only routes with the minimum cost are used in DUE, and that all permitted routes are used in SUE regardless of their costs. We achieve this by combining the advantages of the two principles, namely the definition of unused routes in DUE and mis-perception in SUE, such that the resulting choice sets of used routes are equilibrated. Two model families are formulated to address this issue: the first is a generalised version of SUE permitting bounded and discrete error distributions; the second is a Restricted SUE model with an additional constraint that must be satisfied for unused paths. The overall advantage of these model families consists in their ability to combine the unused routes with the use of random utility models for used routes, without the need to pre-specify the choice set. We present model specifications within these families, show illustrative examples, evaluate their relative merits, and identify key directions for further research.

Keywords: Choice Set; Random Utility; Traffic Assignment; Stochastic User Equilibrium.

## 1. Introduction

The Stochastic User Equilibrium (SUE) traffic assignment model was first proposed as an approach for investigating congested road networks by Daganzo and Sheffi (1977). Though not a necessary requirement, Daganzo and Sheffi gave the model theoretical appeal by basing it on the Random Utility Model (RUM), a well-known approach for modelling the discrete choice behaviour of agents. RUM permits the inclusion of random error structures which may be used to capture the uncertainty of travellers in terms of perception errors, and the uncertainty of the modeller in terms of unobserved attributes or unobserved heterogeneity. In addition, when the variance in the error terms approaches zero, SUE is able to approximate Deterministic User Equilibrium (DUE, Wardrop, 1952) to an arbitrary accuracy, and so it has a claim to be a generalisation of DUE.

Though originally proposed primarily with logit (Fisk, 1980; Miyagi, 1985), probit (Daganzo, 1979, 1982) and potentially nested logit (Williams, 1977) models in mind, the SUE approach has since been extended to accommodate a range of alternative choice models within the RUM family. However, a common feature of both the original logit-/probit-based models and their later developments/generalisations is (as remarked by Damberg et al., 1996, in relation to the logit model) 'that of every route receiving a positive flow in the equilibrium state, regardless of its travel cost'. While only considering actual minimum cost routes, as in DUE, seems difficult to justify, moving to a case where all routes are used seems equally questionable.

To investigate the seriousness of this point, consider the well-known and frequently studied Sioux Falls network (LeBlanc et al., 1975). Using the DUE link travel costs[1], we randomly generated a sample of paths which are unused in the DUE solution by replicating 10,000 times the following procedure for each Origin-Destination (OD) movement: i)

[1] Link flows and travel costs were downloaded on December 10 2012 from the web-page of Professor Hillel Bar-Gera (http://www.bgu.ac.il/~bargera/tntp/): The solution is an approximate DUE solution, with an Average Excess Cost (normalised gap) of $3.9 \cdot 10^{-15}$.

perturbing the DUE link travel costs with a Normally distributed error (with a variance = DUE link cost); ii) performing a shortest path search on the perturbed costs; iii) storing this path if not previously generated, and if its actual (unperturbed) cost is greater than the DUE route cost. For the resulting set of distinct unused paths, we calculated their relative travel cost with respect to the DUE travel cost on the corresponding OD movement. The frequency distribution of the relative travel costs for unused paths, across all OD movements, is illustrated in Figure 1.

[Insert figure 1 about here]

It can be seen that there are a significant number of unused paths with travel cost only a little greater than the travel cost of the used routes from the DUE solution. In reality, travellers may not perceive these paths as more costly (due to their mis-information or the modeller not being able to capture their personal preference structure in the specification of the cost function), and consequently we may expect travellers to use them, which they will not in the DUE model. On the other hand, there is a significant right-hand tail of unused routes which are between two and seven times as costly as the used routes from the DUE solution, yet all such routes should be used in the case of a perfect SUE solution, however circuitous and seemingly implausible is the route. While SUE will only assign small amounts of demand to the most costly routes, their existence means that at least in principle any SUE algorithm has the aim to consider all such routes, resulting in high computational requirements.

These issues are particularly evident in large-scale network models, which are becoming increasingly common, spanning the regional (e.g., Bar-Gera et al., 2012; Ben-Akiva et al., 2012; Kumar et al., 2012), national (see, e.g., Lundqvist and Mattson, 2001), and transnational scales (e.g., Burgess et al., 2008; Hansen, 2009; Petersen et al., 2009), which lead to extremely large feasible route choice sets. It is useful to imagine a network construction process for such a problem, in which we begin with a city network and then 'grow' it to a regional scale. At each step of the growth, new links are added, and every OD movement may have potential new routes which should theoretically attract flow at the SUE solution, however unattractive the route may seem. In this way, SUE is not as 'scaleable' as DUE; we can add unappealing routes to a DUE solution, and as long as the travel costs are higher than any used route, the solution will be unaffected.

In the present paper, and a companion paper (Rasmussen et al., 2014), we are particularly interested in developing a theoretical foundation for SUE-style approaches which does not suffer from such scaleability problems. Of course we are aware that practical solution algorithms, after a finite number of iterations, will typically produce an estimated SUE solution which includes only a subset of the available routes. However, such algorithms are an undesirable way of addressing the problem for several reasons. Firstly, the number of used paths is typically directly connected to the number of iterations, and so we confound the problem of convergence to the desired (SUE) conditions with the issue of which permitted paths it might be reasonable to use. Secondly, it means that the paths that are used are sensitive to the initial conditions and to the algorithm adopted. Thirdly, as we solve the problem only at the algorithmic stage, we do not have a criterion for judging whether one subset of used paths is more reasonable than another (as we do in DUE), since all that we know from SUE is that all permitted paths should be used.

An alternative way of addressing this problem would be to consider it at the model specification stage, in terms of the paths that are permitted. Bovy (2009) argues for the behavioural realism of models in which only a subset of all paths is permitted, stating that people do not choose their path from the full Universal Choice Set of alternatives, but rather from a Master Choice Set of paths considered relevant. To this end, several studies have developed methods to pre-generate a fixed Master Choice Set, such as distance-bounded enumeration (Leurent, 1997), constrained enumeration (Friedrich et al., 2001; Prato and Bekhor, 2006), probabilistic generation techniques (Cascetta and Papola, 2001; Frejinger et al., 2009) and various deterministic or stochastic shortest path algorithms (e.g., Dijkstra, 1959; Sheffi and Powell, 1982; Ben-Akiva et al., 1984).

While the objective of using more limited path sets accords with our present study, we believe there are several disadvantages to SUE models based on a pre-generated Master Choice Set:

- Path generation methods typically require as input an estimate of level-of-service variables in order to generate plausible paths. These might be estimated by observing on-street travel times, using values generated by another model (e.g., DUE travel costs), or using fixed measures as a proxy (e.g., distance, free flow speed). Regardless of the adopted method, an inconsistency problem arises, as the travel costs assumed for choice set generation will not be the same as those arisen from solving for SUE based on that choice set.
- The pre-defined choice set may exclude path options that, having solved for SUE on that choice set, would seem to be attractive based on the link travel costs of that solution. While it is undoubtedly true that travellers in real-life are not aware of all available paths, we have no independent evidence for excluding such apparently viable paths from consideration.
- Several of the methods are based on Monte Carlo simulation, which introduces a lack of repeatability in the final estimated solution due to this additional source of randomness.
- The notion of a fixed Master Choice Set implies that the set of considered paths is independent of any policy measure, whereas plausibly infrastructure improvements or tolling may make attractive some previously unattractive alternative paths or vice versa.

With these comments in mind, the present paper considers how we might consistently integrate the problem of distinguishing used and unused paths within the theoretical framework of SUE. The aim is to define not only an equilibrated flow solution but also an Equilibrated Choice Set in which the equilibrium conditions specify that some available routes should be unused, even at a perfect equilibrium (i.e., it has nothing to do with the limitations of finite iterations of a solution algorithm). In doing so, we aim to retain the basic simplicity and therefore applicability for large-scale problems that has made DUE so attractive in the past, in terms of its distinction between potentially used and definitely unused paths. We do this in a way that allows a connection with RUM and SUE. On the other hand, we aim to avoid an unappealing feature of SUE in assigning some traffic to all feasible paths, however unattractive, which is both behaviourally unrealistic and creates problems in devising convergent algorithms for large-scale networks.

The specific purpose of this paper is to set out two distinct methodological approaches for handling equilibrated choice sets in the framework of SUE, each approach leading to a family of techniques. Section 2 introduces the notation. Section 3 sets out the first approach, effectively by returning to the original foundations of SUE, and setting out the full range of possibilities it admits in terms of probability distributions. Section 4 formulates an alternative approach, namely a Restricted SUE model in which the choice set equilibration is handled through additional constraints that must be satisfied by unused alternatives. In both sections 3 and 4, simple illustrative examples are used to communicate the key concepts. In section 5 we evaluate the relative merits of these approaches, including possible extensions to the basic models, and set out the key areas for development in each. Finally, in section 6, we draw the main conclusions and wider future research directions.

## 2. Notation and Reference Definitions

We first introduce the basic common notation adopted in the paper. Most of this is the standard familiar notation for SUE problems, although two subtle and important variants are introduced and exploited later. We consider a network as a directed graph consisting of links a ($a = 1, 2, \ldots, A$) and origin-destination (OD) pairs m ($m = 1, 2, \ldots, M$). We define the demand $d_m$ for OD-pair m composing a non-negative M-dimensional vector **d**, the index set $R_m$ of all[2] simple paths (without cycles) for each OD-pair m, the number $N_m$ of paths in $R_m$ and the union R of the sets $R_m$. The route index sets are constructed so that $R = \{1,2,\ldots,N\}$, where $N = \sum_{m=1}^{M} N_m$.

Denote the flow on path $r \in R_m$ between OD-pair m as $x_{mr}$ and let $\mathbf{x} = \left( x_{11}, x_{12}, \ldots, x_{1N_1}, x_{21}, x_{22}, \ldots, x_{2N_2}, \ldots, x_{M1}, x_{M2}, \ldots, x_{MN_M} \right)$ be the N-dimensional flow-vector on the universal choice set across all M OD-pairs, so that the notation $x_{mr}$ refers to element number $r + \sum_{k=1}^{m-1} N_m$ in the N-dimensional vector **x.** Denote the flow on link a ($a=1,2,\ldots,A$) as $f_a$ and let $\mathbf{f} = \left( f_1, f_2, \ldots, f_a, \ldots, f_A \right)$ be the A-dimensional link flow-vector where $f_a$ refers to element number a in **f**.

The convex set of demand-feasible non-negative path flow solutions G is given by:

$$G = \left\{ \mathbf{x} \in \mathbf{R}_{+}^{N} : \sum_{r=1}^{N_m} x_{mr} = d_m, m = 1, 2, \ldots, M \right\} \tag{1}$$

where $\mathbf{R}_{+}^{N}$ denotes the N-dimensional, non-negative Euclidean space.

Next, define $\delta_{amr}$ equal to 1 if link a is part of path r for OD-pair m and zero otherwise. Then the convex set of demand-feasible link flows is:

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[2] We shall suppose that there are no pre-defined restrictions on the set of available routes, other than that they are acyclic, but our methods apply equally if $R_m$ is pre-defined such that other routes are excluded, leading to some smaller Master Choice Set. We have avoided referring to this, so as not to confuse the reader between such pre-defined exclusions from the choice set, and those paths that emerge as unused from the equilibration process

$$F = \left\{ \mathbf{f} \in \mathbf{R}_{+}^{A} : f_a = \sum_{m=1}^{M} \sum_{r=1}^{N_m} \delta_{amr} \cdot x_{mr}, a = 1, 2, ..., A, \mathbf{x} \in G \right\} \tag{2}$$

In vector/matrix notation, let $\mathbf{x}$ and $\mathbf{f}$ be column vectors, and define $\mathbf{\Delta}$ as the A×N-dimensional link-path incidence matrix. Then the relationship between link and path flows may be written as $\mathbf{f} = \mathbf{\Delta x}$. We suppose that the travel cost on path r for OD-pair m is additive in the link travel costs of the utilised links:

$$c_{mr}(\mathbf{x}) = \sum_{a=1}^{A} \delta_{amr} \cdot t_a(\Delta \mathbf{x}) \qquad (r \in R_m;\ m = 1,2,\ldots,M;\ \mathbf{x} \in G) \tag{3}$$

Define $\mathbf{t}(\mathbf{f})$ ($\mathbf{t} : \mathbf{R}_{+}^{A} \to \mathbf{R}_{+}^{A}$) as the vector of generalised link travel cost functions, and $\mathbf{c}(\mathbf{x})$ ($\mathbf{t} : \mathbf{R}_{+}^{N} \to \mathbf{R}_{+}^{N}$) as the vector of generalised route travel cost functions. Then the relationships between path and link flows, and between link and path costs, may be succinctly written as:

$$\mathbf{f} = \mathbf{\Delta x} \quad \text{and} \quad \mathbf{c(x)} = \mathbf{\Delta}^{T}\mathbf{t(\Delta x)} \tag{4}$$

Our particular interest is in SUE models, which capture traveller heterogeneity and mis-perceptions through first positing random utilities $U_{mr}$ for each route:

$$U_{mr} = -\theta \cdot c_{mr}(\mathbf{x}) + \xi_{mr} \qquad \left(r \in R_m; m = 1, 2, ..., M\right) \tag{5}$$

where $\boldsymbol{\xi} = \left\{\xi_{mr} : r \in R_m, m = 1, 2, ..., M\right\}$ are random variables following some given joint probability distribution, and $\theta > 0$ is a given parameter. We then define the following functions as the probability relations:

$$P_{mr}(\mathbf{c(x)}) = \Pr(-\theta \cdot c_{mr}(\mathbf{x}) + \xi_{mr} \geq -\theta \cdot c_{ms}(\mathbf{x}) + \xi_{ms}, \forall s \in R_m) \qquad \left(r \in R_m; m = 1, ..., M\right) \tag{6}$$

These relations express the probability that path r between OD-pair m will have a perceived utility greater than or equal to the utilities of all alternative paths in the universal set of routes for that OD-pair, when the random utilites are $-\theta \cdot \mathbf{c(x)} + \boldsymbol{\xi}$ and the generalised path travel costs are $\mathbf{c}(\mathbf{x})$.

The two slight variants to the standard SUE formulation are introduced for their relevance in subsequent sections. Firstly, we define:

$$\mathrm{Q}_{\mathrm{mr}}(\mathbf{c}(\mathbf{x})) = \Pr(-\theta \cdot \mathrm{c}_{\mathrm{mr}}(\mathbf{x}) + \xi_{\mathrm{mr}} > -\theta \cdot \mathrm{c}_{\mathrm{ms}}(\mathbf{x}) + \xi_{\mathrm{ms}}, \forall \mathrm{s} \in \mathrm{R}_{\mathrm{m}}) \qquad \left(\mathrm{r} \in \mathrm{R}_{\mathrm{m}}; \mathrm{m} = 1, \ldots, \mathrm{M}\right) \quad (7)$$

with the only distinction with $\mathrm{P}_{\mathrm{mr}}(\mathbf{c}(\mathbf{x}))$ being the strict inequality inside the probability statement.

Secondly, we define:

$$\mathrm{P}_{\mathrm{mr}}(\mathbf{c}(\mathbf{x}) | \tilde{\mathrm{R}}_{\mathrm{m}}) = \Pr\left(-\theta \cdot \mathrm{c}_{\mathrm{mr}}(\mathbf{x}) + \xi_{\mathrm{mr}} \geq -\theta \cdot \mathrm{c}_{\mathrm{ms}}(\mathbf{x}) + \xi_{\mathrm{ms}}, \forall \mathrm{s} \in \tilde{\mathrm{R}}_{\mathrm{m}}\right) \quad \left(\mathrm{r} \in \tilde{\mathrm{R}}_{\mathrm{m}} \subseteq \mathrm{R}_{\mathrm{m}}; \mathrm{m} = 1, 2, \ldots, \mathrm{M}\right)$$
(8)

for any non-empty subset $\tilde{\mathrm{R}}_{\mathrm{m}}$ of $\mathrm{R}_{\mathrm{m}}$ (m = 1,2,…,M). That is to say, whenever such a subset is not specified, we suppose $\mathrm{P}_{\mathrm{mr}}$ refers to the universal set.

Moreover, we define for completeness the reference concepts in network equilibrium analysis (see, e.g., Sheffi, 1985; Patriksson, 1994):

**Definition 1: Wardrop conditions**

For any OD movement, the generalised travel costs on all paths actually used are equal, and less than or equal to the cost that would be experienced by a traveller on any unused path for that OD movement.

**Definition 2: Deterministic User Equilibrium (DUE)**

The route flow vector $\mathbf{x} \in \mathrm{G}$ is a DUE solution if and only if, for some $\pi_{\mathrm{m}}$ (m = 1,2,…,M):

$$\mathrm{x}_{\mathrm{mr}} > 0 \Rightarrow \mathrm{c}_{\mathrm{mr}}(\mathbf{x}) = \pi_{\mathrm{m}} \qquad \forall \mathrm{r} \in \mathrm{R}_{\mathrm{m}} \quad \mathrm{m} = 1, 2, \ldots, \mathrm{M} \qquad (9)$$

$$\mathrm{x}_{\mathrm{mr}} = 0 \Rightarrow \mathrm{c}_{\mathrm{mr}}(\mathbf{x}) \geq \pi_{\mathrm{m}} \qquad \forall \mathrm{r} \in \mathrm{R}_{\mathrm{m}} \quad \mathrm{m} = 1, 2, \ldots, \mathrm{M} \qquad (10)$$

**Definition 3: Stochastic User Equilibrium (SUE)**

Given probability relations $\mathrm{P}_{\mathrm{mr}}(.)$ ($\mathrm{r} \in \mathrm{R}_{\mathrm{m}}$; m = 1,2,…,M) of the form given above, where $\boldsymbol{\xi}$ is a vector of continuous random variables, the route flow vector $\mathbf{x} \in \mathrm{G}$ is a SUE solution if and only if:

$$x_{mr} = d_m \cdot P_{mr}(\mathbf{c}(\mathbf{x})) \qquad (r \in R_m; m = 1, 2, ..., M)\,. \tag{11}$$

## 3. Stochastic User Equilibrium with General Error Distributions

In the following section, we begin by effectively asking the question: what is "SUE"? The reason to posit this question is that in its original specification, a much wider range of possibilities were possible than has subsequently become adopted in the literature and has been applied in practice. We present these wider possibilities in order to later evaluate them (section 5) in the context of dealing with equilibrated choice sets, since we believe they offer interesting possibilities. An important point to keep in mind, however, is that there are two separate issues which should not be confounded. Namely, in section 1 we discussed the implications of adopting unbounded distributions for the random elements of utility; in the present section we begin by considering a separate issue of whether the random elements follow a continuous or discrete distribution. Thus, we will end up with four possibilities, from the choice of one from {Bounded, Unbounded} and one from {Discrete, Continuous}, and it is such a two-way classification that we shall evaluate when considering the potential for this approach in section 5. While these possibilities can all be seen or inferred from the original paper by Daganzo & Sheffi (1977), we wish to shed light on them again in a context that does not previously seem to have been considered, namely choice set formation. Since the term SUE is now widely understood to refer to a narrower range of possibilities (i.e. those with continuous random elements, as in Definition 3), we propose below to use a different abbreviation SUEGE to emphasise this greater generality.

### 3.1 Specification and definition

The generalised cost may be expressed as a function of several factors, and if travellers were asked about their perceptions of travel time or trip length, then two things are certain:

they would not include $\infty$ as a possibility, and they would give a round (likely whole-number) answer in whatever units they use. For such perceived factors, the reality is thus discrete and bounded. Many real-life phenomena are also discrete (e.g., due to the limitations of measurement equipment) and bounded, yet we choose to represent their likely values - for mathematical convenience - by continuous and unbounded variables. For example, we cannot measure vehicle speeds to a strictly continuous precision and we know that no infinitely fast vehicles exist, yet we might represent their probability distribution as a Lognormal distribution. In a similar way, when using RUM to represent travellers' heterogeneity/mis-perception inside an SUE model, it has become standard practice to use continuous and unbounded distributions for the random error terms. However, this mathematical convenience comes at a price, namely the SUE model must assign some flow to all alternative routes in the Master Choice Set, however unattractive they are.

In order to address this issue, it is useful to first reflect on the behavioural assumptions underlying SUE, which are typically not stated in the way that the Wardrop conditions (Def. 1) are as the foundation for DUE.

**Definition 4: Stochastic User Conditions (continuous distributions)**

For any OD movement, the proportion of travellers on a path is equal to the probability that that path has a perceived utility greater than [or equal to] the perceived utility of all alternative paths.

The Wardrop conditions are written more as a plausibility test, they are not a definitive statement of the flows on alternative routes; even knowing some pattern of travel costs consistent with these conditions, we cannot even definitely state which routes are used - only those that are potentially used (those with equal minimum cost) and definitely unused (those with greater cost) - so we certainly have no chance to say with what proportions the routes are chosen. In contrast, the Stochastic User conditions in (Def. 4) make an explicit statement of the proportion of flow on alternative routes. This level of specificity emerges directly due to the assumption of a continuous error distribution in the RUM, since then the probability is zero of two routes being tied for maximum perceived utility. This also means that it is irrelevant whether we specify the condition with or without the *'[or equal to]'* part of the statement.

We then propose a more general set of conditions, which make no premise on the form of the probability distribution of random error terms:

**Definition 5: General Stochastic User Conditions**

For any OD movement, the proportion of travellers on a path is:

(i) greater than or equal to the probability that {that path has a perceived utility strictly greater than the perceived utility on all alternative paths}, and

(ii) less than or equal to the probability that {that path has a perceived utility greater than or equal to the perceived utility on all alternative paths}.

This definition captures the fact that, when there is a non-zero probability that the perceived utilities could be exactly equal, then there is not an exact equality definition for how proportion and probability may be related. In this way, (Def. 5) thus is more similar to the definition of Wardrop conditions (Def. 1).

**Proposition 1**

*The General Stochastic User conditions contain Wardrop's conditions as a special case,* when travellers make no perception errors (i.e. when discrete probability mass one is placed at zero perception error).

**Proof**

As defined earlier we suppose the deterministic part of utility being equal to $-\theta$ multiplied by the route travel cost, for scale parameter $\theta > 0$. When there are no stochastic elements in utility, comparing routes is invariant to $\theta$ and without loss of generality we may assume $\theta =$ 1. Thus, we can replace maximising perceived utilities in the General Stochastic User conditions with minimising travel costs. The probability that a 'path has a perceived utility strictly greater the perceived utility on all alternative paths' is then either 1 (if the path has a strictly lower cost than all other paths) or 0 (if its cost is greater than or equal to all other costs). Similarly, the second General Stochastic User condition translates to a probability 1 for a path if it has cost lower than or equal to the cost on all other paths, and 0 if is its cost is greater than all other paths. We can then consider three cases: (i) a path that has cost strictly greater than all other paths for that movement has flow proportion bounded below and above by 0, and therefore is zero (i.e., is unused); (ii) a path that has cost less than or equal to that of all other paths has flow proportion bounded below by 0 and above by 1 (i.e., it is potentially used, but may be unused); (iii) a path that has cost strictly less than that of all other paths has flow proportion bounded above and below by 1 (i.e., it is the only used path). Used paths are definitely in case (iii) and potentially in case (ii): in case (iii) there is only one used route, so clearly then all used routes have equal cost; if we identify the subset of routes to which case (ii) applies (which may be both used and unused), then the only way any route in this subset is no worse than any other route in the subset is if all routes in the subset have equal cost, and so certainly all used routes in this subset will have equal cost to one another. Thus, we have verified the first component of Wardrop's conditions ('travel costs on all paths actually used are equal'). Note also that, by the same argument, any unused path in this subset satisfying case (ii) must also have equal cost to the used paths in that subset; this satisfies the second component of Wardrop's conditions ('travel costs on all paths actually used are less than or equal to the cost that would be experienced by a traveller on any unused path for that OD movement'). By construction, cases (i) and (iii) also satisfy this second component, thus completing the proof. □

This result is significant as it implies that, within the framework of the General Stochastic User conditions, we have at least one special case that has unused alternatives, namely the Wardrop case. In contrast, the standard Stochastic User conditions for continuous-only error distributions cannot include this special case, and can only at best "approximate" the Wardrop conditions in some sense. This leads us to posit the following formulation of an equilibrium solution corresponding to the General Stochastic User conditions. In fact, these conditions were mentioned in passing by Daganzo & Sheffi (1977), and in the midst of a proof by Sheffi (1985), but it seems that they were never proposed or explored as a model in their own right, and it seems their possibility has since been forgotten (and certainly their relation to dealing with equilibrated choice sets has not been explored).

**Definition 6: Stochastic User Equilibrium with General Error distribution (SUEGE)**

Given probability relations $P_{mr}(.)$ and $Q_{mr}(.)$ ($r \in R_m$; $m = 1,2,\ldots,M$) of the form defined in section 2, the route flow $\mathbf{x} \in G$ is a SUEGE if and only if:

$$d_m \cdot Q_{mr}(\mathbf{c}(\mathbf{x})) \le x_{mr} \le d_m \cdot P_{mr}(\mathbf{c}(\mathbf{x})) \quad (r \in R_m; m = 1, 2, ..., M) \quad . \tag{12}$$

**Proposition 2**

(i) Supppose that, in the definition of $P_{mr}(.)$ and $Q_{mr}(.)$, the $\{\xi_{mr} : r \in R_m, m = 1, 2, ..., M\}$ are discrete random variables, each with probability mass 1 at the value of zero. Then $\mathbf{x}$ is a SUEGE if and only if $\mathbf{x}$ is a DUE.

(ii) Alternatively, suppose that the $\{\xi_{mr} : r \in R_m, m = 1, 2, ..., M\}$ are continuous random variables. Then $\mathbf{x}$ is a SUEGE if and only if $\mathbf{x}$ is a SUE.

**Proof**

The fact that SUEGE contains DUE as a special case follows directly by applying (Prop. 1) at the route flows **x** and costs **c**(**x**). The fact that SUEGE contains SUE as a special case arises due to the fact that for continuous-only distributions $P_{mr}(\mathbf{c}(\mathbf{x})) = Q_{mr}(\mathbf{c}(\mathbf{x}))$, and so the only solution to the SUEGE inequality (12) is $x_{mr} = d_m \cdot P_{mr}(\mathbf{c}(\mathbf{x}))$, i.e. SUE given by (11). □

3.2 Instances of SUEGE models

SUEGE generalises both the standard DUE and SUE models. However, the existence of examples of SUEGE with unused paths has been shown only for the special case of DUE where the stochastic element disappears, not for SUEGE models with stochastic terms that incorporate unused alternatives. The existence of such examples would satisfy the aim of the present study to have an equilibrated (but non-universal) choice set. The two examples below establish the existence of such examples for the cases of discrete and continuous bounded error distributions.

**Example 1: SUEGE with discrete bounded error distribution**

Consider a network serving an OD demand of $d_1$=100 and consisting of three parallel links/paths with separable link travel cost functions $t_1(\mathbf{f}) = 8 + f_1/10 \quad t_2(\mathbf{f}) = 18 + f_2/15 \quad t_3(\mathbf{f}) = 25 + f_3/50$, and hence route travel cost functions $c_{11}(\mathbf{x}) = 8 + x_{11}/10 \quad c_{12}(\mathbf{x}) = 18 + x_{12}/15 \quad c_{13}(\mathbf{x}) = 25 + x_{13}/50$.

Suppose that $\theta$ = 1, and that the three discrete error distributions in the RUM are statistically independent between routes, and are given by:

$$\begin{aligned}
&\Pr(\xi_{11}=0)=0.7,\ \Pr(\xi_{11}=5)=0.3,\\
&\Pr(\xi_{12}=0)=0.6,\ \Pr(\xi_{12}=10)=0.4,\\
&\Pr(\xi_{13}=-5)=0.2,\ \Pr(\xi_{13}=0)=0.8.
\end{aligned}$$

Consider the flow **x*** = (70, 30, 0). The corresponding travel costs are **c**(**x***) = (15, 20, 25).

Then:

$$\begin{aligned}
\mathrm{Q}_{11}(\mathbf{c}(\mathbf{x}^*)) &= \Pr(-\theta\cdot \mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11} > \max(-\theta\cdot \mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12}, -\theta\cdot \mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13}))\\
&= \Pr(-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11} > \max(-\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12}, -\mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13}))
\end{aligned}$$

and we consider the $2^3$ = 8 states for the three error terms across the routes, adding the appropriate joint probability whenever the condition in the $\mathrm{Q}_{11}$ probability is satisfied. The states 1-8 for $(-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11}, -\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12}, -\mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13})$ are respectively (–15,–20,–30), (–15,–10,–30), (–10,–20,–30), (–10,–10,–30), (–15,–20,–25), (–15,–10,–25), (–10,–20,–25), (–10,–10,–25). State 1 occurs (due to the independence assumption) with probability $0.7\cdot 0.6\cdot 0.2 = 0.084$, and the remaining seven state probabilities are 0.056, 0.036, 0.024, 0.336, 0.224, 0.144,0.096. Only states 1, 3, 5, 7 satisfy the condition $-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11} > \max(-\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12}, -\mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13})$, and so:

$$\begin{aligned}
\mathrm{Q}_{11}(\mathbf{c}(\mathbf{x}^*)) &= \Pr(-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11} > \max(-\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12}, -\mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13}))\\
&= \Pr(-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11} > -\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12})\\
&= 0.084+0.036+0.336+0.144 = 0.6.
\end{aligned}$$

$\mathrm{Q}_{11}(\mathbf{c}(\mathbf{x}^*)) = \Pr(-\mathrm{c}_{11}(\mathbf{x}^*) + \xi_{11} > -\mathrm{c}_{12}(\mathbf{x}^*) + \xi_{12}) = 0.084 + 0.036 + 0.336 + 0.144 = 0.6.$

For $\mathrm{P}_{11}$ we have additionally states 4 and 8 that satisfy the equality condition, hence:

$$\begin{aligned}
\mathrm{P}_{11}(\mathbf{c}(\mathbf{x}^*)) &= \Pr(-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11} \ge \max(-\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12}, -\mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13}))\\
&= \mathrm{Q}_{11}(\mathbf{c}(\mathbf{x}^*))+0.024+0.096 = 0.72.
\end{aligned}$$

By repeating this process:

$$\begin{aligned}
\mathrm{Q}_{12}(\mathbf{c}(\mathbf{x}^*)) &= \Pr(-\mathrm{c}_{12}(\mathbf{x}^*)+\xi_{12} > \max(-\mathrm{c}_{11}(\mathbf{x}^*)+\xi_{11}, -\mathrm{c}_{13}(\mathbf{x}^*)+\xi_{13}))\\
&= 0.056+0.224 = 0.28.
\end{aligned}$$

$$\mathrm{P}_{12}(\mathbf{c}(\mathbf{x}^*)) = \mathrm{Q}_{12}(\mathbf{c}(\mathbf{x}^*))+0.024+0.096 = 0.4$$

and finally:

$$Q_{13}(\mathbf{c}(\mathbf{x}^*)) = P_{13}(\mathbf{c}(\mathbf{x}^*)) = 0$$

Hence, the SUEGE conditions would require the flow **x*** to satisfy (in addition to demand feasibility):

$$100 \cdot 0.6 \le x_{11}^* \le 100 \cdot 0.72;\ 100 \cdot 0.28 \le x_{12}^* \le 100 \cdot 0.4;\ 100 \cdot 0 \le x_{13}^* \le 100 \cdot 0$$

i.e.:

$$60 \le x_{11}^* \le 72;\ 28 \le x_{12}^* \le 40;\ x_{13}^* = 0\,.$$

Since the given **x*** = (70, 30, 0) satisfies this condition, it is indeed a SUEGE solution, and it notably consists of an equilibrated but non-universal choice set. We searched for alternative SUEGE solutions on the network but found that only the **x*** above exists. It is important to note however that while only one SUEGE solution exists for the example, several SUEGE solutions may exist for other networks.

Moving on from discrete to continuous distributions, our second example is motivated by the work by Burrell (1968) that paved the way for Daganzo and Sheffi's (1977) paper. Burrell assumed the random terms to follow a uniform distribution, presumably due to the computational ease with which they may be simulated. The focus of Daganzo and Sheffi's analysis of Burrell's work was on the plausibility of the shape of the distribution, but the remedies considered (gamma, normal, etc.) were the more natural ones arising in statistics which all use unbounded distributions. The effect of Burrell's assumption on choice set formation were not considered.

**Example 2: SUEGE with continuous bounded error distribution**

Again, consider a network serving an OD demand of $d_1$=100 and consisting of three parallel links/paths, again with $\theta = 1$, and now with link cost functions:

$$t_1(\mathbf{f}) = 7.5 + f_1/10 \quad t_2(\mathbf{f}) = 15 + f_2/5 \quad t_3(\mathbf{f}) = 25 + f_3/50 \ ,$$

Now suppose three continuous bounded error distributions in the RUM, again statistically independent between routes:

$$\xi_{11} \sim \text{Uniform}(0,5),\ \xi_{12} \sim \text{Uniform}(0,10),\ \xi_{13} \sim \text{Uniform}(-5,0)\ .$$

Consider the flow allocation $\mathbf{x^*}$ = (75, 25, 0), with $\mathbf{c}(\mathbf{x^*})$ = (15, 20, 25). In this case, the SUEGE conditions reduce to an equality-based (SUE) fixed point problem, and we need consider only the P functions:

$$P_{11}(\mathbf{c}(\mathbf{x}^*)) = \Pr(-c_{11}(\mathbf{x}^*) + \xi_{11} \geq \max(-c_{12}(\mathbf{x}^*) + \xi_{12}, -c_{13}(\mathbf{x}^*) + \xi_{13}))\ .$$

The three random utilities, $-c_{1r}(\mathbf{x}^*) + \xi_{1r}$ (r = 1,2,3), are thus distributed uniformly on the intervals (–15, –10), (–20, –10), and (–30,–25), and so for these random variables it is the case that:

$$\max(-c_{12}(\mathbf{x}^*) + \xi_{12}, -c_{13}(\mathbf{x}^*) + \xi_{13}) = -c_{12}(\mathbf{x}^*) + \xi_{12}$$

since the interval on which $-c_{13}(\mathbf{x}^*) + \xi_{13}$ is defined is greater than the interval on which $-c_{12}(\mathbf{x}^*) + \xi_{12}$ is defined. Thus:

$$\begin{aligned}
P_{11}(\mathbf{c}(\mathbf{x}^*)) &= \Pr(-c_{11}(\mathbf{x}^*) + \xi_{11} \geq -c_{12}(\mathbf{x}^*) + \xi_{12}) \\
&= \Pr(-c_{11}(\mathbf{x}^*) + \xi_{11} \geq -c_{12}(\mathbf{x}^*) + \xi_{12} \,|\, -c_{12}(\mathbf{x}^*) + \xi_{12} \leq -15) \cdot \Pr(-c_{12}(\mathbf{x}^*) + \xi_{12} \leq -15) \\
&+ \Pr(-c_{11}(\mathbf{x}^*) + \xi_{11} \geq -c_{12}(\mathbf{x}^*) + \xi_{12} \,|\, -c_{12}(\mathbf{x}^*) + \xi_{12} \geq -15) \cdot \Pr(-c_{12}(\mathbf{x}^*) + \xi_{12} \geq -15) \\
&= 1 \cdot 0.5 + 0.5 \cdot 0.5 = 0.75
\end{aligned}$$

exploiting the uniformity of the distributions.

Following the same logic:

$$\Pr(-c_{12}(\mathbf{x}^*) + \xi_{12} \geq -c_{11}(\mathbf{x}^*) + \xi_{11}) = 1 - 0.75 = 0.25 = P_{12}(\mathbf{c}(\mathbf{x}^*)) \quad \text{and}$$
$$P_{13}(\mathbf{c}(\mathbf{x}^*)) = 0$$

thus confirming that **x*** = (75, 25, 0) is a SUEGE solution, and it notably consists of an SUE over an equilibrated but non-universal choice set. We have searched numerically (by grid search) for other SUEGE solutions on the network, but did not find any. Therefore we believe the solution **x*** above to be a unique solution in the example.

In conclusion, we have defined in the present section a formulation of SUE that permits both discrete and continuous, bounded and unbounded error distributions. We have shown that includes both Wardrop's conditions/DUE and traditional SUE as special cases, and with simple illustrative examples have shown how it may indeed lead to solutions with equilibrated but non-universal choice sets.

## 4. Stochastic User Equilibrium with Restrictions

### 4.1 Specification and definition

In section 3, we presented formulations of SUE that, by moving away from the continuous/unbounded error distributions traditionally associated with RUM theory, were able to capture equilibrated, non-universal choice sets within an SUE framework. A disadvantage of this approach, however, is precisely the fact that it departs from the well-understood and well-researched range of choice models that incorporate continuous, unbounded distributions, such as logit, probit, path-size logit, and so forth. Therefore, as an alternative to the SUEGE models introduced in section 3, in the present section we explore the formulation of models that on the one hand retain the connection to these well-researched methods of representing choice across the used routes, but on the other hand includes equilibrium conditions that distinguish used and unused routes.

The inspiration for these conditions derives from the Wardrop conditions; in order to understand this connection, it is helpful to first write the Wardrop conditions in a slightly alternative, but equivalent form:

**Definition 7: Wardrop conditions (alternative form)**

For each OD movement:

i) the generalised travel costs on all paths actually used are equal;

ii) *the 'reference cost' is equal to the cost on any used* path;

iii) the cost which would be experienced by a traveller on any unused path is greater than or equal to the reference cost as defined in ii).

This alternative definition introduces the notion of a 'reference cost' as a single value representing all used paths, as the benchmark against which to judge unused alternatives. For the Wardrop conditions, it is very clear what this reference cost must be, since all used paths have the same travel cost, and so introducing it seems an unnecessary complication. However, this turns out to be a key element to defining our new conditions, in a situation where the used paths have unequal travel costs. These conditions are a combination of (Def. 4) and (Def. 7):

**Definition 8: Φ-Restricted Stochastic User Conditions[3]**

For each OD movement:

i) the proportion of travellers on any <u>used</u> path is equal to the probability that that path has a perceived utility greater than or equal to the perceived utility of all alternative <u>used</u> paths;

ii) *the 'reference cost' is a value uniquely defined by some relationship* Φ to the travel costs on the <u>used</u> paths;

[3] Note that in the Restricted Stochastic User models we shall only consider the case of continuous and unbounded error distributions, since the aim is to utilise standard RUM specifications.

iii) the travel cost which would be experienced by a traveller on any unused path is greater than or equal to the reference cost as defined in ii).

In comparison with the Stochastic User conditions (Def. 4), it can be seen that condition (i) above overcomes one of the main limitations, in that they may only apply to a sub-network of the paths available. This is also true of SUE models applied to a pre-defined Master Choice set, but the key difference in (Def. 8) is that, at equilibrium, conditions (ii)/(iii) must be simultaneously satisfied alongside condition (i). That is to say, given that perceived utility is an affine function of travel cost, plus random errors, the three conditions above must be consistently satisfied, at the same travel cost levels. Thus, they do indeed yield an alternative mechanism for defining equilibrated, non-universal choice sets in an SUE framework. It is also worth remarking that the Φ-Restricted Stochastic User conditions, though owing their inspiration partly to the Wardrop conditions (Def. 7), are not as tightly defined, since there exist several alternative, plausible ways for defining the reference costs in condition (ii). That is to say, (Def. 8) defines a class of conditions that is as wide as the ways in which the relationship Φ may be defined.

A final remark on (Def. 8) is that these criteria may, in principle, be applied to a flow allocation produced by any method, and reported as a plausibility measure of the resulting flow pattern. For example, they could be applied to the iterations of a conventional path-based SUE solution algorithm, in which at any iteration typically only a subset of the available paths are used, as a sensible guide to whether some important plausible paths may still need to be included. Alternatively, they could be applied to an estimated SUE solution based on a pre-defined Master Choice Set, in this case as a measure of the extent to which the estimated equilibrium costs on the network support the assumed Master Choice Set. However, henceforth in this section we explore the properties of an equilibrium model in its own right, based on the Φ-Restricted Stochastic User Conditions:

**Definition 9: $\Phi$-Restricted Stochastic User Equilibrium (RSUE($\Phi$))**

Suppose that we are given a collection of continuous, unbounded random variables $\{\xi_{mr} : r \in R_m, m = 1,2,...,M\}$ defined over the whole choice set $R_m$, and that for any non-empty subsets $\tilde{R}_m$ of $R_m$ (m = 1,2,…,M), probability relations $P_{mr}(\mathbf{c} | \tilde{R}_m)$ are given $\tilde{R}_m$ by considering the relevant marginal joint distributions from $\{\xi_{mr} : r \in R_m, m = 1,2,...,M\}$.

The route flow $\mathbf{x} \in G$ is a RSUE($\Phi$) if and only if for all $r \in R_m$ and m = 1,2,…,M:

$$x_{mr} > 0 \quad \Rightarrow \quad r \in \tilde{R}_m \quad \wedge \quad x_{mr} = d_m \cdot P_{mr}(\mathbf{c}(\mathbf{x}) | \tilde{R}_m) \tag{13}$$

$$x_{mr} = 0 \quad \Rightarrow \quad r \notin \tilde{R}_m \quad \wedge \quad c_{mr}(\mathbf{x}) \geq \Phi\left(\left\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\right\}\right) \tag{14}$$

The RSUE($\Phi$) conditions ensure that the restricted choice set contains only the used paths and that the Φ-Restricted Stochastic User conditions hold. Comparing the RSUE formulation (Def. 9) to that for DUE (Def. 2), it can be seen that there are similarities: they both have one statement concerning utilised paths and one statement concerning non-utilised paths, but present several important differences.

Firstly, comparing the conditions on used paths, there is the use of perceived utility in RSUE($\Phi$) rather than actual travel costs in DUE; in this way, RSUE overcomes the main limitation of DUE, as it accounts for perception errors of path attributes by allowing traffic to be distributed to non-minimum cost paths, in order that the SUE conditions are satisfied on the restricted choice set of used paths.

Secondly, in the RSUE conditions we have a choice of how to define the operator Φ, whereas in the DUE model we do not. In fact, in the DUE model the variable $\pi_m$, even if defined as a free variable, must at equilibrium equal the travel cost on any used path for OD movement m, and we do not need to add any additional constraint to ensure that $\pi_m$ is related to the path costs in this way. In the RSUE model, no such condition emerges, and we then need an explicit definition of how the reference OD travel cost is related to the path travel costs of used paths. In addition, since in RSUE used paths will typically not have the same travel cost, there is some leeway in how precisely to define Φ.

It should be noted that, in the RSUE definition (Def. 9), we consider only RUM models with continuous and unbounded error distributions. As noted in section 3, under such an assumption all alternatives in the RUM (in this case, those in $\tilde{R}_m$) will have a non-zero probability of being chosen. Thus, condition (14) will never be relevant for a path that is subject to the RUM, i.e. in $\tilde{R}_m$, since such a path will always attract a positive flow. This makes the separation of used/unused paths coincide with the separation of those paths subject to the RUM and not subject to it.

A final remark is on the relation of the RSUE($\Phi$) model to conventional notions of equilibrium in networks. Unlike the SUEGE model, the RSUE($\Phi$) model does not contain DUE as a special case, in spite of the similarities in the specification of RSUE($\Phi$) and DUE. This is due to the fact that we restrict the attention in RSUE($\Phi$) to choice models which have continuous random utilities on the used paths, and thus the probability of two paths being exactly equal in terms of perceived utility is zero, whatever continuous distribution is adopted for the error terms. RSUE($\Phi$) does, however, contain SUE as a special case (regardless of the specification of $\Phi$). This may be seen by setting $\tilde{R}_m = R_m$ in the RSUE definition (Def. 9), meaning that there are no paths for which condition (14) is tested, and condition (13) is simply an SUE condition on the universal choice set. This is true for any problem, and therefore we can guarantee existence of at least one RSUE($\Phi$) solution by exactly the same conditions as those that guarantee existence of a SUE solution. In particular, Cantarella (1997) proposed a Fixed-Point formulation of SUE, and using Brouwer's theorem he showed that a solution exists if the choice function and the cost-flow functions are continuous, the link flow feasible set F is non-empty (i.e., at least 1 path exists between OD-pairs m for which $d_m>0$), compact and convex, and the link flows resulting from the flow network loading map (expressing link flows in terms of link costs) are always feasible.

4.2 Instances of RSUE($\Phi$)models

A key question that appears is the definition of Φ. Since in condition (14) the actual travel cost on an unused alternative must be compared with the actual travel costs on used alternatives, and since these unused alternatives are not subject to the random utility specification, it seems reasonable that Φ must map to something that makes sense in terms of the actual travel costs (rather than the randomly perceived utilities). Thus, while it might seem a possibility, it is not so sensible that Φ is a satisfaction function (expected maximum perceived utility, such as logsum for multinomial logit) over the used alternatives, as then we are in the 'scale' of perceived utility as opposed to actual travel cost. An alternative, then, might be to define Φ as the average or median travel cost of the used alternatives, but there are surely many possibilities that might be explored. In our case, we focus on two example possibilities (without wishing to rule out others), each seemingly having its own attractive features.

The two particular examples are the RSUE(min) model, obtained by defining for any non-empty set B:

$$\Phi(B) = \min\{b : b \in B\} \tag{15}$$

and the RSUE(max) model, obtained by defining:

$$\Phi(B) = \max\{b : b \in B\} \ . \tag{16}$$

An attraction of the RSUE(min) model is that, apparently, it leads in the direction of a computationally tractable method. The reason for believing this is as follows. With the min operator in equation (14), then if we are given some candidate flow pattern, and wish to verify whether it satisfies condition (14), then we have a form which is relatively easy to verify using standard computational tools for networks. In particular, given some path flow allocation and resulting network link costs, one can use some standard shortest path algorithm (for each OD movement) to identify the minimum cost path of any kind on the network. If the cost on this is (strictly) less than the cost on the currently minimum cost used route (for the corresponding OD-pair), then condition (14) is not satisfied.

Thus, the RSUE(min) model is in some sense the logical combination of DUE and SUE. However, it has a disadvantage in that it allows for traffic to be assigned to paths with actual travel costs greater than the actual travel costs of paths which are not utilised. From a behavioural point of view, one might question the plausibility of this, and in this respect the RSUE(max) model has an advantage.

The RSUE(max) model requires that no path is unutilised if it has an actual travel cost that is lower than or equal to the actual travel cost on the longest utilised path. While this seems behaviourally more defensible, it may lead to a less tractable computational model. Certainly, property (18) is more difficult to verify from a computational perspective for the RSUE(max) model than it is for RSUE(min), yet still there are standard network analysis tools for doing so. In particular, given some path flow allocation and the resulting network link costs, a standard tool can be used (for each OD movement) to identify the current k shortest paths (where k is the number of used paths). If there among these exists any currently unused path on which the cost is (strictly) less than the cost on currently maximum cost used route (for the corresponding OD movement), then condition (14) is violated. Clearly, the computational effort involved in solving k-shortest path problems and identifying any unused paths among these is significantly greater than that required for solving standard shortest path problems, and so verifying that the RSUE(max) conditions are satisfied is much more demanding than the verification of the RSUE(min) conditions.

**Proposition 3**

Any RSUE(max) solution is also a RSUE(min) solution. An RSUE(min) solution may not, however, necessarily fulfil the RSUE(max) conditions.

**Proof**

Suppose a flow allocation satisfies the RSUE(max) conditions. Then from conditions (14) when $\Phi$ is the max operator, any unused path must have a travel cost greater than or equal to the maximum cost used path. By definition, the maximum cost used path must have cost at least as great as the minimum cost used path, and so property (14) is also satisfied when instead $\Phi$ is the min operator. Property (13) of RSUE(max) is the same regardless of the choice of $\Phi$, and so we have shown that the flow allocation must also satisfy the RSUE(min) conditions. For the converse situation, suppose that a flow allocation satisfies the RSUE(min) conditions, and in addition has an unused path which has a cost less than the maximum cost of any used path. Then the RSUE(max) conditions are violated as illustrated in the following Example 3. □

**Example 3**

In this example we explore the multiplicity of solutions in a simple example in which we can exhaustively check the conditions for all non-empty subsets $\tilde{R}_m$ of the universal choice set $R_m$. We illustrate that RSUE solutions do indeed exist with an equilibrated but non-universal choice set and that RSUE(min) solutions may violate the RSUE(max) conditions.

Consider the network also considered in Example 1 and Example 2, serving an OD demand of $d_1$=100 and consisting of three parallel links/paths, now with link cost functions $t_1(\mathbf{f}) = 8 + f_1/10 \quad t_2(\mathbf{f}) = 13 + f_2/15 \quad t_3(\mathbf{f}) = 15 + f_3/50$ , Suppose that the choice model for used routes is a multinomial logit model with $\theta = 1$. With such a small network, it is possible to identify all 7 possible choice sets, and for each choice set to find an SUE solution by some traditional path-based solution method. We may then subsequently check each of these 7 possibilities with respect to the RSUE conditions. Clearly such an exhaustive search of possible choice sets would be infeasible for large-scale networks, but this example allows investigating the existence and multiplicity of RSUE solutions. The solution method is a path-based MSA (Sheffi and Powell, 1982) with 10,000 iterations and the solutions are shown in Table 1.

[Insert table 1 about here]

For all cases, SUE has been found among utilised paths. This means that the first condition (13) of the RSUE(min) as well as the RSUE(max) definition is fulfilled in all cases, conditional on $\tilde{R}_m$ being the set of utilised paths. The second condition is fulfilled if the actual travel cost of paths not in the choice set is not shorter than the actual travel cost on the shortest (longest) utilised path for the RSUE(min) (RSUE(max)). Note that this is always fulfilled in the case where all paths are in the choice set, and the traditional SUE will always be a RSUE solution. From Table 1 we see that there exist unused paths which are shorter than the shortest (longest) used path for configurations 2-4 and 6-7, and these do thus not fulfil the second RSUE(min) (RSUE(max)) condition and do therefore not constitute RSUE(min) (nor RSUE(max)) solutions. The violation of the RSUE(max) conditions could have also been realised by using (Prop. 3) and the knowledge that the RSUE(min) conditions are violated.

However, since $\min\left\{c_{1s}(\mathbf{x}) : s \in \tilde{R}_1\right\} = \min(14.6, 15.3) = 14.6$ for configuration 5 with paths 1 and 2 used (i.e. $\tilde{R}_1 = \{1,2\}$), and since 15.0≥14.6 then the second RSUE(min) condition is fulfilled for configuration 5 that consequently gives a RSUE(min) solution. Assuming instead a max operator for $\Phi$, then the second condition (14) requires that any unused paths have cost at least as great as the maximum cost of a used path (= 15.3 in this case), and since 15.0 < 15.3 the flow solution where paths 1 and 2 are used is not a RSUE(max) solution.

From this example we can see that RSUE solutions exist with equilibrated but non-universal choice sets, and that solutions that satisfy RSUE(min) may not satisfy RSUE(max) for a given problem. In the example we did not find any RSUE(max) solutions using a non-universal choice set. We could however imagine such a solution by adding a fourth non-overlapping route with free-flow travel cost of e.g. 20. In such a case configuration 1 (the current full SUE solution) would be a RSUE(max) with equilibrated but non-universal choice set, as a cost of 20 on the unused route would be higher than the most expensive used route (=15.5) causing the second RSUE(max) condition to be satisfied.

In the example we have focused on the RSUE(min) and RSUE(max), but surely other formulations of $\Phi$ could be investigated. One such could be the RSUE(avg) discussed earlier, where the operator is average travel cost of the used alternatives. This seems to 'mediate' between the RSUE(min) and RSUE(max) by putting a stricter condition on the cost on unused paths than the RSUE(min), however not as strict as the RSUE(max). In the example, configurations 1 and 5 satisfies the RSUE(avg) conditions, thus coinciding with the possible RSUE(min) solutions. It is important to note that RSUE(min) solutions does not always fulfil the RSUE(avg) conditions; Imagine adding a fourth alternative with free-flow travel cost of 14.8. In such a case configuration 5 using paths 1 and 2 would still be a RSUE(min) solution but not a RSUE(avg) solution, as the operator specifies a threshold cost of 14.95.

## 5. Analysis of SUEGE and RSUE Models

After presenting two alternative frameworks and models for representing route choice based on RUM theory, both of which lead to an equilibrated but potentially non-universal choice set, we analyse and compare these approaches in terms of three key areas: (i) applicability and potential for calibration, (ii) theoretical issues, and (iii) potential for devising solution methods for large-scale networks.

5.1 Applicability and Potential for Calibration

The applicability and calibration potential of SUEGE and RSUE models requires considering first what the ‘stochastic’ terms might be useful for representing. In particular, we shall make a case that each model may be suitable for representing quite different kinds of variation.

Considering the SUEGE models, the pertinent cases are those in which the error distribution in the random utilities is bounded (either discrete or continuous). In such cases, the lower and upper bounds of the random utility distribution, and hence those of the error term, play the key role in distinguishing used from unused routes. The important question then arises: in what circumstances might it be (a) reasonable to assume such bounds, and (b) possible to estimate the bounds. This connects directly to what the error term may be capturing, for which there are several possibilities (see Watling et al., 2013, for a further discussion of this issue in a somewhat different context).

Two possibilities are that the random error is capturing (i) unobserved factors that might affect route choice, other than those measured in the generalised travel cost, or (ii) unobserved heterogeneity across the population of travellers. In these cases, we cannot directly estimate the bounds on the distributions of the error term, but supposing for example that we had data on routes actually chosen by OD movement, an indirect method could be (i) specify the error distribution as a bounded parameteric family of distributions, and (ii) estimate the parameters by finding the best fit between the observed route distributions and that distribution of routes that would occur under the SUEGE model with those parameters. Such a problem could, for example, be addressed by using some least squares metric as a goodness-of-fit measure, and then maximising the fit subject to a SUEGE constraint (a kind of Mathematical Program with Equilibrium Constraints, MPEC).

Although this method (for indirectly calibrating unobserved phenomena) sounds potentially feasible, there are several significant difficulties with it. It is not clear what goodness-of-fit measure would truly focus on the boundary of distinguishing lightly used from unused routes. If our focus is on identifying which routes are not used at all, it seems we would need to observe all trips, rather than estimating route proportions from a sample. Moreover, it is still rather unusual to have route-level information available for network problems; with only link flow information, we are one stage further removed from the 'direct observation' described above.

A third possible interpretation of the random error in the utilities would be that they are a representation of perception errors of the travellers. The SUEGE model with discrete bounded distributions seems particularly attractive to represent this, as travellers will typically think of travel times or other measures in whole units (e.g. minutes), or even further rounded. It seems possible to determine the distributions to use through experiments where a sample of travellers are asked to estimate travel times (both pre- and post-trip), which can then be compared with independent measurements of the actual trip times.

Considering the RSUE models, there are two quite distinct mechanisms to consider for calibration: the RUM for dispersing traffic among used routes, and the operator $\Phi$ for distinguishing used from unused routes. The RUM element is effectively the same as for conventional SUE models, and so this element effectively offers no new challenge over-and-above calibrating a conventional SUE model based on RUM models. The new challenge for RSUE is in specifying a sensible $\Phi$ operator that allows the modeller to restrict the assumed choice set to a manageable size, and is not about traveller perception or unobserved factors.

In this respect, we can make comparisons with actual travel costs (e.g. travel times) observed on the network, separately from the issue of perception.

For example, if we suppose that the only element of travel cost were travel time, and that we have observations of link travel times across a network, then we could assume and apply an operator $\Phi$ to the observed travel times, and compare the ranking of routes with the distinction between used and unused routes as predicted by a RSUE($\Phi$) solution. If we also have observations of actual routes chosen by travellers, then we might verify whether the actually chosen routes are all used routes in an RSUE solution, what is the distribution of travel costs across these chosen routes, and how does it compare with the distribution of travel costs on used routes in the RSUE solutionfor a given problem. The answers to these questions vary under different assumed $\Phi$ functions, and under different assumed choice models for a given problem.

Summarising, there is potential for calibrating both SUEGE and RSUE models, especially when more route-level data become available in the future through increasingly popular and ever more precise tracking devices. In both cases, this tracking needs to penetrate to quite a large fraction of travellers, if we are truly to distinguish lightly used from completely unused routes. The SUEGE models seem particularly appropriate for modelling traveller perception errors, whereas RSUE is probably more suited to capturing unobserved factors and unobserved heterogeneity. The RSUE models seem more straightforward to apply in the short term, as extensions of existing and calibrated SUE models, especially if supplemented with some information on routes actually chosen.

## 5.2 Theoretical Considerations

To the best of the authors' knowledge no work exists, establishing the existence and uniqueness of SUEGE solutions, other than that for the traditional SUE case of unbounded and continuous distributions (e.g., Cantarella, 1997). In the case of RSUE models, we can guarantee existence of at least one solution under the same condition as continuous, unbounded SUE solutions exist (Cantarella, 1997), but what we are really interested in is the existence of other RSUE solutions which do not use the full choice set.

Thus while it seems behaviourally implausible that travellers are error-free and identical in their perceptions of travel cost (as in DUE), or use all available paths (as in SUE), the price we pay for the additional plausibility in SUEGE or RSUE is a model with non-unique solutions, which may not be so convenient for cost-benefit analysis. In defence of these new models we would offer several arguments. Firstly, uniqueness of DUE and SUE solutions is only known under quite limited circumstances, which break down when we have problems with non-additive path costs, within-day dynamics, junction interactions, multiple vehicle types, responsive control or non-separable/monotone variable demand models. In fact examples of multiple solutions are known to exist in many such cases (Watling, 1996; Iryo, 2011). Secondly, there has been recent work that has deliberately sought to generate non-unique solutions, through multi-objective route choice modelling (Wang and Ehrgott, 2013). While the intention of the present study is to postulate more realistic models, rather than non-unique solutions, we may see the models as a way of generating reasonable candidate solutions. Thirdly, the SUEGE and RSUE models contain SUE or DUE as special or limit cases and hence, in a calibration process, it is legitimate to consider whether SUE or DUE offer a better fit to observations. If they do, they may be preferred, and so we do not rule out their use. Fourthly, traditional equilibrium models are heavily calibrated on link flow data (even the OD matrix). When the non-uniqueness we are referring to leads to different link flow solutions, then we may (at the calibration stage) choose between alternative candidate SUEGE/RSUE solutions based on such conventional link data. Fifthly, in the future, as it becomes more typical to have access to data from mobile/GPS devices, then the focus of calibration of equilibrium models may switch to a more route-based one. In such a case, there is an even better chance to resolve the non-uniqueness at the calibration stage, given several candidate solutions with different equilibrated choice sets – see the discussion of section 5.1.

A wider issue that has had great influence on the possibility to establish theoretical properties of network equilibrium models (as well as to devise efficient solution methods) is the kind of mathematical formulation adopted (e.g., convex optimisation problem, variational inequality, fixed point problem). It is therefore appropriate to examine the formulations adopted for the models presented. In the case of SUEGE with a continuous bounded error distribution, the resulting problem is also an SUE problem, and so is a fixed point condition. Alternative formulations of SUE (e.g., optimisation problem) have been established under certain assumptions on the error terms, but these are specific to particular models with unbounded errors, and so do not transfer unless specifically proven. The case of a discrete error SUEGE distribution is more complex. It is certainly possible to convert the inequality constraints (12) into equalities, by adding appropriate slack variables (two per route), and thus define a fixed point equality condition. However, since we suspect there may be several solutions, an alternative approach would be to consider the inequalities (12) as constraints to an optimisation problem, namely to minimise g(**x**) subject to **x** satisfying (12). With g(**x**) = constant, solutions to such a problem define the full solution set, while g(**x**) = $x_r$ (or g(**x**) = -$x_r$) would define for some r the lower (or upper) bound on the flow on route r in the solution set, and $g(\mathbf{x}) = \sum_{m=1}^{M} \sum_{r=1}^{R_m} \delta(x_r)$ (where δ(y) = 1 if y > 0, 0 otherwise) would define a SUEGE solution using the fewest number of routes.

In the case of RSUE models, the definitions (13)/(14) appear rather complex, but a more parsimonious formulation can be gained using the δ(.) indicator function introduced above. We further denote $\boldsymbol{\delta}(\mathbf{y}) = (\delta(y_1), \delta(y_2), ..., \delta(y_n))$ and then re-write (14) as:

$$(1-\delta(x_{mr})) \cdot (c_{mr}(\mathbf{x}) - \Phi(\mathbf{c}(\mathbf{x}), \boldsymbol{\delta}(\mathbf{x}))) \geq 0 \qquad \forall r \in R_m, m = 1, 2, ..., M$$

In vector notation:

$$(\mathbf{1}-\boldsymbol{\delta}(\mathbf{x}))\circ(\mathbf{c}(\mathbf{x})-\boldsymbol{\Phi}(\mathbf{c}(\mathbf{x}),\boldsymbol{\delta}(\mathbf{x})))\geq\mathbf{0}$$

where the symbol $\circ$ denotes the Hadamard product (element-wise multiplication):

$$\mathbf{a}\circ\mathbf{b} = \mathrm{diag}(\mathbf{a})\,\mathbf{b}\,.$$

Similarly we can re-write (13) as (with a different definition of $P_{mr}$ note):

$$\delta(x_{mr})\cdot(x_{mr}-d_m\cdot P_{mr}(\mathbf{c}(\mathbf{x}),\boldsymbol{\delta}(\mathbf{x})))=0 \qquad \forall r\in R_m, m=1,2,\ldots,M$$

where for all $r \in R_m$ (i.e. not just those that are subject to the RUM) and m= 1, 2, ..., M

$$P_{mr}(\mathbf{c}(\mathbf{x}),\boldsymbol{\delta}(\mathbf{x}))\equiv\delta(x_{mr})\cdot\Pr(-\theta\cdot c_{mr}(\mathbf{x})+\xi_{mr}\geq-\theta\cdot c_{ms}(\mathbf{x})+\xi_{ms}\quad\forall s\in R_m \text{ such that } \delta(x_{ms})=1)$$

or in vector notation:

$$\boldsymbol{\delta}(\mathbf{x})\circ(\mathbf{x}-\boldsymbol{\Gamma}\mathbf{d}\circ\mathbf{P}(\mathbf{c}(\mathbf{x}),\boldsymbol{\delta}(\mathbf{x})))=\mathbf{0}$$

where $\boldsymbol{\Gamma}$ is the path-OD incidence matrix (i.e a N×M-dimension 0-1 matrix with a 1 only if a path is relevant to an OD movement). Overall, then, we can write RSUE($\boldsymbol{\Phi}$) as the two conditions:

$$\boldsymbol{\delta}(\mathbf{x})\circ(\mathbf{x}-\boldsymbol{\Gamma}\mathbf{d}\circ\mathbf{P}(\mathbf{c}(\mathbf{x}),\boldsymbol{\delta}(\mathbf{x})))=\mathbf{0} \tag{17}$$

$$(\mathbf{1}-\boldsymbol{\delta}(\mathbf{x}))\circ(\mathbf{c}(\mathbf{x})-\boldsymbol{\Phi}(\mathbf{c}(\mathbf{x}),\boldsymbol{\delta}(\mathbf{x})))\geq\mathbf{0} \tag{18}$$

In this formulation, we have a combination of a complementarity kind of condition (as in DUE) and a fixed point condition (as in SUE). We can see special cases as follows:

- If $\boldsymbol{\delta}(\mathbf{x}) = \mathbf{1}$ then the second constraint is redundant, and the first reduces to the SUE condition on the universal path set ($\mathbf{x} = \boldsymbol{\Gamma}\mathbf{d}\circ\mathbf{P}(\mathbf{c}(\mathbf{x}),\mathbf{1})$).
- If $\boldsymbol{\delta}(\mathbf{x}) = \boldsymbol{\delta}_0 \neq \mathbf{1}$ then the first condition above represents a pre-defined (non-equilibrated) restricted choice set.
- If we terminate an SUE solution algorithm (on the universal choice set) after a finite number of steps, at some point $\mathbf{x}_{est}$, then in most practical cases not all routes will be used, which means $\boldsymbol{\delta}(\mathbf{x}_{est}) \neq \mathbf{1}$, i.e. not an actual SUE solution is found. In this case,

similarly to the case of a pre-defined choice set, we only satisfy (approximately) the first condition, there is no analogous condition to the second for those routes not used.

It should be noted that this is a fixed point problem in $\mathbf{x}$, but a rather difficult one to solve given that $\delta(\mathbf{x})$ maps onto integers ($\mathbf{x}$ is real, but the functions are non-smooth, unlike traditional SUE).

5.3 Computational Considerations

A key question is to what extent the proposed models could be applied in large-scale networks. In the case of SUEGE with a continuous, bounded error distribution, an obvious general candidate is the Method of Sucessive Averages (MSA) algorithm (Sheffi and Powell, 1982). However, there appears to be no existing proof of convergence of such an algorithm that covers the case of bounded error distributions, and so the approach would be heuristic. An alternative possibility arises by noting that the set of used routes in a continuous, bounded SUE problem is affected only by the bounds, and these bounds will change only in response to the flows through the travel cost functions; thus, for fixed flows, the set of used routes is fixed. Therefore, it would seem possible to devise an algorithm in which, at each iteration, updates the choice set by dropping or appending new options based on a column generation method (if needed based on the latest bounds), and then a path-based SUE problem is solved on the fixed choice set. However, this latter element would require some developments to avoid Monte Carlo methods, entailing large computational cost, in the case of general bounded distributions.

In the case of SUEGE models with a discrete error distribution, an MSA would again be a potential candidate algorithm. However, given the combinatorial nature of the discrete problem, it would seem more sensible to explore the possibility to employ deterministic algorithms to exploit such a structure, such as those specifically developed for solving shortest path algorithms with discrete distributions (e.g., Mirchandani, 1976; Fajardo and Waller, 2012) or problems associated with connectivity reliability (e.g., Bell and Iida, 1997).

In the case of RSUE models, the reformulation (17)/(18) shows that the problem contains elements of not only SUE but also DUE (complementarity), and for this reason it would seem sensible to consider the possible transfer to the RSUE case of developments in path-based DUE algorithms (e.g., Larsson and Patriksson, 1992; Chen et al., 2002; Carey and Ge, 2012). On the other hand, the structure of the RSUE problem seems to lend itself well too to methods that align with its two separate conditions, which define (a) the dispersion among used paths and (b) the question of which paths are used. In this respect, we would note algorithms for solving path-based SUE problems (e.g. Xu et al., 2012), and especially those that decompose path generation and path loading (e.g., Damberg et al., 1996; Bell et al., 1997). This is an element where the RSUE model is particularly attractive (relative to SUEGE with bounded distributions), in that any sub-problems of allocating flow on restricted choice sets with continuous unbounded stochastic error distributions are problems with which the traffic assignment field has considerable familiarity.

## 6. Conclusions

The commonly used models within traffic assignment, namely DUE and SUE, have some known limitations by either allowing only routes with the minimum cost to be used (DUE) or requiring all routes to be used regardless of their costs (SUE). The paper shows how we might overcome these limitations by consistently integrating the problem of distinguishing used and unused paths within the concept of SUE. This led us to set out two distinct, alternative methodological approaches to addressing this problem. The two approaches define not only an equilibrated flow solution but also an equilibrated choice set in which the equilibrium conditions (and not the solution algorithm adopted) specify that some available routes could be unused at perfect equilibrium. The potential benefits of such approaches are greatest, it would seem, in large-scale regional and trans-national studies, meaning that we no longer have the choice only between DUE (which will tend to assign all-or-nothing to congested parts of such networks) and SUE (which can be computationally demanding and rather implausible, in attempting to assign some traffic to all routes).

The present study justifies, defines and illustrates both approaches as well as discusses their similarities, differences, capabilities and potentials. Also, we have outlined possible approaches for calibration and application of both methods. The RSUE seems more straightforward to apply in short term, as an extension of existing, calibrated SUE models, especially if supplemented with some information on routes actually chosen to aid in the determination of the $\Phi$ operator. In a companion paper (Rasmussen et al., 2014), we develop solution methods for generating RSUE solutions for large-scale networks, and explore the characteristics of the solutions produced. Beyond these two papers, what is required next, we believe, is a further development and study of the capabilities and potential of RSUE and SUEGE. One issue to consider lies within the possible non-uniqueness of solutions. We have outlined and discussed different possibilities for tackling this challenge in real-life applications, and believe that especially the increased availability of observed route choices can be utilised in doing so.

In terms of further applications and extensions of the methods proposed, one natural direction for the RSUE family especially would be to consider other closed-form choice models, such as the weibit (Castillo et al, 2008; Kitthamkesorn & Chen, 2014), paired combinatorial logit (Chen et al, 2014) and q-logit (Nakayama, 2013).

Finally, we would like to draw attention to the way in which our research began, namely with a re-thinking and formulation of ‘user conditions’ analogous to Wardrop’s, and we found this especially helpful in the development of our model. Indeed, in doing so we found that the oft-quoted informal description of the behaviour underlying SUE as “minimising perceived cost” was not especially helpful, yet a more formal articulation did not apparently exist (analogous to Wardrop’s very clear conditions). We believe such an approach, beginning with a development of the user conditions, is especially suitable for the traffic assignment community to exploit the insights from empirical work and behavioural studies. This is especially relevant as nowadays we have the potential to track routes through GPS or mobile phone devices, and as a result there is a rapidly growing body of evidence. While many phenomena may be location-specific, it is also interesting to look across such data sets for transferable phenomena which may be included as (potentially adjustable) rules within a new set of user conditions. The original developers of the route choice conditions underlying DUE or SUE could not have envisaged the wealth of explicit route-based data to which we now have access, and so it seems timely to reconsider these conditions in the light of such a new evidence-base.

**Acknowledgements**

The financial support of the Danish Council for Strategic Research to the project “Analyses of activity-based travel chains and sustainable mobility” (ACTUM) is gratefully acknowledged. We also thank three anonymous referees for their insightful comments, which helped us to improve two earlier drafts of this paper.

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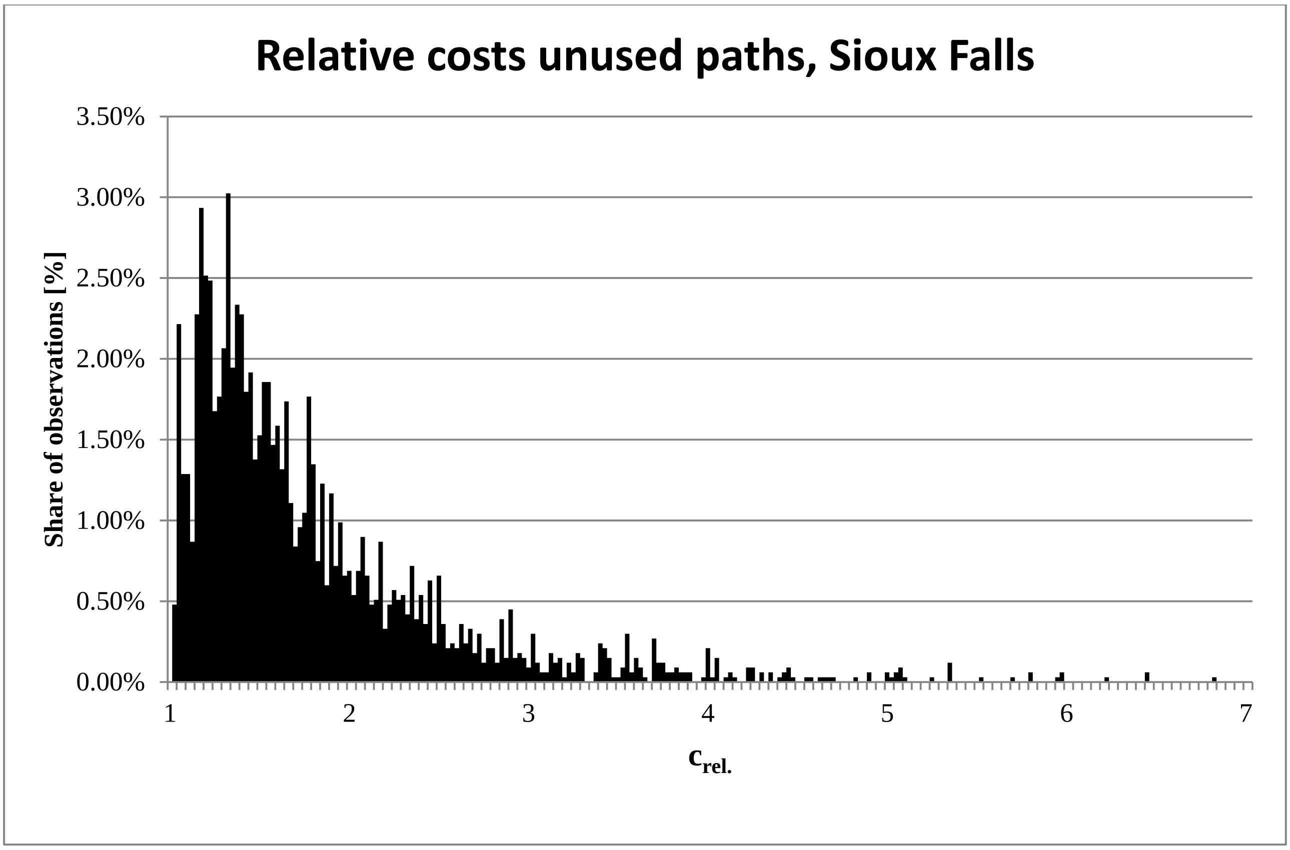


Figure 1. Distribution of relative travel costs of random sample of unused DUE paths. Vertical axis is share of sampled paths as function of path cost relative to cost on minimum cost path for that OD movement

Table 1 SUE solutions for all seven possible choice sets.

| | Choice set (included paths) | | | | | | |
|---|---|---|---|---|---|---|---|
| | {1,2,3} | {1} | {2} | {3} | {1,2} | {1,3} | {2,3} |
| | cost/flow | cost/flow | cost/flow | cost/flow | cost/flow | cost/flow | cost/flow |
| Path 1 | 13.9/59.1 | 18.0/100.0 | 8.0/0 | 8.0/0 | 14.6/66.0 | 14.9/68.5 | 8.0/0 |
| Path 2 | 14.7/26.0 | 13.0/0 | 19.7/100.0 | 13.0/0 | 15.3/34.0 | 13.0/0 | 16.2/47.4 |
| Path 3 | 15.5/14.8 | 15.0/0 | 15.0/0 | 17.0/100.0 | 15.0/0 | 15.6/31.5 | 16.1/52.6 |
| RSUE(min) | YES (=SUE) | NO | NO | NO | YES | NO | NO |
| RSUE(max) | YES (=SUE) | NO | NO | NO | NO | NO | NO |